Generating r.v.s for simulation (Ch. 4)

4.1 Inverse-Transform Method

4.1.2 Discrete case

\[ X = \begin{cases} x_0 & \text{with prob } p_0 \\ x_1 & \text{with prob } p_1 \\ \vdots & \vdots \\ x_k & \text{with prob } p_k \end{cases} \]

1. Arrange \( x_0, \ldots, x_k \) in order
2. Generate \( u \in U(0,1) \)
3. (If \( 0 < u \leq p_1 \), then output \( x_0 \))
   \[ \text{If } \sum_{i=0}^{k-1} p_i < u \leq \sum_{i=0}^{k} p_i, \text{ then output } x_k \]
4.2.1 The continuous case

A value in $(0, x)$ should be output with probability $f_X(x)$

We want $P(0 < U < u) = P(0 < X < x)$

But $u = \frac{1}{F_X(x)}$

and $F_X(x)$

(where $f_X(\cdot)$ is the pdf of rv $X$)
So, we want \( u = F_X(x) \) or equivalently \( x = F_X^{-1}(u) \).

This gives the Inverse-Transform method to generate r.v. \( X \):

1. Generate \( u \sim U(0, 1) \)
2. Return \( x = F_X^{-1}(u) \)

Example: Generate \( X \sim \text{Exp}(\lambda) \)
\[
F_X(x) = 1 - e^{-\lambda x}
\]

So, we want \( x = F_X^{-1}(u) \) \( \Rightarrow F(x) = u \) \( \Rightarrow 1 - e^{-\lambda x} = u \) \( \Rightarrow \)
\[ -Ax = \ln(1-u) \implies x = -\frac{1}{A} \ln(1-u) \]

Given \( u \sim U(0,1) \), setting \( x = -\frac{1}{A} \ln(1-u) \) produces an instance of \( X \sim \text{Exp}(\lambda) \).
The above result can be obtained as an application of the following theorem (3.1, Trivedi). Let $X$ be a continuous r.v. with density $f_X$ which is non-zero on a subset $I$ of real numbers (i.e., $f_X(x) > 0$, $x \in I$ and $f_X(x) = 0$, $x \notin I$). Let $\Phi$ be a differentiable monotone (invertible) function whose domain is $I$ and whose range is the set of reals. Then, $Y = \Phi(X)$ is a continuous r.v. with the density $f_Y$ given by
\[ f_y(y) = \begin{cases} 
\int_x f_x(x) \left[ \Phi^{-1}(y) \right] \left[ 1/(\Phi^{-1})'(y) \right] dx, & y \in \Phi(I) \smallskip \\
0, & \text{otherwise} 
\end{cases} \]

where \( \Phi^{-1} \) is the uniquely defined inverse of \( \Phi \) and \( (\Phi^{-1})' \) is the derivative of the inverse function.

**Proof (omitted)**

**Example.**

Let \( \Phi \) be the distribution function, \( F \), of an r.v. \( X \).
with density $f$. Applying the above theorem, $Y = F(X)$ and $F_Y(y) = F_X(F_X^{-1}(y)) = y$. Therefore, the r.v. $Y = F(X)$ has the density given by

$$f_Y(y) = \begin{cases} 1 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

In other words, if $X$ is a continuous r.v. with CDF $F$, then the new r.v. $Y = F(X)$ is uniformly distributed over the interval $(0,1)$. 
4.2 Accept-Reject Method

4.2.1 Discrete case

1. Find r.v. \( Q \) s.t. \( q_j \rightarrow 0 \Leftrightarrow p_j \rightarrow 0 \)

2. Generate an instance of \( Q \), and call it \( j \)

3. Generate instance \( z \) of r.v. \( U(0,1) \)

4. If \( u < \frac{p_j}{q_j} \), return \( P = j \) and stop; otherwise, return to step 2.

Here is a proof of correctness. We want to show that \( P \) ends up being set to \( j \) (as opposed to some other value) if \( \frac{p_j}{q_j} \).
\[ P \{ \text{Pends up being set to } j \} = \frac{\text{Fraction of time } j \text{ is generated & accepted}}{\text{Fraction of time any value is accepted}} \]

\[ = \frac{q_j \cdot \frac{\mu_j}{c}}{\sum_j \frac{\mu_j}{c}} = \frac{\mu_j}{c} \]

\[ \Rightarrow \frac{\mu_j}{c} = \frac{1}{c} \]

So, \[ P \{ \text{Pends up being set to } j \} = \frac{\mu_j}{c} = \mu_j \cdot \frac{1}{c} \]
Note that, from (2), we can derive that c values are generated on average before one is accepted.
4.2.2. Continuous case (of Accept–Reject method)

1. Find continuous r.v. \( Y \) s.t. \( f_Y(t) > 0 \implies f_X(t) > 0 \).
   
   Let \( c \) be a constant s.t.
   
   \[
   \frac{f_X(t)}{f_Y(t)} \leq c \quad \forall t \text{ s.t. } f_X(t) > 0
   \]

2. Generate an instance \( t \) of \( Y \).

3. With prob. \( \frac{f_X(t)}{c \cdot f_Y(t)} \), return \( X = t \) (i.e., "accept \( t \) and stop").
   
   Else reject \( t \) and return to step 2.
Question: How does one return "with prob p"?

Answer: As in the discrete case, i.e., generate an instance, u, of U(0,1) (using a random number generator). If $u \leq p$, accept. If $u > p$, reject.
We want to generate an r.v. \( X \) with pdf \( f_X(t) = 20 + (1-t)^3 \), \( 0 < t < 1 \).

\[ Y \sim U(0,1) \]

\[ \frac{f_X(t)}{f_Y(t)} = f_X(t) \leq \frac{135}{64} \]

\[ \max_{0 \leq t \leq 1} f_X(t) \]

\[ \max_{0 \leq t \leq 1} f_X(t) \]
Generate $N \sim \text{Normal}(0,1)$. We will instead generate $X = \lfloor N \rfloor$ and multiply by $-1$ with probability $\frac{1}{2}$.

\[
f_X(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, -\infty < t < \infty
\]

\[
f_X(t) / f_{\lfloor N \rfloor}(t) = 2 f_X(t) = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}
\]

Choose $Y \sim \text{Exp}(1)$, so $f_Y(t) = e^{-t}, 0 < t < \infty$.

\[
f_Y(t) = \frac{f_X(t)}{f_Y(t)} = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} / e^{-t} = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2} + t} = \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \square
\]
\[
\text{max } g(t) \text{ occurs for } \max \left( t - \frac{t^2}{2} \right)
\]
\[
\frac{d}{dt} \left( t - \frac{t^2}{2} \right) = 1 - t \implies t = 1 \text{ maximizes } g(t)
\]

The corresponding value of \( g(1) = \frac{f_x(1)}{f_Y(1)} = \sqrt{\frac{2e}{\pi}} \approx 1.3 \)

\[
\phi(1) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}} > \sqrt{\frac{2}{\pi}} e^{\frac{1}{2}} = \sqrt{\frac{2e}{\pi}}
\]