HW1: Exercises 3.2, 3.3, 3.4, 3.5 [H] due on January 19, 2017

A sample space

$$\Omega = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
                     (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
                     (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
                     (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
                     (5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
                     (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

Defn. 3.1 An event, $E_i$, is a subset of $\Omega$.

Here, as in most of the text, we adopt the classical interpretation of probability and assume that each outcome is equally possible. So,

$$P(E_i) = \frac{\text{# of outcomes in } E_i}{\text{total # of outcomes}} \in \{\frac{1}{12}, \frac{1}{36}, \ldots \}$$

$$P(E_1) = \frac{3}{36} = \frac{1}{12}$$

$$P(E_2) = \frac{6}{36} = \frac{1}{6}$$

$E_1 = \{ (1,1), (2,2), (3,1), (4,2), (5,2), (6,2) \}$; $E_2 = \{ (1,4), (1,5), (1,6) \}$

Defn. 3.2 If $E_1 \cap E_2 = \emptyset$ then $E_1$ and $E_2$ are mutually exclusive.
Def. 3.3 If $E_1, E_2, \ldots, E_n$ are events s.t. $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$, and s.t. $\bigcup_{i=1}^n E_i = \Omega$, then we say that $E_1, E_2, \ldots, E_n$ partition $\Omega$; we also say that they partition $\Omega$. We also say that $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive.

Thm 3.4:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Thm 3.5:

$$P(E \cup F) \leq P(E) + P(F)$$

When is $P(E \cup F) = P(E) + P(F)$?

When $E_1$ and $E_2$ are mutually exclusive.
Defn 3.1: The conditional probability of event $E$ given event $F$ is written as $P(E|F)$ and given by the following, where we assume $P(F) > 0$:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$\Omega$

$E$

$F$
Table 1. My sandwich choices

<table>
<thead>
<tr>
<th>Mon</th>
<th>Tue</th>
<th>Wed</th>
<th>Thu</th>
<th>Fri</th>
<th>Sat</th>
<th>Sun</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jelly</td>
<td>Cheese</td>
<td>Turkey</td>
<td>Cheese</td>
<td>Turkey</td>
<td>Cheese</td>
<td>None</td>
</tr>
</tbody>
</table>

\[ P(\text{Cheese} \mid \text{Second Half of the Week}) \]

In the second half of the week, 4 outcomes: (Thu, Fri, Sun)

Cheese occurs twice, so we have 4 total outcomes, so \( \frac{2}{4} = \frac{1}{2} \).

We could instead use Defn. 3.6:

\[ P(\text{Cheese} \mid \text{Second Half}) = P(\text{Cheese} \mid \text{Second Half}) = \frac{2/7}{4/7} = \frac{2}{4} = \frac{1}{2} \]
Def. 3.7  Events E and F are independent if \( P(\text{E} \cap \text{F}) = P(\text{E}) \cdot P(\text{F}) \).

Then if E and F are independent, then \( P(\text{E} | \text{F}) = P(\text{E}) \).

Assume \( P(\text{F}) > 0 \).
\[
P(\text{E} | \text{F}) = \frac{P(\text{E} \cap \text{F})}{P(\text{F})} = \frac{P(\text{E}) \cdot P(\text{F})}{P(\text{F})} = P(\text{E}) \]

The converse also holds: if \( P(\text{E} | \text{F}) = P(\text{E}) \), then \( P(\text{E} \cap \text{F}) = P(\text{E}) \cdot P(\text{F}) \).

Also, note that independence is symmetric.

Can two mutually exclusive and non-null events be independent?

Let E and F be such events. Then \( P(\text{E} \cap \text{F}) = 0 \). Then
\[
P(\text{E} | \text{F}) = \frac{P(\text{E} \cap \text{F})}{P(\text{F})} = \frac{0}{P(\text{F})} = 0 = P(\text{E})
\]

Note.
I ignore the grey events in the figure. Let $E_1$ be "First roll is 6" and $E_2$ be "Second roll is 6."

Then $E_1 = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$E_2 = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$

$P(E_1) = \frac{6}{36} = \frac{1}{6} = P(E_2)$

$P(E_1 \cap E_2) = P\{(6,6)\} = \frac{1}{36}$

$P(E_1)P(E_2) = P(E_1 \cap E_2)$?

$\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \checkmark$ YES

Is $E_1$ "Sum of the rolls is 7" independent of $E_2$ "Second roll is 6."

$E_1 = \{16, 25, 34, 43, 52, 61\}$

$E_2 = \{14, 24, 34, 44, 54, 64\}$

$P(E_1) = \frac{6}{36} = \frac{1}{6}$

$P(E_2) = \frac{6}{36} = \frac{1}{6}$

$P(E_1 \cap E_2) = P\{(1,6)\} = \frac{1}{36} \checkmark$ YES

$P(E_1)P(E_2) = P(E_1 \cap E_2)$?
15. \( E_1 \): sum of the rolls is 8, independent of \( E_2 \): second roll is 4.

\[ E_1 = \{2, 3, 4, 5, 6\} \quad E_2 = \{1, 4, 2, 4, 3, 4, 4, 5, 4, 6\} \]

\[ E_1 \cap E_2 = \{4\} \]

\[ P(E_1) = \frac{5}{36} \quad P(E_2) = \frac{6}{36} = \frac{1}{6} \] 

\[ P(E_1 \cap E_2) = \frac{1}{36} \neq P(E_1)P(E_2) \]

**Def 3.8:** Two events \( E \) and \( F \) are said to be **conditionally independent given event \( A \)** if, when \( P(A) > 0 \),

\[ P(E \cap F | A) = P(E | A)P(F | A) \]

Holmes cracks Watson crisis

\( \text{Hand} \) and \( \text{Wear} \) are not independent, but they are independent given \( \text{Ice Road} \) (Bayesian network)