317 2016-02-18 extra

Note Title 2016-02-22

Some notes on reliability. (Based on Trivedi's notes)

Let the r.v. X be the lifetime of a component. So,

$F_X(t) = P\{-\infty < X < t\}$ is the probability that the lifetime of the component is less than t.

The <u>reliability</u> $R(t)$ of a component is the probability that the component survives until some time t.

So, $R(t) = 1 - F_X(t)$ (and $F_X(t)$ is sometimes called the unreliability.)

The conditional reliability $R_t(y)$ is the probability that the component survives an additional time y given that it has survived until time t. So,

$$R_t(y) = \frac{R(t+y)}{R(t)} = \frac{P\{X > t+y\}}{P\{X > t\}}$$

A related quantity is the probability that a component fails between t and $t+y$, given that it survived until t:

$$G_t(y) = \frac{P\{t < X < t+y\}}{P\{X > t\}}.$$ Therefore, $R_t(y) = 1 - G_t(y)$.

Consider a component that does not age. Then,

$$R_t(y) = R(y) \quad \text{for all } y, t > 0$$

$$R(y+t) = R(t)\, R_t(y) = R(t)\, R(y)$$

$$\frac{R(y+t) - R(y)}{t} = \frac{R(t)R(y) - R(y)}{t} = \frac{[R(t)-1]R(y)}{t}.$$

Taking limits for $t \to 0$ and noting that $R(0) = 0$,

$$R'(y) = R'(0)\, R(y).$$

Solving this differential equation gives:

$R(y) = e^{yR'(0)}$. Letting $R'(0) = \lambda$, we get:

$R(y) = e^{-\lambda y}$, $y > 0$, which implies that the lifetime $(X = 1 - Y)$

$X \sim EXP(\lambda)$.

It can also be shown that the exponential distribution is the only one that has the memoryless (Markov) property.

<u>Defn.</u> The <u>instantaneous failure rate</u> at time t, $h(t)$ is

$$h(t) = \lim_{x\to 0} \frac{1}{x} \frac{F(t+x)-F(t)}{R(t)} = \lim_{x\to 0} \frac{F(t+x)-F(t)}{x} \cdot \frac{1}{R(t)} =$$

$$= \frac{1}{R(t)} \lim_{x\to 0} \frac{F(t+x)-F(t)}{x} = \frac{1}{R(t)} F'(t) = \frac{f(t)}{R(t)}$$

This is also called the hazard rate, the force of mortality, intensity rate, conditional failure rate, or simply failure rate.

If $X \sim Exp(\lambda)$, then $h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{1-F_X(t)} = \frac{\lambda e^{-\lambda t}}{1-(1-e^{-\lambda t})} = \lambda$: constant!

Observe: $h(t) \geq f(t)$, because $R(t) \leq 1$

$f_X(t)\Delta t$ is the unconditional prob that a component will fail in the interval $[t, t+\Delta t]$.

$h(t)\Delta t$ is the conditional prob that a component will fail in the interval $[t, t+\Delta t]$ given that it has survived until time t.