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Discrete-Time Markov Chains

A *stochastic process* is simply a sequence of random variables.

Definition 8.1 A *DTMC* (discrete-time Markov chain) is a stochastic process $\{X_n, n = 0, 1, 2, \ldots\}$, where X_n denotes the state at (discrete) time step n and such that, $\forall n \geq 0$, $\forall i, j$, and $\forall i_0, \ldots, i_{n-1}$,

Markovian (memoryless) property

$$\mathbf{P}\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = \mathbf{P}\{X_{n+1} = j \mid X_n = i\}$$
$$= P_{ij} \text{ (by stationarity)},$$

where P_{ij} is independent of the time step and of past history.

(Memoryless)

Definition 8.2 The ***Markovian Property*** states that the conditional distribution of any future state X_{n+1}, given past states $X_0, X_1, \ldots, X_{n-1}$, and given the present state X_n, is independent of past states and depends only on the present state X_n.

The second equality in the definition of a DTMC follows from the "stationary" property, which indicates that the transition probability is independent of time.

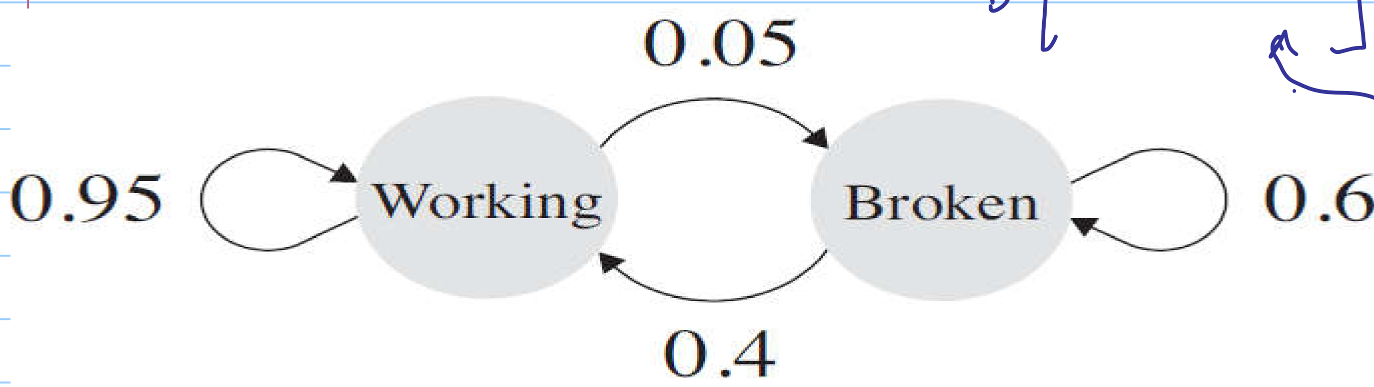
Definition 8.3 The ***transition probability matrix*** associated with any DTMC is a matrix, $\mathbf{P}$, whose (i, j)th entry, P_{ij}, represents the probability of moving to state j on the next transition, given that the current state is i.

We begin with DTMC w/ finite # states.

Repair Facility Problem

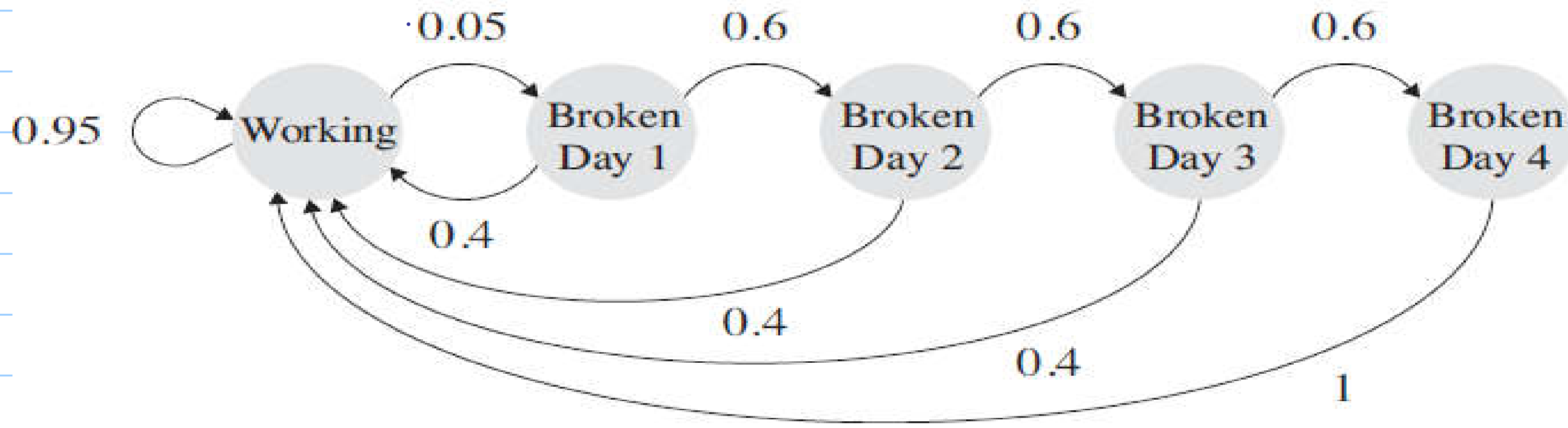
$$P = \begin{array}{c|cc} i \backslash i+1 & W & B \\ \hline W & 0.95 & 0.05 \\ B & 0.4 & 0.6 \end{array}$$

$P\{X_{i+1} = W \mid X_i = W\} +$
$P\{X_{i+1} = B \mid X_i = B\}$



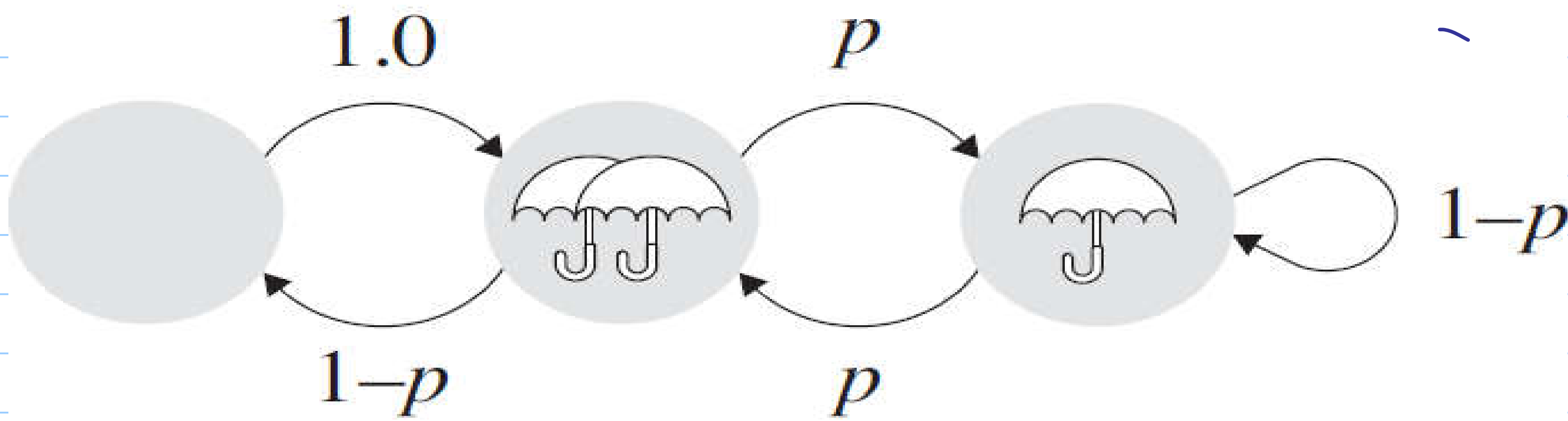
$P\{X_{i+1} = B \mid X_i = W\} *$
$P\{X_{i+1} = B \mid X_i = B\}$

Modification of above; if a machine is broken for four days, it is replaced.



Umbrella Problem

$$
\begin{array}{c|ccc}
n \backslash n+1 & 0 \text{ umbr.} & 1 & 2 \\
\hline
0 & & & \\
1 & & \left[\begin{matrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{matrix}\right] & \\
2 & & &
\end{array}
$$



Program Analysis Problem

CPU instructions (C)
Memory Instructions (M)
User interaction Instruction (U)

$$\mathbf{P} = \begin{array}{c} \\ C \\ M \\ U \end{array} \overset{\begin{array}{ccc} C & M & U \end{array}}{\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.9 & 0.1 & 0 \end{bmatrix}}$$

$i \backslash i+1$

Typical questions: How frequent are CPU instructions? What is the mean length of the instruction sequence between consecutive memory instructions?

The answer for the first question is part of the answer to exercise 8.1 (one of the exercises of HW 9).

8.4 Powers of P; n-step Transition Probabilities

Let $\mathbf{P}^n = \mathbf{P} \cdot \mathbf{P} \cdots \mathbf{P}$, multiplied n times. We will use the notation P_{ij}^n to denote $(\mathbf{P}^n)_{ij}$.

$$\underset{(n,n)}{A} \cdot \underset{(n,n)}{B} = \underset{(n,n)}{C}, \qquad C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$A \cdot B = C$$

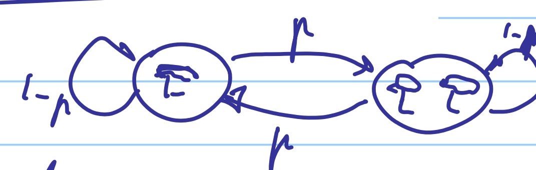
Umbrella Problem

Consider the umbrella problem from before where the chance of rain on any given day is $p = 0.4$. We then have

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 \end{bmatrix}, \quad \mathbf{P}^5 = \begin{bmatrix} .06 & .30 & .64 \\ .18 & .38 & .44 \\ .38 & .44 & .18 \end{bmatrix}, \quad \mathbf{P}^{30} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} .230 & .385 & .385 \\ .230 & .385 & .385 \\ .230 & .385 & .385 \end{bmatrix} \end{array}$$

Observe that all the rows become the *same*! Note also that, for all the above powers, each row sums to 1.

$$\boxed{= \lim_{n \to \infty} P^n}$$



What is the limiting prob. that the prof. gets wet? It is:

$0.23 \times 0.4 = 0.092$ (9%) Not too bad!

Repair Facility Problem

$$P = \begin{array}{c c} & \begin{array}{cc} W & B \end{array} \\ \begin{array}{c} W \\ B \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array}, \quad 0 < a < 1, \quad 0 < b < 1$$

a: probab. of getting broken

b: probab. of getting repaired

$$P^n = \begin{bmatrix} \frac{b + a(1-a-b)^n}{a+b}, & \frac{a - a(1-a-b)^n}{a+b} \\ \frac{b - b(1-a-b)^n}{a+b} & \frac{a - b(1-a-b)^n}{a+b} \end{bmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{array}{c c} & \begin{array}{cc} W & B \end{array} \\ \begin{array}{c} W \\ B \end{array} & \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} \end{array}$$

$$\left(\overset{W}{\frac{b}{a+b}}, \overset{B}{\frac{a}{a+b}} \right) = \pi$$

$$p^2_{ij} = \sum_{k=1}^{M} p_{ik} \cdot p_{kj} =$$

$\sum_k$ P{end at j | start at i & go through k} x

P{go through k | start at i}

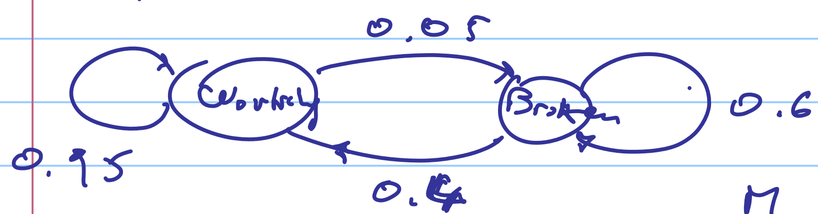
$$p^n_{ij} = \sum_{k=1}^{M} p^{n-1}_{ik} \cdot p_{kj}$$ = prob. of ending at state j in n steps, given that one started from state i.

$\pi_j = \lim_{n \to \infty} P^n_{ij}$ limiting probability (of being in state j)

$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ limiting distribution

Solving stationary equation. (8.7)

Repair Facility Problem with cost.



$$P = \begin{bmatrix} .95 & .05 \\ .4 & .6 \end{bmatrix}$$

$$\vec{\pi} = \vec{\pi} \cdot P \qquad \sum_{i=1}^{M} \pi_i = 1$$

"steady-state"

$$[\pi_W, \pi_B] = [\pi_W, \pi_B] \cdot P$$

$$\underset{\substack{1 \times M \\ 1 \times 2}}{[\pi_W, \pi_B]} = \underset{\substack{1 \times M \\ 1 \times 2}}{[\pi_W, \pi_B]} \cdot \underset{\substack{M \times M \\ 2 \times 2}}{\begin{bmatrix} .95 & .05 \\ .4 & .6 \end{bmatrix}}$$

$$\begin{cases} \pi_w = \pi_w(.95) + \pi_B(.4) \\ \pi_B = \pi_w(.05) + \pi_B(.6) \\ \pi_w + \pi_B = 1 \end{cases}$$

$$\begin{cases} \pi_w(-.05) + \pi_B(.4) = 0 \\ \pi(.05) + \pi_B(-.4) = 0 \\ \pi_w + \pi_B = 1 \end{cases}$$

equivalent

$$\begin{cases} \pi_w(-.05) + \pi_B(.4) = 0 \\ \pi_w + \pi_B = 1 \end{cases} \Rightarrow \pi_B = 1 - \pi_w$$

$$\pi_w(-.05) + 0.4 + 0.4(-\pi_w) = 0$$

$$\pi_w(-.05 - 0.4) + 0.4 = 0 \Rightarrow \pi_w = \frac{0.4}{0.45} = \frac{8}{9}$$

$$\pi_B = 1 - \frac{8}{9} = \frac{1}{9}$$

I have a machine. There is a charge of \$300 for each day the machine is in repair. The repair model is the DTMC described above.

What is my expected charge for a year?

Per day, $\$300 * \frac{1}{9} = \$33.33$

Per year, $\$33.33 \times 365 \approx \$12{,}000$.

Definition 8.4 Let

$$\pi_j = \lim_{n \to \infty} P_{ij}^n.$$

π_j represents the ***limiting probability*** that the chain is in state j (independent of the starting state i). For an M-state DTMC, with states $0, 1, \ldots, M-1$,

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \text{where} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

represents the ***limiting distribution*** of being in each state.

8.5 Stationary Equations

Definition 8.5 A probability distribution $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ is said to be ***stationary*** for the Markov chain if

$$\vec{\pi} \cdot \mathrm{P} = \vec{\pi} \quad \text{and} \quad \sum_{i=0}^{M-1} \pi_i = 1.$$

These are the stationary equations.

So, $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$ is stationary if

$$\underline{\underline{\sum_{i=0}^{M-1} \pi_i P_{ij} = \pi_j, \ \forall j}}, \quad \text{and} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

Verify the first part:

$$\underset{(1\times M)}{\vec{\pi}} \cdot \underset{(M\times M)}{\underline{P}} = \underset{(1\times M)}{\vec{\pi}} \Rightarrow [\ldots, \pi_j, \ldots] \begin{array}{c} j\text{-th column} \\ \downarrow \\ \left[\quad P_{ij} \quad \right] \end{array} = [\ldots, \pi_j, \ldots]$$

$$\Rightarrow \underbrace{\pi_j = (\text{row} \times j\text{-th column}) = \sum_{i=0}^{M-1} \pi_i P_{ij}}_{8.1 \text{ (first part)}} \quad \checkmark$$

Question: What does the left-hand-side (LHS) of the first equation in (8.1) represent?

Think about it a moment!

$$\vec{\pi} P = \vec{\pi}$$

Answer: The LHS represents the probability of being in state j one transition from now, given that the current probability distribution on the states is $\vec{\pi}$. So equation (8.1) says that if we start out distributed according to $\vec{\pi}$, then one step later our probability of being in each state will still follow distribution $\vec{\pi}$. Thus from then on we will always have the same probability distribution on the states. Hence we call the distribution "stationary."

8.6 The Stationary Distribution Equals The Limiting Distribution

Theorem 8.6 (Stationary distribution = Limiting distribution) *Given a finite-state DTMC with M states, let*

$$\pi_j = \lim_{n \to \infty} P_{ij}^n > 0$$

be the limiting probability of being in state j and let

$$\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1}), \quad \textit{where} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and no other stationary distribution exists.

The proof is in two parts

1. We will prove that $\{\pi_j,\ j = 0, 1, 2, \ldots, M-1\}$ is a stationary distribution. Hence at least one stationary distribution exists.
2. We will prove that any stationary distribution must be equal to the limiting distribution.

$$1.\ \underbrace{\pi_j = \lim_{n\to\infty} P_{ij}^{n+1} = \lim_{n\to\infty} \sum_{k=0}^{M-1} P_{ik}^{n} \cdot P_{kj} = \sum_{k=0}^{M-1} \underbrace{\lim_{n\to\infty} P_{ik}^{n}} \cdot P_{kj} = \sum_{k=0}^{M-1} \pi_k P_{kj}}_{=}$$

(stationarity equation

2. Let $\vec{\pi}'$ be any stationary prob distr. (Let $\vec{\pi}$ is the limiting prob distr.) We are going to show that $\vec{\pi}' = \vec{\pi}$. We will do so by showing that $\pi_j' = \pi_j$.

We assume that at time ϕ we have distr. $\vec{\pi}'$

$$\pi_j = P\{X_0 = j\} = P\{X_n = j\}. \quad \text{So,}$$

$$\pi_j' = P\{X_n = j\}, \quad \forall n \quad (\text{stationarity})$$

$$= \sum_{i=0}^{M} P\{X_n = j \mid X_0 = i\} \cdot P\{X_0 = i\} \quad \forall n$$

$$= \sum_{i=0}^{M} P_{ij}^{n} \quad \pi_i' \quad \forall n$$

So,

$$\pi_j' = \lim_{n\to\infty} \pi_j' = \lim_{n\to\infty} \sum_{i=0}^{M} P^n_{ij}\,\pi_i' = \sum_{i=0}^{M} \underbrace{\lim_{n\to\infty} P^n_{ij}}_{\text{j-th entry in any row of the } P^n \text{ matrix}}\,\pi_i' = \sum_{i=0}^{M} \pi_j\,\pi_i' =$$

$$= \pi_j \sum_{i=0}^{M} \pi_i' = \pi_j \cdot 1 = \pi_j \quad ✓$$

Definition 8.7 A Markov chain for which the limiting probabilities exist is said to be ***stationary*** or in ***steady state*** if the initial state is chosen according to the stationary probabilities.

Summary: Finding the Limiting Probabilities in a Finite-State DTMC:

By Theorem 8.6, given the limiting distribution $\{\pi_j, \ j = 0, 1, 2, \ldots, M-1\}$ exists, we can obtain it by solving the stationary equations

$$\vec{\pi} \cdot \mathbf{P} = \vec{\pi} \quad \text{and} \quad \sum_{i=0}^{M-1} \pi_i = 1$$

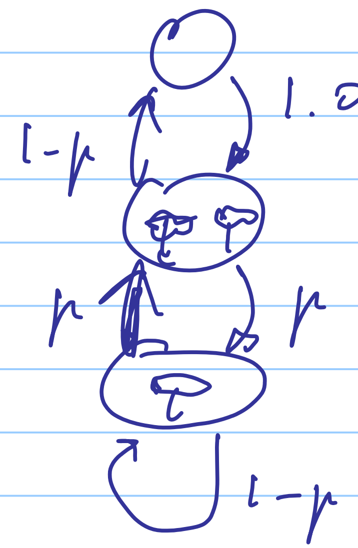
where $\vec{\pi} = (\pi_0, \pi_1, \ldots, \pi_{M-1})$.

8.7 Examples of Solving Stationary Equations

- Already done for Repair Facility Problem (8.7.1)

- 8.7.2. Umbrella Problem

$$[\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2]$$



$$\begin{cases} \pi_0 = \pi_2(1-p) \\ \pi_1 = \pi_1(1-p) + \pi_2 p \\ \pi_2 = \pi_0 + \pi_1 p \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \qquad \begin{cases} \pi_0(-1) + \pi_2(1-p) = 0 \quad (1) \\ \pi_1(-p) + \pi_2(p) = 0 \quad (2) \\ \pi_0 + \pi_1(p) + \pi_2(-1) = 0 \quad (3) \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

Note that $-(1)-(2)=(3)$, so we can remove (3)

$$\begin{cases} \pi_0(-1) + \pi_2(1-p) = 0 \\ \pi_1(-p) + \pi_2(p) = 0 \Rightarrow \pi_2 = \pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$\pi_0 = \pi_2(1-p)$

$\pi_2(1-p) + \pi_2 + \pi_2 = 1 \Rightarrow \pi_2 = \dfrac{1}{3-p}$

$\pi_2 = \frac{1}{3-p}$, $\pi_1 = \frac{1}{3-p}$, $\pi_0 = \frac{1-p}{3-p}$.

Question: Suppose the probability of rain is $p = 0.6$. What fraction of days does the professor get soaked?

Answer: The professor gets wet if she has zero umbrellas and it is raining: $\pi_0 \cdot p = \frac{0.4}{2.4} \cdot 0.6 = 0.1$. Not too bad!

8.8 Infinite-State DTMCs

The limiting distribution is:

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \quad \text{where} \quad \pi_j = \lim_{n \to \infty} P_{ij}^n \quad \text{and} \quad \sum_{j=0}^{\infty} \pi_j = 1.$$

8.9 Infinite-State Stationarity Result

Theorem 8.8 (Stationary distribution = Limiting distribution) *Given an infinite-state DTMC, let*

$$\pi_j = \lim_{n\to\infty} P^n_{ij} > 0$$

be the limiting probability of being in state j and let

$$\vec{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \quad \textit{where} \quad \sum_{i=0}^{\infty} \pi_i = 1$$

be the limiting distribution. Assuming that the limiting distribution exists, then $\vec{\pi}$ is also a stationary distribution and no other *stationary distribution exists.*

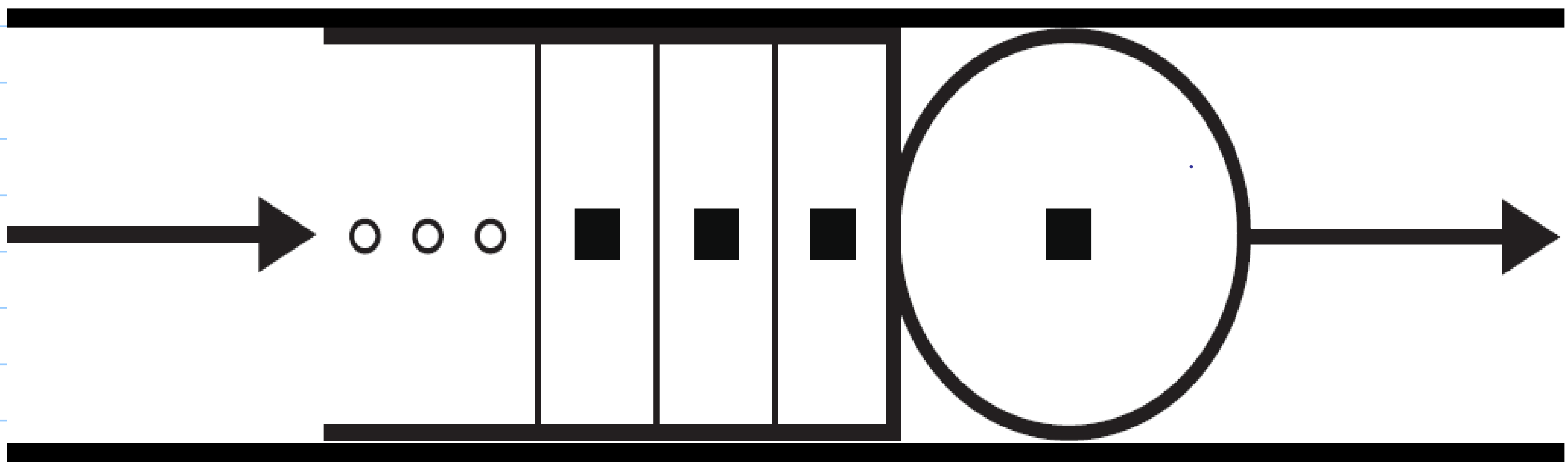
As for the finite-state case, the proof is in two parts:

1. We will prove that $\{\pi_j,\ j = 0, 1, 2, \ldots\}$ is a stationary distribution. Hence at least one stationary distribution exists.
2. We will prove that any stationary distribution must be equal to the limiting distribution.

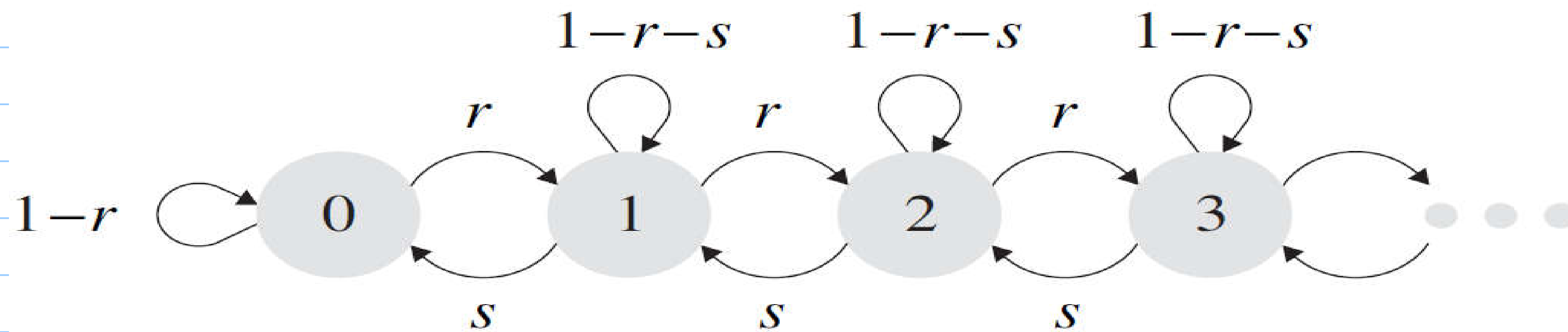
We will follow the book directly.

8.10 Solving Stationary Equations in Infinite-State DTMCs

Server with unbounded queue



DTMC for server with unbounded queue:



Transition probability matrix (infinite)

$$\mathbf{P} = \begin{pmatrix} 1-r & r & 0 & 0 & \cdots \\ s & 1-r-s & r & 0 & \cdots \\ 0 & s & 1-r-s & r & \cdots \\ 0 & 0 & s & 1-r-s & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Stationarity equations:

$$\begin{aligned}
\pi_0 &= \pi_0(1-r) + \pi_1 s \\
\pi_1 &= \pi_0 r + \pi_1(1-r-s) + \pi_2 s \\
\pi_2 &= \pi_1 r + \pi_2(1-r-s) + \pi_3 s \\
\pi_3 &= \pi_2 r + \pi_3(1-r-s) + \pi_4 s \\
&\vdots
\end{aligned}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \cdots = 1$$