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Note Title 2016-01-26

HW1: Exercises 3.2, 3.3, 3.4, 3.5 [H] due on Tuesday, February 2, 2016.

$\Omega =$

	E_1		E_2		
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$P(E_1) = \frac{6}{36} = \frac{1}{6}$

$P(E_2) = \frac{3}{36} = \frac{1}{12}$

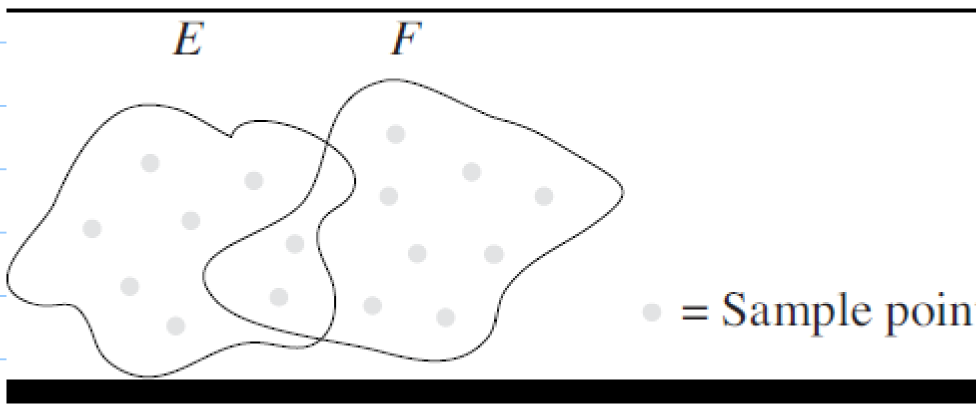
$E_1 = \{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2)\}$

$E_2 = \{(1,4), (1,5), (1,6)\}$

$E_1 \cap E_2 = \{\}$

Defn 3.2: If $E_1 \cap E_2 = \{\}$, then E_1 and E_2 are mutually exclusive.

Defn 3.3. If $E_1, E_2, \ldots E_n$ are events such that $E_i \cap E_j = \{\}$, $i \neq j$, $i, j \in 1..n$ and such that $\bigcup_{i=1}^{n} E_i = \Omega$, then we say that $E_1, \ldots E_n$ partition Ω (and $\mathcal{F} = \{E_1, \ldots E_n\}$). We also say that $E_1 \ldots E_n$ are mutually exclusive and exhaustive.

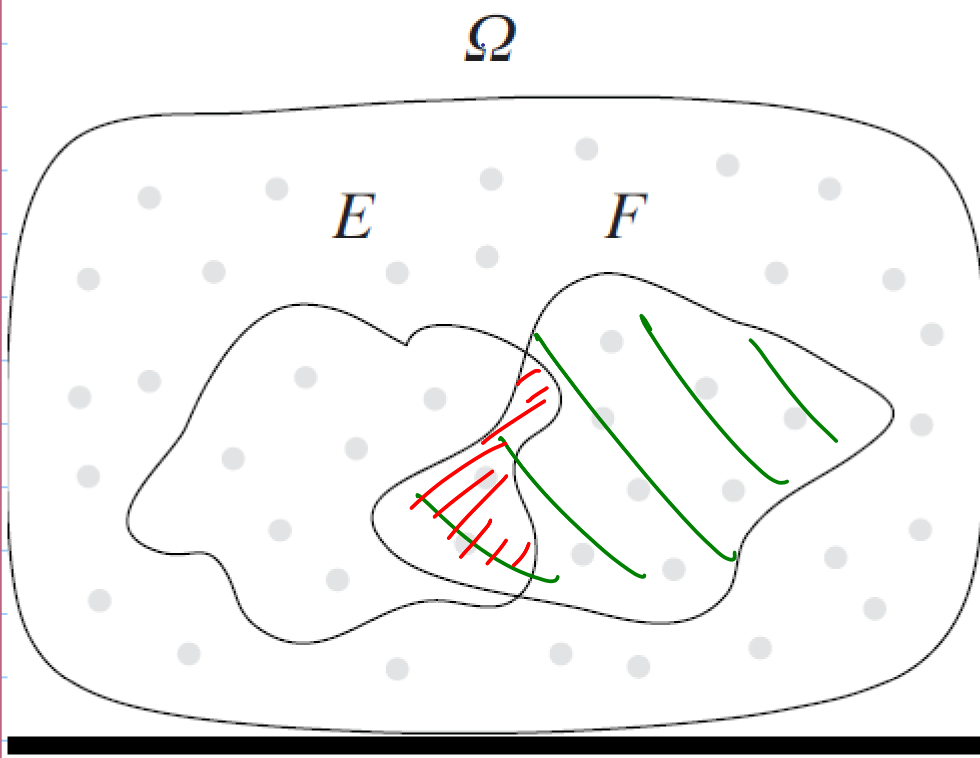


$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\}$.

Thm 3.5 $P\{E \cup F\} \leq P(E) + F(F)$

Equality holds when E and F are mutually exclusive.



$$P(E|F) = \frac{P(E\cap F)}{P(F)} =$$

$$= \frac{\text{number of points in } E \cap F}{\text{number of points in } F}$$

Table 3.1 My sandwich choices

Mon	Tue	Wed	Thu	Fri	Sat	Sun
Jelly	Cheese	Turkey	Cheese	Turkey	Cheese	None

What is P{Cheese | Second Half of the Week} ?

$$= \frac{\text{\# Cheese Sandwiches}}{\text{Total \# Sandwiches in 2nd half of the week}} = \frac{2}{4} = \frac{1}{2}$$

$$= \frac{P(\text{Cheese Sandwich \& 2nd Half of the week})}{P(\text{2nd Half of the Week})} = \frac{\frac{2}{7}}{\frac{4}{7}} = \frac{2}{4} = \frac{1}{2}$$

Defn 3.7 Events E and F are independent if

$$P\{E \cap F\} = P(E) \cdot P(F)$$

Thm If E and F are independent events, then

$$P(E|F) = P(E).$$

Proof $P(E|F) = \dfrac{P(E \cap F)}{P(F)} = (\text{indep}) = \dfrac{P(E) \cdot P(F)}{P(F)} = P(E)$

[One can show the converse.]

Can two mutually exclusive (and non-null) events ever be independent?

Let E and F be " " " " . Then

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0}{P(F)} = 0 \neq P(E).$$

No

$\cancel{E_1}$ $\cancel{E_2}$

$$\Omega = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right.$$

[redacted]

Is E_1 = "First roll is 6" independent from E_2 = "Second roll is 6" ?

$P\{E_1\} = P\{61, 62, 63, 64, 65, 66\} = \frac{6}{36} = \frac{1}{6}$

$P\{E_2\} = P\{16, 26, 36, 46, 56, 66\} = \frac{6}{36} = \frac{1}{6}$

$P\{E_1 \cap E_2\} = P\{66\} = \frac{1}{36} \stackrel{?}{=} P(E_1)\,P(E_2) = \frac{1}{6}\cdot\frac{1}{6}$ ✓ Yes

Is E_1 = "Sum of the rolls is 7" independent of E_2 = "Second roll is 4"

$P\{E_1\} = P\{16, 25, 34, 43, 52, 16\} = \frac{6}{36} = \frac{1}{6}$

$P(E_2) = P\{14, 24, 34, 44, 54, 64\} = \frac{6}{36} = \frac{1}{6}$

$P\{E_1 \cap E_2\} = P\{34\} = \frac{1}{36} = \frac{1}{6}\cdot\frac{1}{6} = P(E_1)\,P(E_2)$ ✓ Yes!

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E_1 = "Sum of the rolls is 8", E_2 = "Second roll is 4"

Are E_1 and E_2 independent?

$E_1 = \{26, 35, 44, 53, 62\}$ $\quad E_2 = \{14, 24, 34, 44, 54, 64\}$

$P(E_1) = \frac{5}{36}$ $\quad P(E_2) = \frac{6}{36} = \frac{1}{6}$

$E_1 \cap E_2 = \{44\}$ $\quad P(E_1 \cap E_2) = \frac{1}{36} \neq \frac{5}{36}\cdot\frac{1}{6} = P(E_1)\cdot P(E_2)$

No, E_1 & E_2 are not independent.

Also check that $P(E_1 | E_2) \neq P(E_1)$

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{1/36}{1/6} = \frac{1}{6} \neq P(E_1) = \frac{5}{36}$$

Independence is symmetric (b/c $P(E_1) \cdot P(E_2) = P(E_2) \cdot P(E_1)$)
and, in fact,

$$P(E_2|E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{\frac{1}{36}}{5/36} = \frac{1}{5} \neq P(E)_2 = \frac{1}{6}$$

$P(E_1 | E_2) =$ the probability of E_1 when the outcome space (Ω) is replaced by E_2 = $\dfrac{\text{the number of outcomes in } E_1 \text{ that are also in } E_2}{\text{the number of outcomes in the new outcome space } (E_2)}$

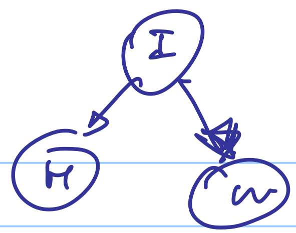
$$= \frac{1\ [44 \text{ is the only such outcome}]}{6} = \frac{1}{6} \neq \frac{5}{36} = P(E_1)$$

Also, $P(E_2 | E_1) = \dfrac{1\ [44]}{5\ [\text{all outcomes in } E_1]} = \frac{1}{5} \neq \frac{1}{6} = P(E_2)$

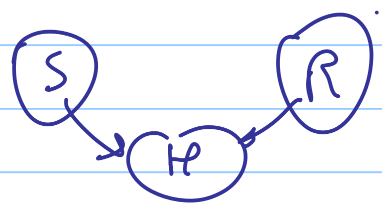
Defn. 3.8 Two events E & F are said to be conditionally independent given event G if, where $P\{G\} > 0$, $P\{E \cap F | G\} = P\{E|G\} \cdot P\{F|G\}$

Thm (w/out proof) E & F are conditionally independent given G if and only if $P\{E \cap F|G\} = P\{E|G\}$

Thm (w/ proof) E & F are conditionally independent given G iff $P\{E \cap F|G\} = P\{F|G\}$



Two events may be (unconditionally) dependent and conditionally independent



Or, two events may be unconditionally independent and conditionally dependent

(Side issues)

3.5 Law of Total Probability

Let $F_1, F_2, \dots, F_n$ partition the state space Ω. Thus,

$$P\{E\} = \sum_{i=1}^{n} P\{E \cap F_i\} = \sum_{i=1}^{n} P\{E \mid F_i\} P\{F_i\}$$

3.6 Bayes Law [Bayes' Rule]

Theorem 3.10 (Bayes Law) [The inversion formula]

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)}.$$

Disease | Symptom

Proof $P(F|E) = \frac{P(E\cap F)}{P(E)} = \frac{P(E|F)\,P(F)}{P(E)}$

Or: $P(E\cap F) = \begin{cases} P(E|F)\,P(F) \\ P(F|E)\,P(E) \end{cases}$

Theorem 3.11 Extended Bayes Law. (This combines Bayes Law & the Law of Total probability)

$$P(F|E) = \frac{P(E|F)\,P(F)}{P(E)} = \frac{P(E|F)\,P(F)}{\sum_{i=1}^{n} P(E|F_i)\,P(F_i)}$$, where

$F_1, \dots, F_n$ partition the outcome (state) space.

$$P\{\text{Disease} \mid \text{Test positive}\} =$$

$$\frac{1}{10{,}000}$$

$$= \frac{P\{\text{Test Positive} \mid \text{Disease}\}\, P\{\text{Disease}\}}{P\{\text{Test positive}\}} =$$

$$= \frac{\quad\quad}{P\{\text{Test positive} \mid \text{Disease}\}\, P\{\text{Disease}\} + P\{\text{Test Positive} \mid \neg\text{Disease}\}\, P\{\neg\text{Disease}\}} =$$

$$= \frac{0.95 \times \frac{1}{10{,}000}}{0.95 \times \frac{1}{10{,}000} + 0.05 \times \frac{9999}{10{,}000}} \simeq 0.0019 \approx \frac{1}{500}$$

3.7 Discrete vs. Continuous Random Variable

[Trivedi] – quote:

A random variable is a rule that assigns a numerical value to each possible outcome of an experiment.

The term "random variable" is actually a misnomer, since a r.v. X is really a function whose domain is the sample space S (Ω) and whose range is the set of all real numbers, $\mathbb{R}$. The image of the function is therefore a subset of the real numbers.

"A random variable is neither"

Example

Let Ω be the set of all cards that are on top of a well shuffled deck of 40 cards (no face cards).

The numerical value of a card is a random variable. This random variable assigns to each outcome (a card) the number on the card.

The r.v. partitions Ω into 10 events, consisting of all cards of a given value

$\Omega =$

1 ♧	2 ♧	....	10 ♧
1 ♢	2 ♢		10 ♢
1 ♡	2 ♡		10 ♡
1 ♤	2 ♤		10 ♤
↓	↓		↓
1	2		10

The r.v. "number value" maps like this:

; a subset of the real numbers

This function is a "random variable"

♧	1
♢	2
♡	3
♤	4

In general, a r.v. X partitions Ω, b/c it is a (well-defined) function, and therefore it cannot assign different numbers to the same outcome.

(added 2/2)

For a r.v. X and a real number x, we define the event A_x to be the subset of all sample points in Ω to which the r.v. X assigns the value x. So,

$$A_x = \{ s \in \Omega \mid X(s) = x \}$$

$$A_x \cap A_y = \{\} \text{ if } x \neq y$$

$$\bigcup_{x \in \mathbb{R}} A_x = \Omega$$

Therefore the collection of events A_x for all x defines an event space (as required by the Kolmogorov axioms.

This is why we often with variables as if we were working with events.