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Note Title 2016-02-23

Midterm will be March 1, 2016.
Closed book, except for a one-side formula page (8.5 x 11 in or A4). handwritten in blue ink. 8.27 x 11.69

The normal (Gaussian) distribution: $X \sim N(\mu, \sigma^2)$

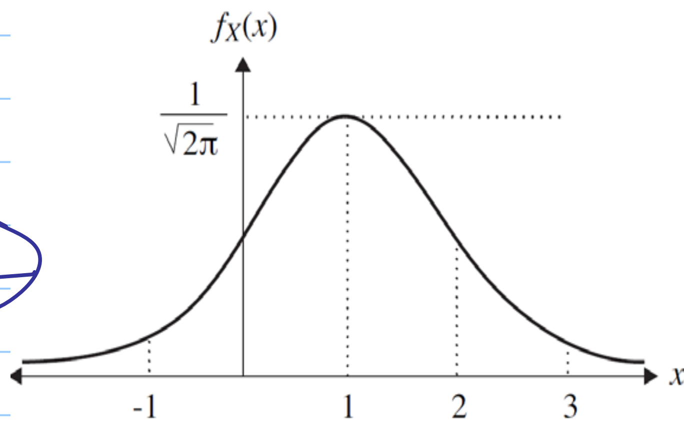
μ is the mean of X
σ^2 is the variance of X

These are facts that can be shown.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\frac{1}{1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-1}{1}\right)^2}$$

$f_X(x)$, when $X \sim N(1,1)$



Theorem 3.30 [H]. If $X \sim N(\mu, \sigma^2)$, then
$E[X] = \mu$ and $Var[X] = \sigma^2$.
(see text.)
There is no closed form solution for $F_X(x)$, where $X \sim N(\mu, \sigma^2)$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

We (instead) use tables with the values of $F_Z(x)$, where $Z \sim N(0,1)$
The r.v. Z is called the standard normal r.v.

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \underbrace{e^{-\frac{1}{2}t^2}}_{f_Z(t)} dt \quad ; \quad F_Z(-z) = \int_{-\infty}^{-z} f_Z(t)\, dt = \text{(always true)},$$

$$= \int_z^\infty f_Z(-t)\,dt = (\text{symmetry}) = \int_z^\infty f_Z(t)\,dt =$$

$$= \int_{-\infty}^\infty f_Z(t)\,dt - \int_{-\infty}^z f_Z(t)\,dt = 1 - \int_{-\infty}^z f_Z(t)\,dt =$$

$= 1 - F_Z(z)$, So, we usually find tables of $F_Z(z)$, $z \geq 0$

Theorem 3.31 [~~#~~] Linear Transformation Property.

Let $X \sim N(\mu, \sigma^2)$. Let $Y = aX + b$, where $a > 0$. Then,
$Y \sim N(a\mu + b, a^2\sigma^2)$

First, we show that $E[Y] = a\mu + b$

$E[Y] = a\,E[X] + b = a\mu + b$

$Var[Y] = a^2 Var[X]$, b/c of a general result (discrete version below)

Let $Y = aX + b$, with X a r.v. (and of course, Y too). Then,

$$Var[Y] = \sum_y (y - E[Y])^2 p_y = \sum_x ((ax+b) - (aE[X]+b))^2 p_x =$$

$$= \sum_x (ax - aE[X])^2 p_x = \sum_x [a(x - E[X])]^2 p_x =$$

$$= a^2 \sum_x (x - E[X])^2 p_x = a^2 Var[X]$$

Now, show that Y is normally distributed.

$$F_Y(y) = P\{Y \leq y\} = P\{aX + b \leq y\} = P\left\{X \leq \frac{y-b}{a}\right\} = F_X\left(\frac{y-b}{a}\right)$$

See book for the proof that $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}(a\sigma)} e^{-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}}$

Z

Tables give the cdf of a std. normal $Y \sim N(0,1)$

What if we have a r.v. $X \sim N(\mu, \sigma^2)$?

$X \sim N(\mu, \sigma^2)$ iff $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$ by the linear transf. property.

Easier to show it the other way around:

if $Y \sim N(0,1)$ and $X = \sigma Y + \mu$, then
$X \sim N(\sigma \cdot 0 + \mu, \sigma^2 \cdot 1) = N(\mu, \sigma^2)$.

In your book, the pdf of the std. normal is called Φ (instead of F_Z or F_Y). So, if $X \sim N(\mu, \sigma^2)$

$$P\{X < k\} = P\left\{\frac{X-\mu}{\sigma} < \frac{k-\mu}{\sigma}\right\} = P\left\{Y < \frac{k-\mu}{\sigma}\right\} = \Phi\left(\frac{k-\mu}{\sigma}\right)$$

Theorem 3.22 [H]

If $X \sim N(\mu, \sigma^2)$, then the probability that X deviates from its mean (μ) by less than k std. deviations ($k\sigma$) is the same as the prob. that $Y \sim N(0,1)$ deviates from its mean by less than k.

Proof

$$P\{-k\sigma < X-\mu < k\sigma\} = P\left\{-k < \frac{X-\mu}{\sigma} < k\right\} = P\{-k < Y < k\}$$

Ex. (p. 60 [H]). $IQ \sim N(100, 15)$.

What is the prob. that someone's $IQ > 130$?

130 is 2 standard deviations above the mean for IQ.

~~$F_Y(2)$ is the prob. that IQ is within 2 std. deviations from its mean.~~ $1 - F_Y(2)$ is the prob. that it is ~~outside~~ more than 2 std. deviations from its mean. By symmetry, $1 - F_Y(2)$ is the prob. that it is greater than 2 std. deviations from its mean.

$$1 - F_Y(2) = 1 - \Phi(2) = 1 - 0.9772 = 0.023.$$

your textbook's notation

Example 3.8 [T] (Trivedi's notes)