317 2016-02-11

Note Title 2016-02-11

If $X \sim \text{Poisson}(\lambda)$, what is $E[X]$

$$p_X(i) = \frac{e^{-\lambda}\lambda^i}{i!} \quad (\text{pdf of } X \sim \text{Poisson}(\lambda)) \quad (0 \le i < \infty)$$

$$E[X] = \sum_{i=0}^{\infty} i\, p_X(i) = \sum_{i=0}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} = \sum_{i=1}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} =$$

$$= e^{-\lambda} \sum_{i=1}^{\infty} i \frac{\lambda^i}{i!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\cancel{i}\,\lambda^{i-1}}{\cancel{i}(i-1)\times(i-2)\times\cdots 1} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} =$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Moment of an r.v. The i-th moment of r.v. X, denoted $E[X^i]$ is

$$E[X^i] = \sum_x x^i p_X(x) \quad \text{(discrete case)}$$

x X

$$E[X^i] = \int_{-\infty}^{\infty} x^i f_X(x)\,dx \quad \text{(continuous case)}$$

More generally, the expectation of function $g(\cdot)$ of the r.v. X is:

$$E[g(X)] = \sum_x g(x) p_X(x) \quad \text{(discrete case)}$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

Example (p.46 [H])

$$X = \begin{cases} 0 & \text{w/ prob. } 0.2 \\ 1 & \text{w/ prob } 0.5 \\ 2 & \text{w/ prob } 0.3 \end{cases}$$

$$E[2X^2+3] = \sum_x (2x^2+3)\, p_X(x) = (2\cdot 0^2+3)\, p_X(0) + (2\cdot 1^2+3)\, p_X(1) +$$
$$+ (2\cdot 2^2+3)\, p_X(2) = 3\times 0.2 + 5\times 0.5 + 11\times 0.3 =$$
$$= 0.6 + 2.5 + 3.3 = 6.4$$

Defn. 3.17 Variance

The square of and we expect X to differ from its mean, $E[X]$.

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right] = \sum_x (x - E[X])^2\, p_X(x) \quad \text{(discrete r.v.)}$$

$$= \int_{-\infty}^{\infty} (x - E(x))^2 f_X(x)\,dx \quad \text{(continuous case)}$$

If $X \sim \text{Bernoulli}(p)$, what is $\text{Var}(X)$?

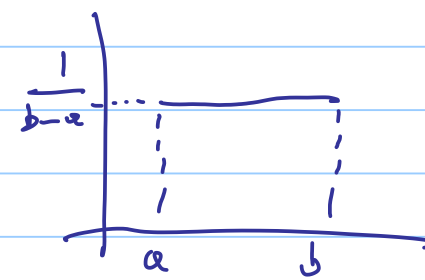
$$\text{Var}(X) = E\left[(X - E[X])^2\right] = (\text{recall, } E[X] = p) =$$

$$= E\left[(X-p)^2\right] = \sum_i (i-p)^2 p_X(i) = (0-p)^2 p_X(0) + (1-p)^2 p_X(1) =$$

$$= (0-p)^2(1-p) + (1-p)^2 p = (1-p)\left[p^2 + (1-p)p\right] = (1-p)(\cancel{p^2} + p - \cancel{p^2}) =$$

$$= p - p^2 = p(1-p)$$

If $X \sim \text{Uniform}(a,b)$, what is $\text{Var}[X]$?



$$\text{Var}[X] = E\left[(X - E[X])^2\right] = *$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) =$$

$$= \frac{1}{\cancel{(b-a)}}\,\frac{\cancel{(b-a)}(b+a)}{2} = \frac{b+a}{2} = \frac{a+b}{2}$$

$$\circledast \quad E\left[\left(X - \frac{a+b}{2}\right)^2\right] = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx = \cdots = \frac{(b-a)^2}{12}$$

Joint probability & independence [3.10 [H]]

The joint probability mass function between discrete r.v.s X and Y (of) is

$$p_{XY}(x, y) = P\underbrace{\{X = x, Y = y\}}_{\text{an event}}$$

The joint probability density function .....

$$\int_c^d \int_a^b f_{XY}(x, y)\, dx\, dy = P\{a < X < b,\ c < Y < d\}$$