Thm 3.25
For discrete r.v.s., \( E[X] = \sum_y E[X|Y=y] P\{Y=y\} \) (*)

For continuous r.v.s., \( E[X] = \int_{-\infty}^{\infty} E[X|Y=y] f_y(y) \, dy \)

Based on the law of total probability

\[ P\{X=k\} = \sum_y P\{X=k|Y=y\} P\{Y=y\} \]

This allows "case analysis."

We use (*) to compute the mean of \( X \) in geometric (\( \mu \)).
We condition on $Y$, the value of the first flip.

$$E[X] = \underbrace{E[X | Y=1] P(Y=1)} + \underbrace{E[X | Y=0] P(Y=0)} =$$

$$= \frac{1}{p} + \left(1 + E[X]\right)(1-p)$$

$$E[X] = \frac{1}{p} \quad \text{for any two r.v.s } X \text{ and } Y.$$
Let $X \sim \text{Binomial}(n, p)$

$$E[X] = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$  

Sorry! But we can think of $X$ as a sum of $n$ independent identically distributed r.v.s, which we will $X_i, i \in \mathbb{N}$,

$$X_i = \begin{cases} 1 & \text{if flip } i \text{ is successful} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X_i] = p$$

$$E[X] = E[X_1] + \ldots + E[X_n] = np$$

$n$ times

$X_i$ are indicator random variables.
Example: Hats

... game ... hats ... n players ... n hats ...

The r.v. $X$ is the number of players who get back their own hats.

$$I_i = \begin{cases} 1 & \text{if person } i \text{ gets back his/her own hat} \\ 0 & \text{otherwise} \end{cases}$$

$$X = I_1 + I_2 + \cdots + I_n$$

Note that the $I_i$'s are not independent r.v.s. However, linearity of expectation still holds, so

$$E[X] = E[I_1] + E[I_2] + \cdots + E[I_n] = n E[I_i] = n \left( 0 \cdot \frac{n - 1}{n} + 1 \cdot \frac{1}{n} \right) = n \cdot \frac{1}{n} = 1$$