Midterm will be March 1, 2016.
Clips book, except for a one-side formula page (8.5 x 11 in or A4), handwritten in blue ink.

The normal distribution \( X \sim N(\mu, \sigma^2) \)

\( \mu \) is the mean of \( X \)
\( \sigma^2 \) is the variance of \( X \)
There are facts that can be shown.

\[
f_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}
\]

\[
f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}
\]

\( f_X(x) \), when \( X \sim N(1, 1) \)
Theorem 3.30 [14]. If \( X \sim N(\mu, \sigma^2) \), then
\[
E[X] = \mu \quad \text{and} \quad \text{Var}[X] = \sigma^2.
\]
(see text)

There is no closed form solution for \( F_X(x) \), where \( X \sim N(\mu, \sigma^2) \)
\[
F_X(x) = \int_{-\infty}^{x} f_X(x) \, dx = \int_{-\infty}^{x} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} (\frac{x-\mu}{\sigma})^2} \, dx.
\]

We instead use tables with the values of \( F_Z(z) \), where \( Z \sim N(0, 1) \)
The r.v. \( Z \) is called the **standard normal** r.v.
\[
F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2} t^2} \, dt ; \quad F_Z(-z) = \int_{-\infty}^{0} f_Z(t) \, dt = (\text{always 0 r.v.)},
\]
\[
F_Z(z) = 1 - \int_{0}^{z} f_Z(t) \, dt.
\]
\[
\begin{align*}
= \int_{-\infty}^{\infty} f_z(t) dt &= (\text{symmetry}) = \int_{-\infty}^{\infty} f_z(-t) dt = \\
= \int_{-\infty}^{\infty} f_z(t) dt - \int_{-\infty}^{\infty} f_z(t) dt &= 1 - \int_{-\infty}^{\infty} f_z(t) dt \\
= 1 - F_2(\beta), &\text{ so we usually find tables of } F_2(\beta), \beta > 0
\end{align*}
\]

**Theorem 3.31** (**)Linear Transformation Property.**

Let \( X \sim \mathcal{N}(\mu, \sigma^2) \), let \( Y = aX + b \), where \( a > 0 \). Then, \( Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2) \)

First, we show that \( E[Y] = a\mu + b \)

\[
E[Y] = aE[X] + b = a\mu + b
\]
\[ \text{Var}[Y] = a^2 \text{Var}[X], \quad \text{b/c of a general result (discussed below)} \]

Let \( Y = aX + b \), with \( X \) a r.v. (and of course, \( Y \) f.r.v.). Then,

\[ \text{Var}[Y] = \sum_y (y - \mathbb{E}[Y])^2 p_y = \sum_y (aX + b - a \mathbb{E}[X] - b)^2 p_y = \]

\[ \sum_x (aX - a \mathbb{E}[X])^2 p_x = \sum_x [a(X - \mathbb{E}[X])]^2 p_x = \]

\[ a^2 \sum_x (X - \mathbb{E}[X])^2 p_x = a^2 \text{Var}[X] \]

Now, show that \( Y \) is normally distributed.

\[ F_Y(y) = P\{Y \leq y\} = P\{aX + b \leq y\} = P\{X \leq \frac{y - b}{a}\} = F_X\left(\frac{y - b}{a}\right) \]

See book for the proof that \( \frac{1}{a} f_Y(y) = \frac{1}{\sqrt{2\pi a^2 \sigma^2}} e^{-\frac{(y-b)^2}{2a^2 \sigma^2}} \)
Tables give the cdf of a std. normal \( Y \sim N(0,1) \)

What if we have a r.v. \( X \sim N(\mu, \sigma^2) \)?

\( X \sim N(\mu, \sigma^2) \) iff \( Y = \frac{X - \mu}{\sigma} \sim N(0,1) \) by the linear transform property.

Easier to show it the other way around:

If \( Y \sim N(0,1) \) and \( X = \sigma Y + \mu \), then

\( X \sim N(\sigma \cdot 0 + \mu, \sigma^2 \cdot 1) = N(\mu, \sigma^2) \).

In your book, the pdf of the std. normal is called \( \Phi \) (instead of \( F_2 \) or \( F_Y \)). So, if \( X \sim N(\mu, \sigma^2) \)

\[
P\{X < k\} = P\left\{\frac{X - \mu}{\sigma} < \frac{k - \mu}{\sigma}\right\} = P\{Y < \frac{k - \mu}{\sigma}\} = \Phi\left(\frac{k - \mu}{\sigma}\right)
\]
Theorem 3.22 [H]

If $X \sim N(\mu, \sigma^2)$, then the probability that $X$ deviates from its mean ($\mu$) by less than $k$ standard deviations ($k\sigma$) is the same as the prob. that $Y \sim N(0, 1)$ deviates from its mean by less than $k$.

Proof

$P \left\{ -k \sigma < X - \mu < k \sigma \right\} = P \left\{ -k < \frac{X - \mu}{\sigma} < k \right\} = P \left\{ -k < Y < k \right\}$

Ex. (p. 60 [H]). $IQ \sim N(100, 15)$.

What is the prob. that someone’s IQ > 130?

130 is 2 standard deviations above the mean for $IQ$. 
\( F_Y(2) \) is the prob. that \( Y \) is within 2 std. deviations from its mean.  
\( 1 - F_Y(2) \) is the prob. that it is outside 2 std. deviations from its mean.  
By symmetry, 
\( 1 - F_Y(2) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228 \). 

Example 3.6 [T] (Towdi's notes)