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Note Title 2015-01-29

Random variable

"A random variable is neither."

A random variable (in the classical interpretation) is a rule that assigns a numerical value to each outcome of an experiment.

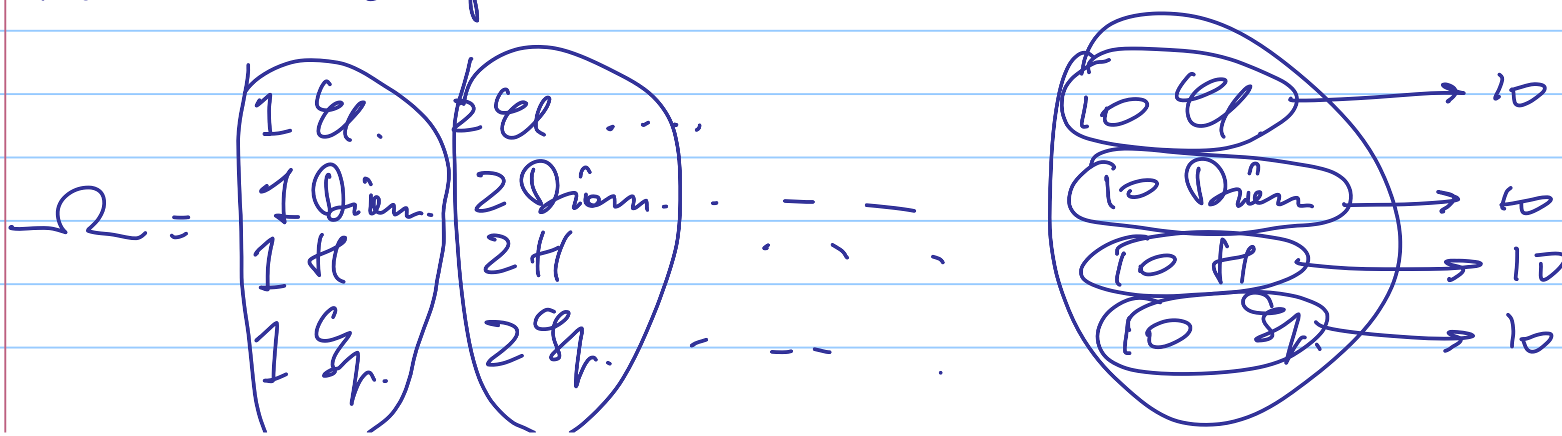
More formally (and for any model of the axioms of K.), it is a function from Ω to the real numbers.

Example.

Let Ω be the set of all cards from a 40-card deck (no face cards).

Then, the <u>value of a card</u> is a random variable.

This random variable maps the 4 cards that have a particular value to that value



(Each $s \in \Omega$ is mapped to an integer between 1 and 10)

$$\downarrow \qquad \downarrow \qquad \qquad \downarrow$$
$$1 \qquad 2 \qquad \qquad 10$$

In general, a random variable partitions Ω.

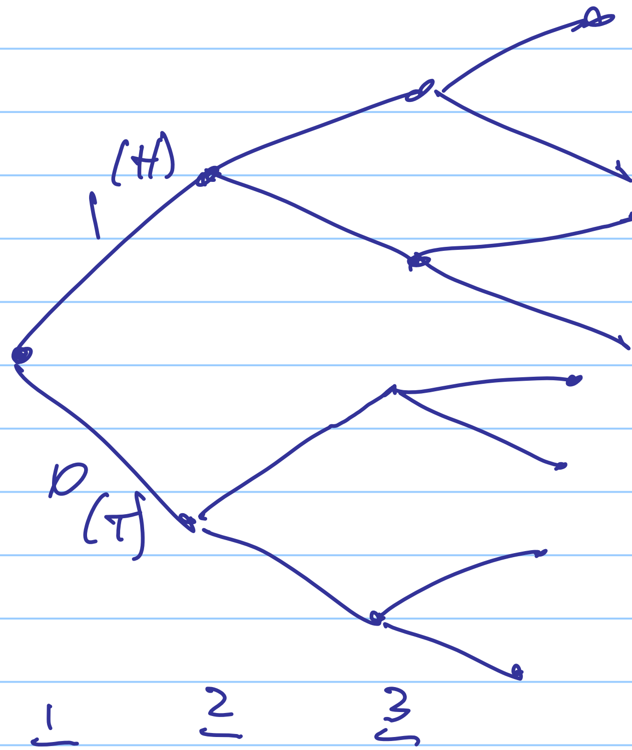
There is a generalization of the defn, which is called a propositional variable.

A propositional variable partitions the state space Ω.

Each partition corresponds to a value of the propositional variable.

Example

Fair coin toss



Experimental Outcomes Sample Points (s)		Random Variable X (# of heads) X(s)
111	0.125	3
110	0.125	2
101	0.125	2
100	0.125	1
011	0.125	2
010	0.125	1
001	0.125	1
000	0.125	0

"A sequence of three Bernoulli trials with $p = 0.5$"

"A binomial $(3, 0.5)$"

The random variable defines 4 events, one per value

$$E_0 = \{ s \in \Omega \mid X(s) = 0 \} = \{ 000 \}$$

$$E_1 = \{ s \in \Omega \mid X(s) = 1 \} = \{ 001, 010, 100 \}$$

$$E_2 = \{ s \in \Omega \mid X(s) = 2 \} = \{ 011, 101, 110 \}$$

$$E_3 = \{ s \in \Omega \mid X(s) = 3 \} = \{ 111 \}$$

A discrete r.v. can take values from a set of discrete numbers (finite or countable).

[Examples up to now are discrete.]

A continuous r.v can take on uncountable set of possible values.

$$
\Omega = \left\{ \begin{array}{cccccc}
 & \cancel{E_1} & & & \cancel{E_2} & \\
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array} \right.
$$

Sum of values is a discrete r.v. whose values are 2, 3, ... 12; let's call it T

$E_2 = \{ s \in \Omega \mid T(s) = 2 \} = \{ (1,1) \}$

$E_{11} = \{ s \in \Omega \mid T(s) = 11 \} = \{ (6,5), (5,6) \}$

Probability Mass Function (p.m.f.)

The pmf of an r.v. X, written $p_X(\cdot)$ is

$$p_X(a) = P(X=a), \text{ where } \sum_x p_X(x) = 1$$

We take $P(X=a)$ to be the probability of the event defined by $X=a$.

Ex. (Three coin tosses) $P(X=3) = 0.125 = P(\{111\})$

$P(X=1) = 0.375 = P(\{100, 010, 001\})$

Ex. (two dice) $P(7) = P(\{(1\,6), (2\,5), (3,4), (4,3), (5\,2), (6\,1)\} = \frac{6}{36} = \frac{1}{6}$

Cumulative Distribution Function (cdf) of a rv X is

$$F_X(a) = P(\{X \le a\}) = \sum_{x \le a} p_X(x)$$

Ex. (sum of two dice) $F_X(4) = P(X \le 4) = \sum_{x \le 4} p_X(x) =$

$= \cancel{p_X(1)} + p_X(2) + p_X(3) + p_X(4) = P(\{(1\,1)\}) + P(\{(1\,2),(2\,1)\}) +$

$+ P(\{(3\,1), (2\,2), (1\,3)\}) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36} = \frac{1}{6}$

We also write

$$\overline{F}_X(a) = P(\{X > a\}) = \sum_{x > a} p_X(x) = 1 - F_X(a)$$

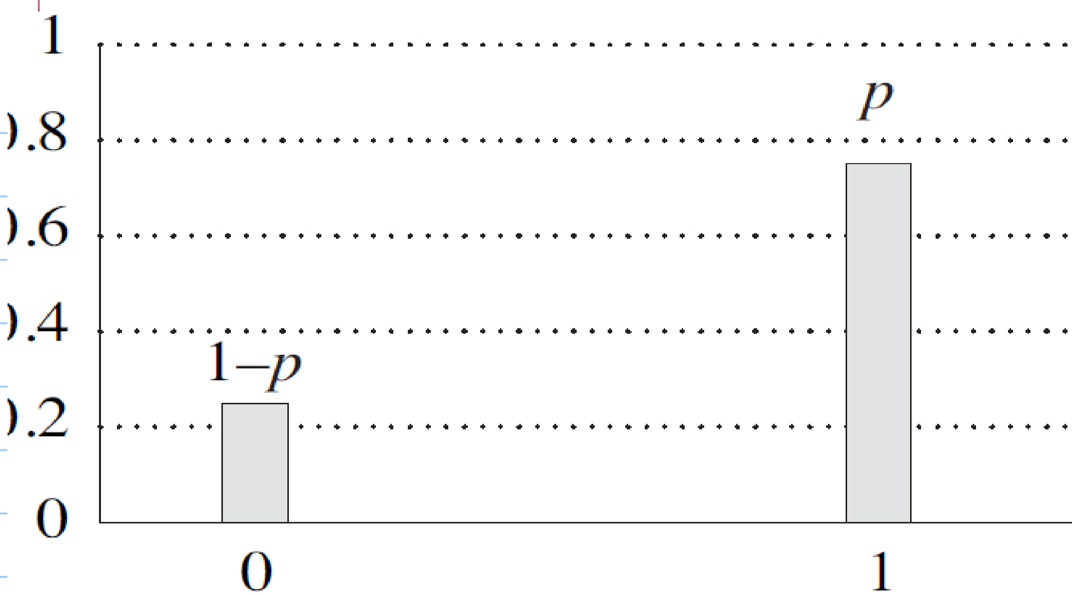
Some common discrete distributions.

Bernoulli (p) parameter   Corresponds to

flipping one coin biased w/ prob. p.

The pmf (probability mass function) of r.v. X is

$$\begin{cases} p_X(1) = p \\ p_X(0) = 1-p \end{cases} \qquad \Bigg\| \qquad X = \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{otherwise} \end{cases}$$



the pmf of the Bernoulli distribution

the cdf

Binomial(n, p) (two parameters)

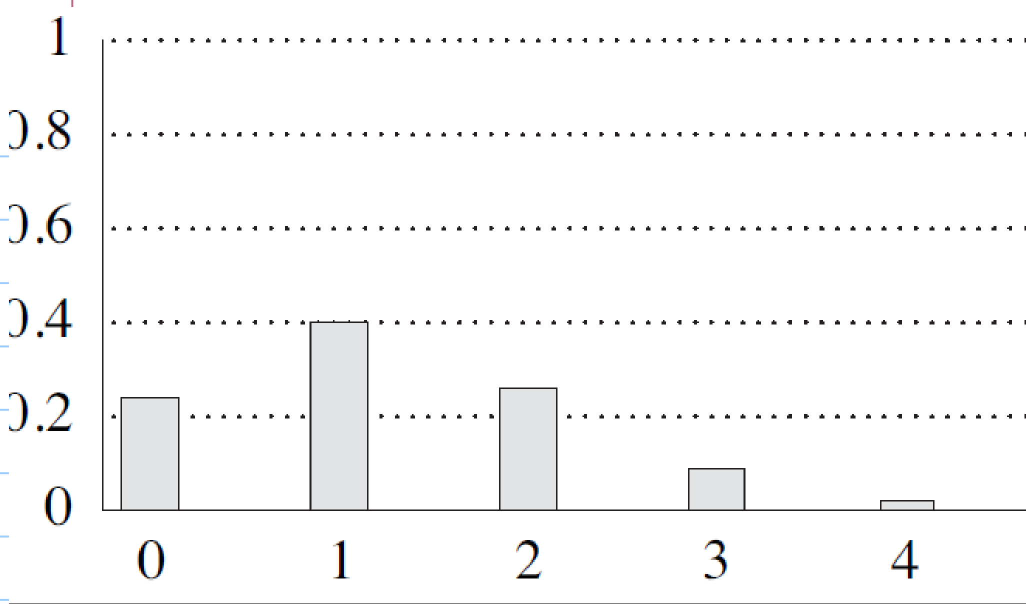
If $X \sim$ Binomial(n,p), [X is a r.v. drawn from a Binomial(n,p)]

then X can take values 0 through n.

Each value represents the number of heads (successes, 1s).

The p.m.f. of r.v. X is defined as follows:

$$p_X(i) = P(\{X = i\}) = \binom{n}{i} p^i (1-p)^{n-i}, \text{ for } 0 \le i \le n$$



pmf of the Binomial$(4, 0.3)$ distribution

$n = 4$

$p = 0.3$

Geometric (p) distribution

$$p_X(i) = P(\{X = i\}) = (1-p)^{i-1}\, p, \quad \text{where } i = 1, 2, 3, \ldots$$