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If $X \sim \text{Poisson}(\lambda)$, what is $E[X]$

$$p_X(i) = \frac{e^{-\lambda}\lambda^i}{i!} \quad (\text{pdf of } X \sim \text{Poisson}(\lambda)) \quad (0 \le i < \infty)$$

$$E[X] = \sum_{i=0}^{\infty} i\, p_X(i) = \sum_{i=0}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} = \sum_{i=1}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} =$$

$$= e^{-\lambda} \sum_{i=1}^{\infty} i \frac{\lambda^i}{i!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\cancel{i}\,\lambda^{i-1}}{\cancel{i}(i-1)\times(i-2)\times\cdots 1} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} =$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Moment of an r.v. The i-th moment of r.v. X, denoted $E[X^i]$ is

$$E[X^i] = \sum_x x^i p_X(x) \quad \text{(discrete case)}$$

x X

$$E[X^i] = \int_{-\infty}^{\infty} x^i f_X(x)\,dx \quad \text{(continuous case)}$$

More generally, the expectation of function $g(\cdot)$ of the r.v. X is:

$$E[g(X)] = \sum_x g(x) p_X(x) \quad \text{(discrete case)}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Example (p.46 [H])

$$X = \begin{cases} 0 & \text{w/ prob. } 0.2 \\ 1 & \text{w/ prob } 0.5 \\ 2 & \text{w/ prob } 0.3 \end{cases}$$

$$E[2X^2+3] = \sum_x (2x^2+3)\,p_X(x) = (2\cdot 0^2+3)\,p_X(0) + (2\cdot 1^2+3)\,p_X(1) +$$

$$+ (2\cdot 2^2+3)\,p_X(2) = 3\times 0.2 + 5\times 0.5 + 11\times 0.3 =$$

$$= 0.6 + 2.5 + 3.3 = 6.4$$

Defn. 3.17 Variance

The square of how much we expect X to differ from its mean, $E[X]$.

$$Var(X) = E\left[(X - E[X])^2\right] = \sum_x (x - E[X])^2\, p_X(x) \quad \text{(discrete r.v.)}$$

$$= \int_{-\infty}^{\infty} (x - E(X))^2 f_X(x)\, dx \quad \text{(continuous case)}$$

If $X \sim \text{Bernoulli}(p)$, what is $Var(X)$?

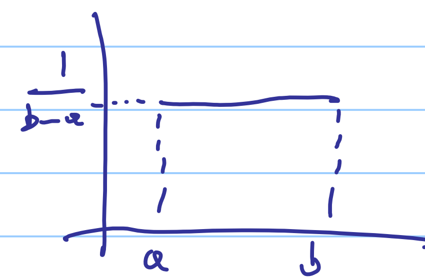
$$Var(X) = E\left[(X - E[X])^2\right] = (\text{recall, } E[X] = p) =$$

$$= E\left[(X - p)^2\right] = \sum_i (i-p)^2 p_X(i) = (0-p)^2 p_X(0) + (1-p)^2 p_X(1) =$$

$$= (0-p)^2 (1-p) + (1-p)^2 p = (1-p)\left[p^2 + (1-p)p\right] = (1-p)(\cancel{p^2} + p - \cancel{p^2}) =$$

$$= p - p^2 = p(1-p)$$

If $X \sim \text{Uniform}(a, b)$, what is $Var[X]$?



$$Var[X] = E\left[(X - E[X])^2\right] = (*)$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{1}{\cancel{(b-a)}}\;\frac{\cancel{(b-a)}(b+a)}{2} = \frac{b+a}{2} = \frac{a+b}{2}$$

$$E\left[\left(X-\frac{a+b}{2}\right)^2\right] = \int_a^b \left(x-\frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx = \cdots = \frac{(b-a)^2}{12}$$

Joint probability & independence (3.10 [H])

Def. 3.8 The joint probability mass function between discrete r.v.s X and Y (of) is

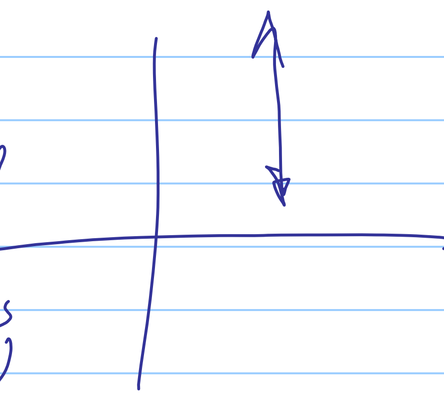
$$p_{XY}(x,y) = P\underbrace{\{X=x,\ Y=y\}}_{\text{an event}}$$

The joint probability density function .....

$$\int_c^d \int_a^b f_{XY}(x,y)\,dx\,dy = P\{a < X < b,\ c < Y < d\}$$

Def. 3.19 r.v. X and Y are independent
(written $X \perp Y$)
if $p_{XY}(x,y) = p_X(y) \cdot p_Y(y)$ (discrete case)

or if $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ (continuous case)



Thm. 3.20 If $X \perp Y$, then $E[XY] = E[X] \cdot E[Y]$

Proof (discrete): $\sum_x \sum_y xy \, P\{X=x, Y=y\}$ = (defn. of joint prob. mass funct)

$$= \sum_x \sum_y xy \, p_{XY}(x,y) \underset{3.19}{=} \sum_x \sum_y xy \, p_X(x) p_Y(y) =$$

$$= \sum_x \sum_y x\, p_X(x)\, y\, p_Y(y) = \sum_x x\, p_X(x) \sum_y y\, p_Y(y) =$$

$$= \sum_x x\, p_X(x) \cdot E[Y] = E[X]\, E[Y]$$

$a \cdot b + a \cdot c = a(b+c)$ (distributivity of product over sum)

This generalizes to the continuous ~~case~~ & ~~to~~ this:

If $X \perp Y$, then $E[g(X) f(Y)] = E[g(X)]\, E[f(Y)]$

The converse is not true:

If $E[XY] = E[X]\, E[Y]$ it is not necessary that $X \perp Y$.

Defn. 3.21 r.v. X with p.m.f $p_X(\cdot)$. Let A be an event.

$p_{X|A}(\cdot)$ is the conditional p.m.f. of X given A,

$$p_{X|A}(\cdot) = P\{X=x \mid A\} = \frac{P\{(X=x) \cap A\}}{P\{A\}}$$

A conditional probability involves a narrowing of the outcome space (from Ω to A).

Defn. 3.22 Let X be a discrete r.v. The conditional expectation of X given event A is:

$$E[X|A] = \sum_x x\, p_{X|A}(x) = \frac{p_X\{X=x \cap A\}}{P\{A\}}$$

The event A may be the event that corresponds to another r.v. having a particular value.

$$E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y) = \frac{\sum_x p_{X,Y}(x,y)}{p\{Y=y\}} = \frac{\sum_x p_{X,Y}(x,y)}{p_Y(y)}$$

Ex. (pp. 50-51 [H])

$p_{X,Y}(x,y)$

	$X=0$	$X=1$	$X=2$
$Y=2$	0	$1/6$	$1/8$
$Y=1$	$1/8$	$1/6$	$1/8$
$Y=0$	$1/6$	$\frac{1}{8}$	0

$$E[X \mid Y=2] = \sum_x x\, p_{X|Y}(x \mid 2) = \sum_x x \cdot P\{X=x \mid Y=2\} =$$

$$= 0 \cdot \frac{P\{X=0, Y=2\}}{P\{Y=2\}} + 1 \cdot \frac{P\{X=1, Y=2\}}{P\{Y=2\}} + 2 \cdot \frac{P\{X=2, Y=2\}}{P\{Y=2\}} =$$

$$= 0 + 1 \cdot \frac{1/6}{1/6 + \frac{1}{8}} + 2 \cdot \frac{1/8}{1/6 + \frac{1}{8}} = \frac{\frac{1}{6}}{\frac{14}{48}} + 2 \cdot \frac{1/8}{\frac{14}{48}} = \ldots = \frac{10}{7}.$$

Defn. 3.23 Conditional p.d.f.

$$f_{X|A}(A) = \begin{cases} \dfrac{f_X(x)}{P\{X \in A\}} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Defn. 3.24 Conditional expectation of X given A

$$E[X|A] = \int_{-\infty}^{\infty} x \, f_{X|A}(x)\,dx = \int_A x \, f_{X|A}(x)\,dx = \frac{1}{P\{X \in A\}} \int_A x \, f_X(x)\,dx$$