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Note Title 2015-01-27

		E_1			E_2	
$\Omega =$	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Mutually exclusive events;

$E_1 \cap E_2 = \phi$

$P(E_1 \cap E_2) \overset{def}{=}$

E_1 and E_2 are <u>independent</u> if $P(E_1 \cap E_2) \overset{!}{=} P(E_1) \cdot P(E_2)$

Are these E_1 and E_2 independent?

$P(E_1) = \frac{6}{36} = \frac{1}{6}$ $P(E_2) = \frac{3}{36} = \frac{1}{12}$. $P(E_1 \cap E_2) = 0 \neq \frac{1}{6} \cdot \frac{1}{12}$

So, E_1 and E_2 are not independent.

Thm. If E and F are independent, then

$P(E \mid F) = P(E)$.

Proof.

$$P(E \mid F) \underset{\text{defn. (4th axiom)}}{=} \frac{P(E \cap F)}{P(F)} \overset{\text{b/c } E \text{ and } F \text{ are independent}}{=} \frac{P(E) \cdot P(F)}{P(F)} = P(E).$$

Note that E and F can be exchanged, so if E and F are independent then both

$P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$.

E_1 = first roll is 6

E_2 = second roll is 6

$\cancel{E_1}$ $\cancel{E_2}$

E_2

$$\Omega = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right.$$

E_1

$E_1 \cap E_2$

$P(E_1) = \frac{6}{36} = \frac{1}{6}$

$P(E_2) = \frac{6}{36} = \frac{1}{6}$

$P(E_1 \cap E_2) = \frac{1}{36} = P(E_1) \cdot P(E_2)$ ✓

So, E_1 and E_2 are independent

E_1 = sum of the rolls is 7

E_2: second roll is 4

$$\begin{array}{cccccccc} & & & \cancel{E_1} & & E_2 & \cancel{E_2} & \\ \Omega = & \left\{ \right. & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ & & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ & & (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ & & (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ & & (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ & & (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}$$

E_1

$E_1 \cap E_2$

$$P(E_1) = \frac{6}{36} = \frac{1}{6}$$

$$P(E_2) = \frac{6}{36} = \frac{1}{6}$$

$$P(E_1 \cap E_2) = \frac{1}{36} \overset{?}{=} P(E_1) \cdot P(E_2)$$

<u>Yes</u>

So, E_1 and E_2 are independent

E_1: sum of the rolls is 8

E_2: second roll is 4

$$\Omega = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & \boxed{(4,4)} & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right.$$

$\cancel{E_1}$ $\quad E_2 \quad$ $\cancel{E_2}$

E_1

$E_1 \cap E_2$

$P(E_1) = \frac{5}{36}$

$P(E_2) = \frac{6}{36} = \frac{1}{6}$

$P(E_1 \cap E_2) = \frac{1}{36} \overset{?}{=} P(E_1) \cdot P(E_2)$

$\frac{1}{36} \overset{?}{=} \frac{5}{36} \cdot \frac{1}{6}$

E_1 and E_2 are <u>not</u> independent.

$P(E_1 \mid E_2) \neq P(E_1)$

$P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} \overset{?}{=} \frac{5}{36}$ <u>No</u>.

$$P(E_2 \mid E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{\frac{1}{36}}{\frac{5}{36}} = \frac{1}{5} \overset{?}{=} \frac{1}{6} \quad \underline{\text{No}}$$

Law of total probability (thm)

Let $F_1, F_2, \ldots, F_n$ (be events that) partition the state space Ω. Then,

$$\underline{\underline{P(E)}} = \sum_{i=1}^{n} P(E \cap F_i) = \underline{\underline{\sum P(E \mid F_i) \cdot P(F_i)}}$$

(where E is a subset of Ω) ("analysis by cases").

Example top p. 37.

$P(T|D) =$

T \ D	present	absent
pos	.95	.05
neg	.05	.95

(accuracy of test)

$$P(D) = \frac{1}{10{,}000}$$

$$P(D = \text{present} \mid T = \text{pos}) = \frac{P(T = \text{pos} \mid D = \text{pres})\, P(D = \text{pres})}{P(T = \text{pos})} =$$

$$= \frac{P(T = \text{pos} \mid D = \text{pres})\, P(D = \text{pres})}{P(T = \text{pos} \mid D = \text{pres})\, P(D = \text{pres}) + P(T = \text{pos} \mid D = \text{absent})\, P(D = \text{absent})} =$$

$$= \frac{.95 \times \frac{1}{10,000}}{.95 \times \frac{1}{10,000} + 0.05 \times \frac{9,999}{10,000}} = 0.0019$$