Give a loop invariant for this program fragment:

\[
\begin{align*}
&x := 2; \\
i := 1; \\
(*\text{What is the precondition here?}*) \\
&\text{while } (i \leq n) \text{ do} \\
&\hspace{1em} \text{begin} \\
&\hspace{2em} x := x \times x; \\
&\hspace{2em} i := i+1 \\
&\text{end}
\end{align*}
\]

with precondition \(n \geq 1\) and postcondition \(x = 2^{2^n}\).

**Answer:** \(x = 2^{2^i-1} \land i \leq n + 1\).

Also answer the following questions.

1. What is the precondition before the loop? **Answer:** \(x = 2 \land i = 1 \land n \geq 1\)

2. Your invariant should consist of the conjunction of two formulas. One of them should be very similar to the condition of the while loop. Explain why that formula is needed. **Answer:** To insure that \(i = n + 1\) (rather than just \(i > n\)) when the loop is exited.

3. Show that the precondition at the line with asterisks implies the invariant. **Answer:** In short: (1) Since \(i = 1\) and \(x = 2\), then \(x = 2^{2^i-1}\). (2) Since \(i = 1\) and \(n \geq 1\), then \(i \leq n + 1\).

4. Show that the invariant together with the negation of the loop implies the postcondition. **Answer:** In short: (1) Since \(i \leq n + 1\) and \(i > n\) then \(i = n + 1\). (2) Since \(x = 2^{2^i-1}\) and \(i = n + 1\), then \(x = 2^{2^n}\).

5. Let \(x\) be the value of the variable \(x\) before executing the body of the loop and \(x'\) be the value of the variable \(x\) after executing the body of the loop. Write an equation that relates \(x\) and \(x'\). **Answer:** \(x' = x \times x\).

6. Let \(i\) be the value of the variable \(i\) before executing the body of the loop and \(i'\) be the value of the variable \(i\) after executing the body of the loop. Write an equation that relates \(i\) and \(i'\). **Answer:** \(i' = i + 1\).