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Note Title 2016-02-09

Three common <u>continuous</u> distribution (3.8.2 [H])

Recall:

The pdf (probability density function) of an r.v. X is a non-negative function $f_X(\cdot)$ s.t.

$$P\{a \le X \le b\} = \int_a^b f_X(x)\,dx, \text{ and where } \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

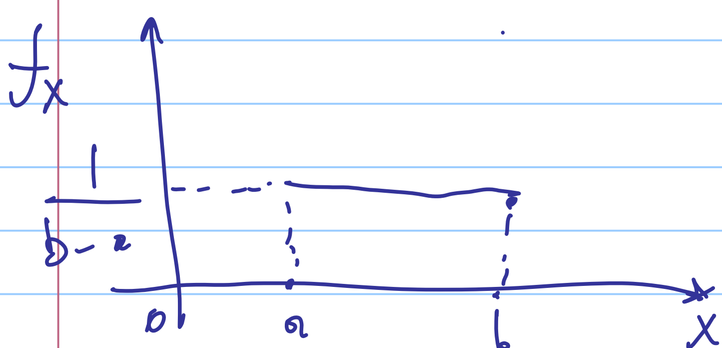
The cdf (cumulative distribution function) of an r.v. X is the function $F_X(\cdot)$ defined as

$$F_X(a) = P\{-\infty < X \le a\} = \int_{-\infty}^{a} f_X(x)\,dx.$$

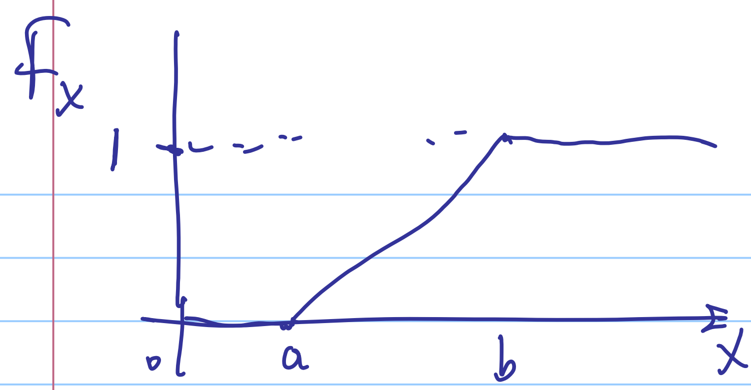
Uniform (a, b)

If $X \sim \text{Uniform}(a,b)$, $P\{c \le X \le d\}$, where $c \ge a$, $d \le b$ is the same for all equal $(d-c)$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



$$\underbrace{\int_{-\infty}^{\infty} f_X \, dx}_{=1} = \int_a^b c \, dx = c(b-a) \Rightarrow c = \frac{1}{b-a}$$

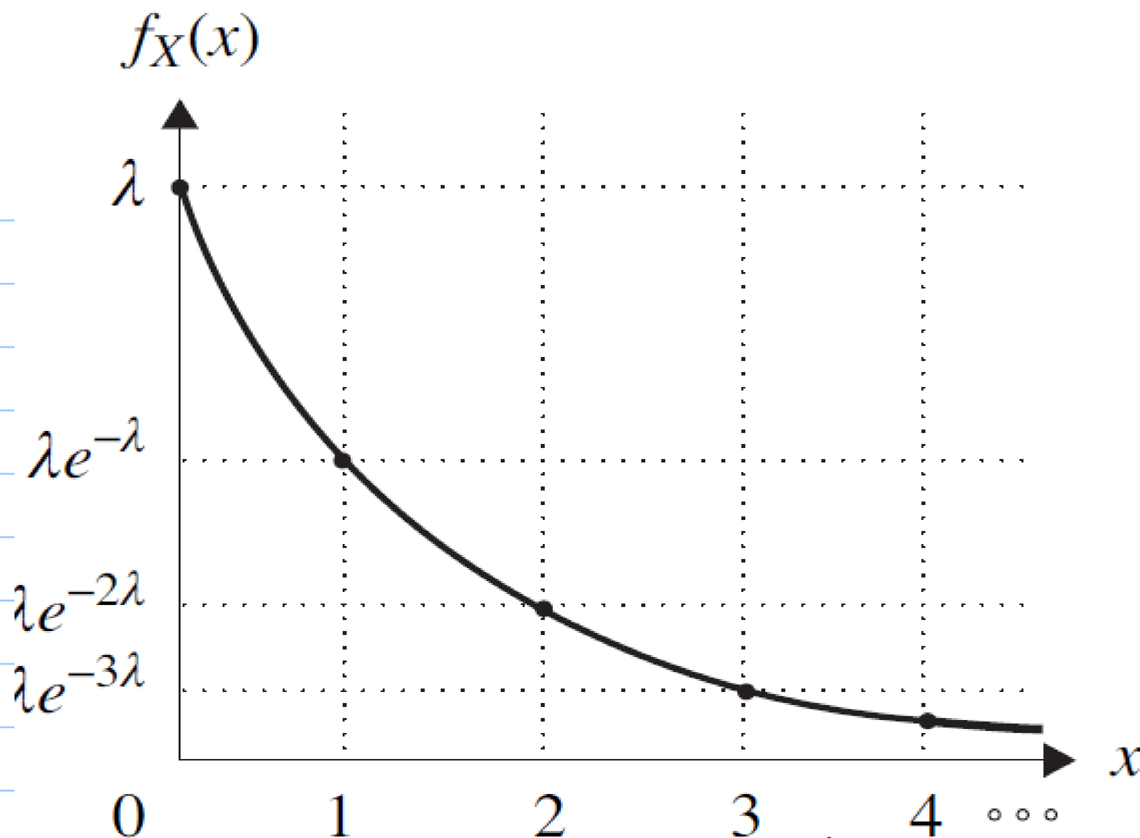


$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{-\infty}^{x} \frac{1}{b-a}\,dy =$$

$$\int_{a}^{x} \frac{1}{b-a}\,dy = \frac{1}{b-a}(x-a)$$

$\mathrm{Exp}(\lambda)$ (Exponential (λ))

$X \sim \mathrm{Exp}(\lambda)$ if $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$



$\lambda e^{-\lambda x}$

The cdf of $X \sim Exp(\lambda)$ $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \int_{0}^{x} \lambda e^{-\lambda y}\,dy = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

$\overline{F}_X(x) = e^{-\lambda x}$

Wilfried Pareto, a Swiss economist

Pareto (α)

This distribution decays as a polynomial in x, rather than an exponential, so it has a "heavy tail".
("fat tail")

$$f_X(x) = \begin{cases} \alpha x^{-\alpha-1} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = 1 - x^{-\alpha}$$

$$\bar{F}_X(x) = x^{-\alpha}$$

Another important continuous distribution is the Normal or Gaussian – we'll see it in a bit!

Expectation & Variance (3.9 (H))

<u>Discrete case</u>. Defn. $E[X] = \sum_x x\, p_X(x)$ is the <u>expected value</u> of the r.v. X

<u>Continuous case</u> Defn. $E[X] = \int_{-\infty}^{\infty} x\, p_X(x)\, dx$ is the <u>expected value</u> of the r. v. X

In the discrete case, you may consider $E[X]$ as the sum of each value of X multiplied by its probability

$$E[X] = \sum_x x\, P\{X = x\}$$

If $X \sim \text{Bernoulli}(p)$, what is $E[X]$?

$$E[X] = \sum_i i\, p_X(i) = 0 \cdot p_X(0) + 1 \cdot p_X(1) = 0 \times (1-p) + 1(p) =$$

$$= p$$

What is $E[X]$, where $X \sim \text{Geometric}(p)$?

$$E[X] = \sum_{n=1}^{\infty} \underbrace{n(1-p)^{n-1} p}_{p_X(n)} = p \sum_{n=1}^{\infty} n \cdot \underbrace{q^{n-1}}_{\substack{\text{defn.} \\ q = 1-p}} = p \sum_{n=1}^{\infty} \frac{d}{dq} q^n =$$

$$= p \frac{d}{dq} \sum_{n=1}^{\infty} q^n = p \frac{d}{dq} \left(\frac{q}{1-q} \right) = p \frac{d}{dq}\, q(1-q)^{-1} =$$

$$= p\left[1\,(1-q)^{-1} + q(-1)^{-1}(1-q)^{-2}\right] = p\,\frac{(1-q)+q}{(1-q)^2} = p\cdot\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$