330 2011-10-25

Note Title 2011-10-25

FP has primitive functions
and functional forms.

Functional forms (also known as combining forms) are higher-order functions, i.e., functions that take other functions as arguments. FP programs are built from primitive functions using functional forms.

Some primitive functions of Backus's FP:

tl (tail) tl : < 1 2 3 4 > ⇒ < 2 3 4 >

\+ (add)

\- (subtract)

* (times)

/ (divide)

More, like selection (i), append, etc. are on the handout

The first combining form we used is

Construction

$$[f_1, f_2, \cdots f_n] : x \equiv$$

$$\langle f_1 : x,\ f_2 : x,\ \ldots,\ f_n : x \rangle$$

Composition $f \circledcirc g$ "composition of f and g"

$$f \circledcirc g : x \equiv f : (g : x)$$

Constant $\%.x$ "the constant x"

$$\%.x : y \equiv x$$

$!f$ is defined for sequences of length one or greater.

Insert $\quad !$ $\quad$ ($!f$ "insert f")

$!f : x \equiv x$ if x is $\langle y \rangle$ (a sequence of length one)

else $(x = \langle x_1, x_2, \ldots, x_n \rangle)$

$f : \langle x_1 \;\; !f : \langle x_2, \ldots, x_n \rangle \rangle$

Example: $!+$

$!+ : \langle 1 \;\; 2 \;\; 3 \rangle \equiv +: \langle 1 \;\; !+: \langle 2 \;\; 3 \rangle \rangle \equiv$

$$\equiv +:\langle 1 \quad +:\langle 2 \quad !+:\langle 3\rangle\rangle\rangle \equiv +:\langle 1 \quad +:\langle 2 \quad 3\rangle\rangle \equiv$$

$$\equiv +:\langle 1 \quad 5\rangle \equiv 6$$

Insert is typically used to obtain a function of an arbitrary number of arguments from a function of two arguments

!+ is the function that sums the entries in a sequence of arbitrary length.

Apply to all α

the empty sequence ("nil")

$\alpha f : x \equiv$ if x is $\langle\,\rangle$ then nil

else $(x = \langle x_1, \dots x_n\rangle)$ $\langle f:x_1, \dots\dots, f:x_n\rangle$

$\{subone \equiv - \circ [id, \%1]\}$

$id : x \Rightarrow x$

$\%1 : x \Rightarrow 1$

$[id, \%1] : x \Rightarrow \langle id:x \quad \%1:x \rangle \Rightarrow \langle x \quad 1 \rangle$

$- : \langle x \quad y \rangle \Rightarrow x - y$

$- : \langle x \quad 1 \rangle \Rightarrow x - 1$

$subone : x \Rightarrow x - 1$

$\{ \text{eqzero} \equiv @ \; [id, \%\phi] \}$

$[id, \%\phi] : x \Rightarrow \langle id : x \quad \%\phi : x \rangle = \langle x \quad \phi \rangle$

$= : \langle x \quad y \rangle \Rightarrow \begin{cases} T & \text{if } x = y \\ F & \text{otherwise} \end{cases}$

$\text{eqzero} : x \Rightarrow = @ \; [id, \%\phi] : x \Rightarrow$

$\Rightarrow = : \langle x, 0 \rangle \Rightarrow \begin{cases} T & \text{if } x \text{ is zero} \\ F & \text{otherwise} \end{cases}$

$$\overbrace{(\text{eqzero} \to \%1 \,;\; \%3)}^{\text{a conditional expression}} : 0 \Rightarrow 1$$

$$(\text{eqzero} \to \%1 \,;\; \%3) : 6 \Rightarrow 3$$

$$\{\text{fact}\ (\text{eqzero} \to \%1 \,;\; * \circ [\text{id}, \text{fact} \circ \text{subone}])\}$$

Matrix multiplication

$M \times N = P$ $\quad$ M of size $a \times b$ $\quad$ $r \times s$

N of size $b \times c$

P of size $a \times c$

$$p_{i,j} = \sum_{k=1}^{b} m_{ik} \times n_{kj}$$

We represent matrices in row-major format, so

$M = \langle m_1, \ldots, m_r \rangle, \quad m_i = \langle m_{i1}, \ldots, m_{is} \rangle$

matrix multiplication $\qquad$ inner product

$$\{\mathrm{mm} \quad (\alpha\alpha\ \mathrm{ip}) \circ (\alpha\ \mathrm{distl}) \circ [1,\ \mathrm{trans} \circ 2]\}$$

i is the index function (in FP) $\quad i: \langle x_1, \ldots x_i, \ldots x_n \rangle \Rightarrow x_i$

$$[1,\ \mathrm{trans} \circ 2] : \langle m, n \rangle \Rightarrow \langle m,\ \mathrm{trans} : n \rangle$$

Ex.: $\quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \quad \mathrm{trans}\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}^T$

$$[1,\ \mathrm{trans} \circ 2] : \langle\langle\langle 1\ 2\rangle \langle 1\ 2\rangle\rangle \langle\langle 1\ 3\rangle \langle 1\ 3\rangle\rangle\rangle \Rightarrow$$

$$\langle\langle\langle 1\ 2\rangle \langle 1\ 2\rangle\rangle\ \mathrm{trans} : \langle\langle 1\ 3\rangle \langle 1\ 3\rangle\rangle\rangle \Rightarrow$$

$$\langle\langle \underbrace{\langle 1\ 2\rangle}_{m_1} \underbrace{\langle 1\ 2\rangle}_{m_2}\rangle \underbrace{\langle\langle 1\ 1\rangle \langle 3\ 3\rangle\rangle}_{n'}\rangle$$

the m_ith row of M

the 1st column of matrix N

p_i = distl : $\langle m_i, n' \rangle \Rightarrow \langle\langle m_i\ n'_1\rangle\langle m_i\ n'_2\rangle \ldots \langle m_i\ n'_s\rangle\rangle$

&distl will apply distl to every row of m.

So, p_i represents "row i of m associated with all columns of n.

Finally &&ip (= & (&ip)) applies &ip to each p_i, which in turn means that it is applied to each "row, column" pair $\langle m_i, n_j' \rangle$

Ex. $mm : \overbrace{\langle\langle\langle 1\ 2\rangle\ \langle 1\ 2\rangle\rangle\ \langle\langle 1\ 3\rangle\ \langle 1\ 3\rangle\rangle\rangle}^{\langle M, N\rangle} \Rightarrow$

$$\underset{M}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} \underset{N}{\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}} = \begin{bmatrix} 1+2 & 3+6 \\ 1+2 & 3+6 \end{bmatrix} = \underset{P}{\begin{bmatrix} 3 & 9 \\ 3 & 9 \end{bmatrix}} \quad \begin{matrix} r=s=t=2 \\ a=b=c=2 \end{matrix}$$

$\Rightarrow \ldots\ldots\ @ [1, \text{trans}@2] : \langle M, N \rangle$

$[1, \mathrm{trans} \circ 2] : \langle M, N \rangle =$

$[1, \mathrm{trans} \circ 2] : \langle\langle\langle 1 \; 2\rangle\langle 1 \; 2\rangle\rangle\langle\langle 1 \; 3\rangle\langle 1 \; 3\rangle\rangle\rangle \Rightarrow$

$\langle\langle\langle 1 \; 2\rangle\langle 1 \; 2\rangle\rangle \; \mathrm{trans} : \langle\langle 1 \; 3\rangle\langle 1 \; 3\rangle\rangle\rangle \Rightarrow$

$\langle\langle\langle 1 \; 2\rangle\langle 1 \; 2\rangle\rangle\langle\langle 1 \; 1\rangle\langle 3 \; 3\rangle\rangle\rangle$

$(\alpha \; \mathrm{distl}) : \langle\langle\langle 1 \; 2\rangle\langle 1 \; 2\rangle\rangle\langle\langle 1 \; 1\rangle\langle 3 \; 3\rangle\rangle\rangle \Rightarrow$

$$\Rightarrow \langle\langle\langle \overbrace{\langle 1\ 2\rangle}^{\text{row 1 of } m}\ \overbrace{\langle\langle 1\ 1\rangle\langle 3\ 3\rangle\rangle}^{n'}\rangle\ \langle \overbrace{\langle 1\ 2\rangle}^{\text{row 2 of } m}\ \overbrace{\langle\langle 1\ 1\rangle\langle 3\ 3\rangle\rangle}^{n'}\rangle\rangle\rangle$$

$$\Rightarrow \langle\langle\langle\langle \overset{\text{row 1 of } m}{\langle 1\ 2\rangle}\ \overset{\text{col 1 of } n}{\langle 1\ 1\rangle}\rangle\langle \overset{\text{row 1 of } m}{\langle 1\ 2\rangle}\ \overset{\text{col 2 of } n}{\langle 3\ 3\rangle}\rangle\rangle$$

$$\langle\langle\langle \underset{\text{row 2 of } m}{\langle 1\ 2\rangle}\ \underset{\text{col 1 of } n}{\langle 1\ 1\rangle}\rangle\langle \underset{\text{row 2 of } m}{\langle 1\ 2\rangle}\ \underset{\text{col 2 of } n}{\langle 3\ 3\rangle}\rangle\rangle\ \rangle \qquad C$$

$$(\&\ (\Delta\ ip)) : C \Rightarrow \langle \Delta ip : \langle\langle\langle 1\ 2\rangle\langle 1\ 1\rangle\rangle\langle\langle 1\ 2\rangle\langle 3\ 3\rangle\rangle\rangle$$

$$\Delta ip : \langle\langle\langle 1\ 2\rangle\langle 1\ 1\rangle\rangle\langle\langle 1\ 2\rangle\langle 3\ 3\rangle\rangle\rangle\rangle \Rightarrow$$

$\Rightarrow \langle\langle ip : \langle\langle 1\ 2\rangle\langle 1\ 1\rangle\rangle\ \ ip : \langle\langle 1\ 2\rangle\langle 3\ 3\rangle\rangle\rangle$

$\langle ip : \langle\langle 1\ 2\rangle\langle 1\ 1\rangle\rangle\ \ ip : \langle\langle 1\ 2\rangle\langle 3\ 3\rangle\rangle\rangle \Rightarrow$

$\langle\langle 3\ \ \ 9\rangle$

$\langle 3\ \ \ 9\rangle\rangle$