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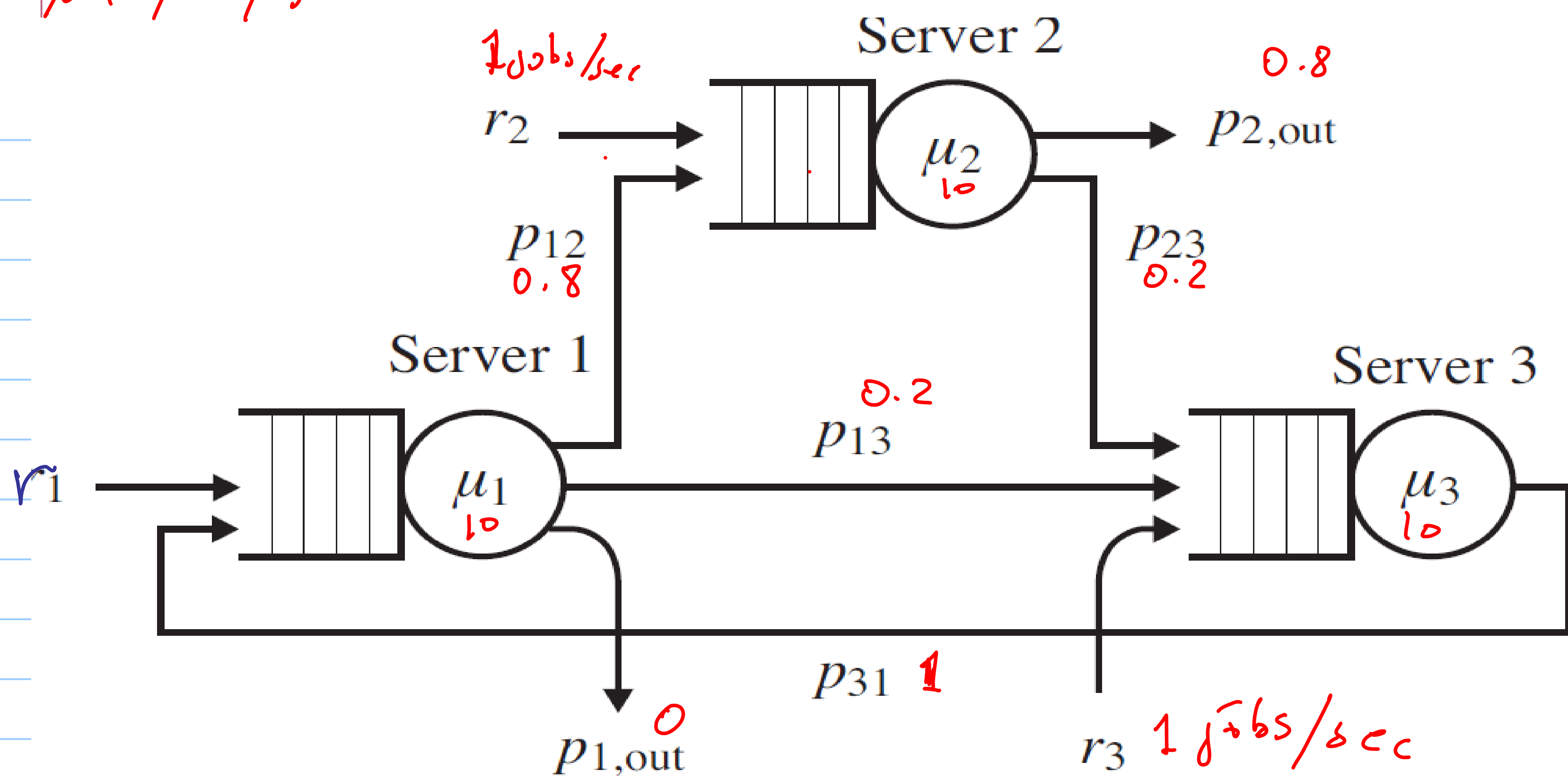
Note Title 2016-01-19

Classification of queueing networks (Sect. 2.3)

open vs. closed

Open networks (Sect. 2.4)

$\mu_1 = \mu_2 = \mu_3 = 10$ jobs/sec



Exercise 2.1 [H] What is the max r_1 to keep the system stable?

At server i, $\lambda_i = r_i + \sum_j \lambda_j P_{ji}$

in a stable open system
the arrival rate is equal to the departure rate

In order to have equilibrium at each server, we want $\lambda_i < \mu_i$ $(1 \le i \le 3)$.

$\lambda_1 = r_1 + \sum_{j=1}^{3} \lambda_j P_{j1} = r_1 + \lambda_1 P_{11} + \lambda_2 P_{21} + \lambda_3 P_{31} =$

$= r_1 + \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \lambda_3 \cdot P_{31} = r_1 + \lambda_3$

$\lambda_2 = r_2 + \sum_{j=1}^{3} \lambda_j P_{j2} = r_2 + \lambda_1 \times 0.8 + 0 + \lambda_3 \times 0 = r_2 + 0.8\lambda_1$

$$d_3 = r_3 + \sum_{j=1}^{3} d_j\, p_{j3} = r_3 + d_1 \times 0.2 + d_2 \times 0.2 + 0 =$$

$$= r_3 + 0.2\, d_1 + 0.2\, d_2$$

$$\begin{cases} d_1 = r_1 + d_3 \\ d_2 = r_2 + 0.8\, d_1 \\ d_3 = r_3 + 0.2\, d_1 + 0.2\, d_2 \end{cases} \qquad \begin{cases} d_1 = r_1 + d_3 \\ d_2 = 1 + 0.8\, d_1 \\ d_3 = 1 + 0.2\, d_1 + 0.2\, d_2 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ -0.8 & 1 & 0 \\ -0.2 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ 1 \\ 1 \end{bmatrix}$$

$$a x = b$$
$$a^{-1} a x = a^{-1} b$$
$$x = a^{-1} b$$

For stability, $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} < \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -0.8 & 1 & 0 \\ -0.2 & -0.2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ 1 \\ 1 \end{bmatrix} < \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

The inverse of a matrix A, called A^{-1}, is the matrix whose elements are $a_{ij} = \frac{A_{ji}}{|A|}$ ← minor, ← determinant

cofactor

A_{ji} is the determinant of the matrix obtained from

A by removing row j and column i and multiplying by −1 if i+j is odd.

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ -.8 & 1 & 0 \\ -0.2 & -0.2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ -0.2 & 1 \end{vmatrix} - 0 + (-1) \begin{vmatrix} -0.8 & 1 \\ -0.2 & -0.2 \end{vmatrix} =$$

$$= 1 - 0.36 = 0.64$$

$$A_{11} = \begin{vmatrix} 1 & 0 \\ -0.2 & 1 \end{vmatrix} = 1 \quad A_{21} = \begin{vmatrix} 0 & -1 \\ -0.2 & 1 \end{vmatrix} \times (-1) = -0.2$$

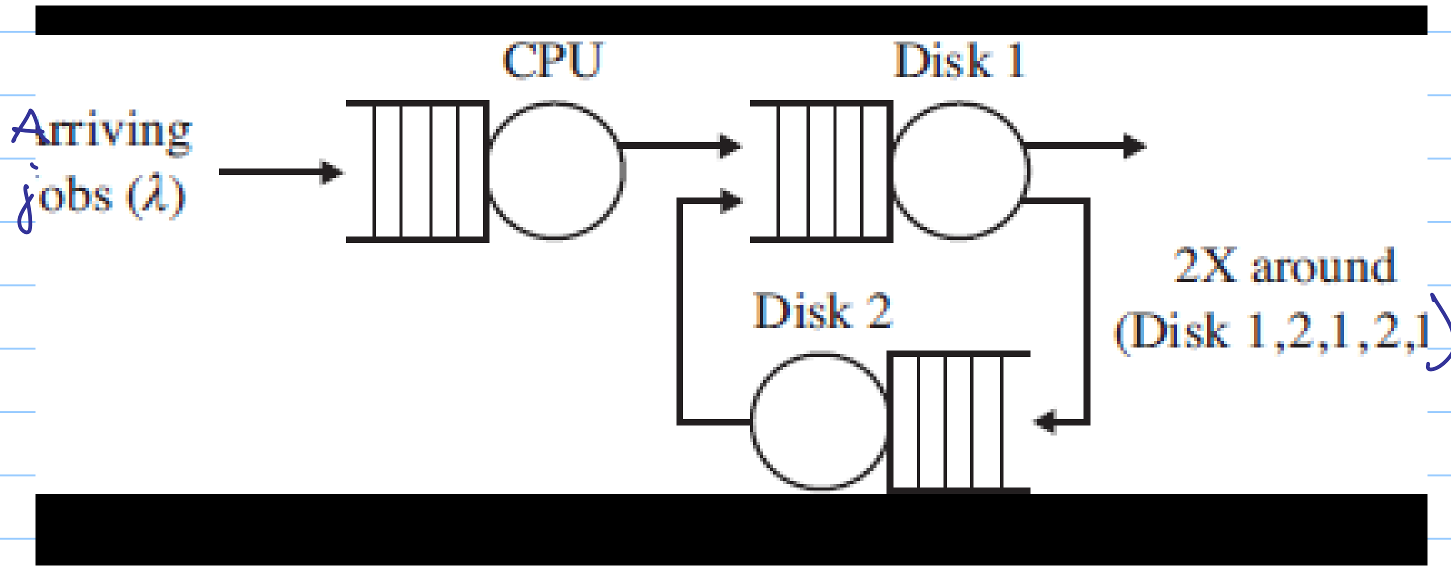
After similar computations, you get:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \underbrace{\frac{1}{16} \begin{bmatrix} 25 & 5 & 25 \\ 20 & 20 & 20 \\ 9 & 5 & 25 \end{bmatrix} \begin{bmatrix} r_1 \\ 1 \\ 1 \end{bmatrix} < \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}}$$

$\left(\frac{1}{.64}\right)$

$$\begin{cases} \frac{1}{16}(25r_1 + 5 + 25) < 10 \\ \frac{1}{16}(20r_1 + 20 + 20) < 10 \\ \frac{1}{16}(9r_1 + 5 + 25) < 10 \end{cases} \quad \begin{cases} r_1 < \frac{130}{25} = \frac{26}{5} \\ r_2 < 6 \\ r_3 < \frac{130}{9} \end{cases} \Rightarrow r_1 < \frac{26}{5}$$

Network of queues with non-probabilistic routing



Example: Finite Buffer