# 3/7 Ch 5 [H]

Note Title 2015-03-05

## Convergence of a sequence of numbers
(Defn. 5.1)

$\{a_n : n = 1, 2, \dots\}$ converges to $b$ as $n \to \infty$, written

$a_n \to b$, as $n \to \infty$, or equivalently,

$\lim_{n \to \infty} a_n = b$ if

$\forall \varepsilon > 0$, $\exists n_0(\varepsilon)$, such that $\forall n > n_0(\varepsilon)$, we have

$$|a_n - b| < \varepsilon$$

Convergence almost surely (i.e., with probability 1) (Defn. 5.2)

**Definition 5.2** The sequence of random variables $\{Y_n : n = 1, 2, \ldots\}$ ***converges almost surely*** to $\mu$, written

$$Y_n \xrightarrow{a.s.} \mu, \text{ as } n \to \infty$$

or equivalently, the sequence ***converges with probability 1***, written

$$Y_n \longrightarrow \mu, \text{ as } n \to \infty \text{ w.p. } 1$$

if

$$\forall k > 0, \mathrm{P}\left\{\lim_{i \to \infty} |Y_n - \mu| > k\right\} = 0.$$

The $\mathbf{P}\{\ldots\}$ in the previous expression is over the set of sample paths. More precisely we might write

$$\forall k > 0, \mathbf{P}\left\{\omega : \lim_{n\to\infty} |Y_n(\omega) - \mu| > k\right\} = 0$$

I.e., sample paths that do not converge to their mean become very rare.

# Convergence in probability

**Definition 5.3** The sequence of random variables $\{Y_n : n = 1, 2, \ldots\}$ ***converges in probability*** to $\mu$, written

$$Y_n \xrightarrow{P} \mu, \text{ as } n \to \infty$$

if

$$\forall k > 0, \lim_{n\to\infty} \mathbf{P}\{|Y_n - \mu| > k\} = 0.$$

The $\mathbf{P}\{\ldots\}$ in Definition 5.3 is over the set of possible sample paths, $\omega$. More precisely we might write

$$\forall k > 0, \lim_{n\to\infty} \mathbf{P}\{\omega : |Y_n(\omega) - \mu| > k\} = 0 \qquad (5.1)$$

It may be no sample path converges in the limit, but the values that are different from the mean on each sample path get rarer and rarer, so that their total mass goes to zero

# Example of convergence in probability from wikepedia

http://en.wikipedia.org/wiki/Convergence_of_random_variables

"Archer"

Suppose a person takes a bow and starts shooting arrows at a target. Let Xn be his score in n-th shot. Initially he will be very likely to score zeros, but as the time goes and his archery skill increases, he will become more and more likely to hit the bullseye and score 10 points. After the years of practice the probability that he hit anything but 10 will be getting increasingly smaller and smaller and will converge to 0. Thus, the sequence Xn converges in probability to X = 10.

Note that Xn does not converge almost surely however. No matter how professional the archer becomes, there will always be a small probability of making an error. Thus the sequence {Xn} will never turn stationary: there will always be non-perfect scores in it, even if they are becoming increasingly less frequent.

Examples of almost sure convergence (equivalently, convergence with probability 1) from Wikipedia

http://en.wikipedia.org/wiki/Convergence_of_random_variables

Consider an animal of some short-lived species. We record the amount of food that this animal consumes per day. This sequence of numbers will be unpredictable, but we may be quite certain that one day the number will become zero, and will stay zero forever after.

Consider a man who tosses seven coins every morning. Each afternoon, he donates one pound to a charity for each head that appeared. The first time the result is all tails, however, he will stop permanently.

Let X1, X2, … be the daily amounts the charity received from him.

We may be almost sure that one day this amount will be zero, and stay zero forever after that.

However, when we consider any finite number of days, there is a nonzero probability the terminating condition will not occur.

# Weak Law of Large Numbers

**Theorem 5.4 (Weak Law of Large Numbers)** *Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with mean $\mathbf{E}[X]$. Let*

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad Y_n = \frac{S_n}{n}.$$

*Then*

$$Y_n \xrightarrow{P} \mathbf{E}[X], \text{ as } n \to \infty.$$

*This is read as "$Y_n$ converges in probability to $\mathbf{E}[X]$," which is shorthand for the following:*

$$\forall k > 0, \lim_{n \to \infty} \mathbf{P}\{|Y_n - \mathbf{E}[X]| > k\} = 0.$$

Comments on Exercise 5.1. (Prove WLLNs)

Markov's inequality. Do example 4.15 [Trivedi]

If $X$ is non-negative then

$$P\{X \geq t\} \leq \frac{E[X]}{t}, \quad \forall\, t \geq 0$$

Chebychev's inequality

Do Example 4.16 [Trivedi]

$$P\{|X - \underbrace{E[X]}_{\mu}| \geq t\} \leq \frac{\sigma_X^2}{t^2}$$

Complete proof of WLLNs.

# Strong Law of Large Numbers

**Theorem 5.5 (Strong Law of Large Numbers)** *Let $X_1, X_2, X_3, \ldots$ be i.i.d random variables with mean $\mathbf{E}[X]$. Let*

$$S_n = \sum_{i=1}^{n} X_i \quad \text{and} \quad Y_n = \frac{S_n}{n}.$$

*Then*

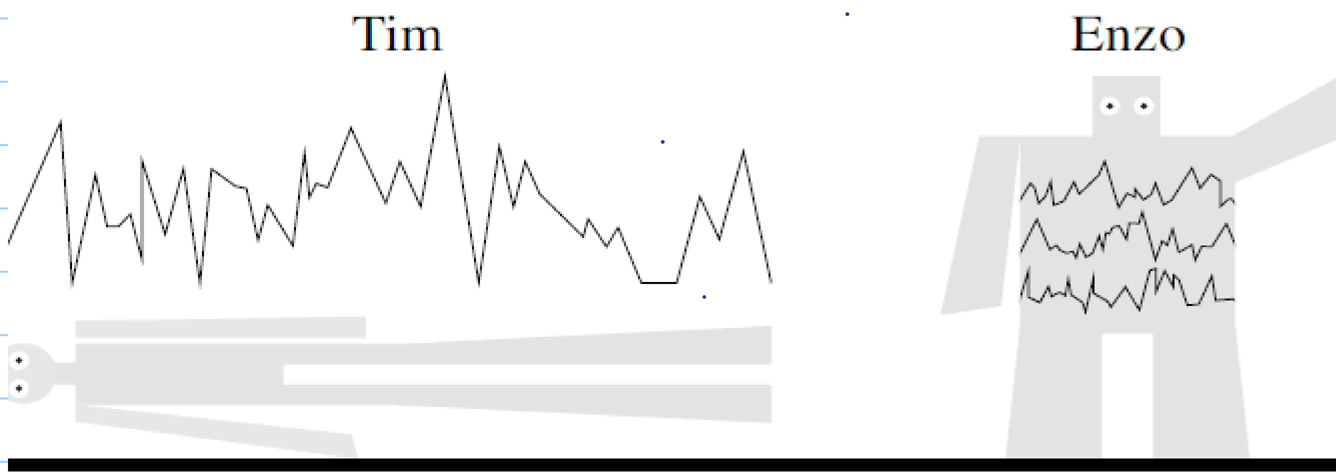
$$Y_n \xrightarrow{a.s.} \mathbf{E}[X], \text{ as } n \to \infty.$$

*This is read as "$Y_n$ converges almost surely to $\mathbf{E}[X]$" or "$Y_n$ converges to $\mathbf{E}[X]$ with probability 1," which is shorthand for the following:*

$$\forall k > 0, \mathbf{P}\left\{\lim_{n\to\infty} |Y_n - \mathbf{E}[X]| \geq k\right\} = 0.$$

# Time Average vs. Ensemble Average

Tim & Enzo



**Definition 5.6**

$$\overline{N}^{\text{Time Avg}} = \lim_{t\to\infty} \frac{\int_0^t N(v)dv}{t}.$$

**Definition 5.7**

$$\overline{N}^{\text{Ensemble}} = \lim_{t\to\infty} \mathbf{E}[N(t)] = \sum_{i=0}^{\infty} i p_i$$

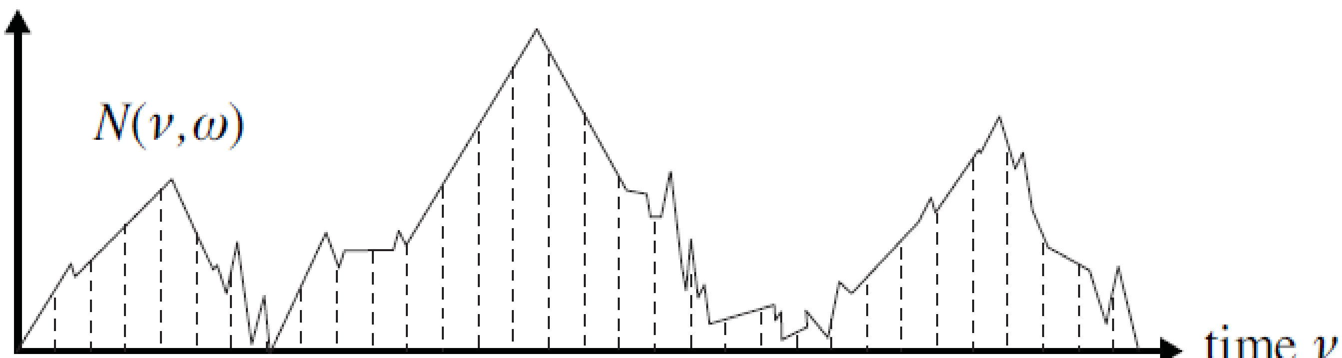
where

$$\begin{aligned} p_i &= \lim_{t\to\infty} \mathbf{P}\{N(t)=i\} \\ &= \text{mass of sample paths with value } i \text{ at time } t. \end{aligned}$$

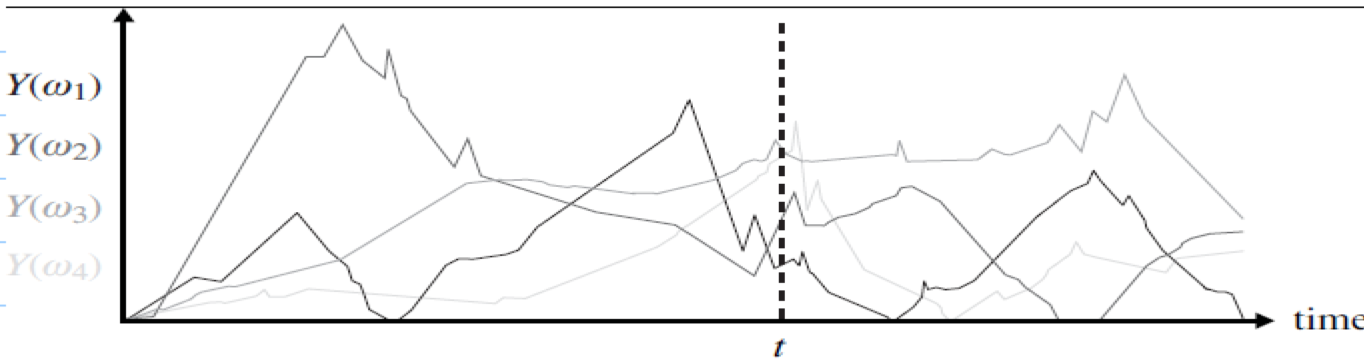
Definition 5.8 (restatement of 5.6)

$$\overline{N}^{\text{Time Avg}}(\omega) = \lim_{t \to \infty} \frac{\int_0^t N(v,\omega)\, dv}{t},$$

where $N(v,\omega)$ represents the number of jobs in the system at time $v$ under sample path $\omega$.




Time average (Fig. 5.5)




Ensemble average (Fig. 5.6)

# Equivalence of averages & ergodicity

**Theorem 5.9** *For an "ergodic" system (see Definition 5.10), the ensemble average exists and, with probability 1,*

$$\overline{N}^{\text{Time Avg}} = \overline{N}^{\text{Ensemble}}.$$

*That is, for (almost) all sample paths, the time average along that sample path converges to the ensemble average.*

**Definition 5.10** An ***ergodic*** system is one that is positive recurrent, aperiodic, and irreducible.

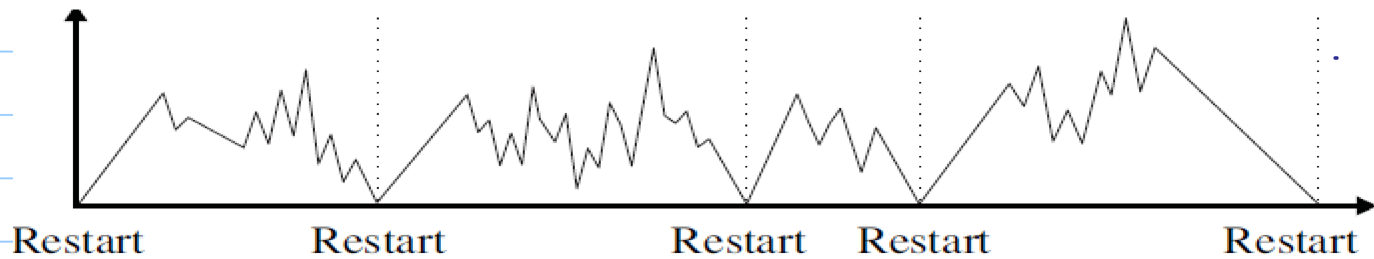



Fig. 5.7 Single Process Restarting Itself

- Simulation

- Average Time In System