7.1.3

a) Eliminate $E$-productions.

b) Eliminate unit productions

c) Eliminate any useless symbols

d) Put in Chomsky Normal Form

\[ S \rightarrow OA0 \mid IB1 \mid BB \]
\[ A \rightarrow C \]
\[ B \rightarrow S \mid A \]
\[ C \rightarrow S \mid \varepsilon \]

Note: $L(\varepsilon)$ contains $\varepsilon$ so we will find an equivalent grammar for $L(\varepsilon) - \varepsilon$

a) Eliminate $E$-productions

\[ C \rightarrow \varepsilon \quad \text{so} \quad C \text{ is nullable.} \]
\[ A \rightarrow C \Rightarrow \varepsilon \quad \text{so} \quad A \text{ is nullable.} \]
\[ B \rightarrow A \rightarrow \varepsilon \quad \text{so} \quad B \text{ is nullable.} \]
\[ S \rightarrow BB \Rightarrow \varepsilon \quad \text{so} \quad S \text{ is nullable.} \]

\[ E_1 \text{ productions} \]

\[ S \rightarrow OA0 \]
\[ S \rightarrow 00 \quad \text{since} \ A \text{ is nullable} \]
\[ S \rightarrow IB1 \quad \text{since} \ B \text{ is nullable} \]
\[ S \rightarrow 11 \quad \text{since} \ B \text{ is nullable} \]
\[ S \rightarrow BB \quad \text{since} \ B \text{ is nullable} \]
\[ S \rightarrow B \quad \text{since} \ B \text{ is nullable} \]
\[ A \rightarrow C \quad \text{note} \ A \rightarrow \varepsilon \text{ should not be added} \]
\[ B \rightarrow S \mid A \quad \text{even though} \ C \text{ is nullable.} \]
\[ C \rightarrow S \]
b) Remove unit production:

1) First find all unit pairs
2) For each unit pair (A, B) add A → x for each non-unit production B → x

S → B ⇒ A → C ⇒ S each of these derives in one step

First (S, S), (A, A), (C, C) and (B, B)

So (B, S), (S, A), (S, C) are all unit pairs

Since B → A ⇒ C ⇒ S ⇒ B so (B, A) (B, C) (B, S)
Similarly for A → C so (A, C) (A, S) (A, B)
and (C, S), (C, B), (C, A) are also unit pairs.

G₂ productions

(S, B) adds nothing because all B productions are unit productions

Similarly for (S, A) and (S, C)

(S, S) adds S → OAO | 00 | 1B1 | 11 | BB
(A, A) adds nothing because all A productions are unit
(B, B) and (C, C) add nothing also because all productions are unit

Similarly for (X, A), (X, B) and (X, C) nothing gets added because all A productions, B productions, and C productions are unit productions

Now (B, S) is a unit pair so we add productions

B → OAO | 00 | 1B1 | 11 | BB
(A, S) is a unit pair so we add
A → OAO | 00 | 1B1 | 11 | BB
(C, S)
C → OAO | 00 | 1B1 | 11 | BB
So $G_2$

\[
S \Rightarrow OA_0 \mid 00 \mid 1B_1 \mid 11 \mid BB
\]
\[
A \Rightarrow OA_0 \mid 00 \mid 1B_1 \mid 11 \mid BB
\]
\[
B \Rightarrow OA_0 \mid 00 \mid 1B_1 \mid 11 \mid BB
\]
\[
C \Rightarrow OA_0 \mid 00 \mid 1B_1 \mid 11 \mid BB
\]

Now to eliminate useless symbols

\[
S \Rightarrow 11
\]
\[
A \Rightarrow 11
\]
\[
B \Rightarrow 11
\]
\[
C \Rightarrow 11 \quad \text{So all are generating}
\]

Also
\[
S \Rightarrow OA_0 \quad \text{so } A \text{ is reachable.}
\]
\[
S \Rightarrow 1B_1 \quad \text{so } B \text{ is reachable.}
\]

\[
C \text{ is not reachable, so it is not an RHS of any production.}
\]

So $G_3$ is $G_2$ minus the $C$ productions

d) put in Chomsky Normal Form

First add Nonterminals $U$ and $Z$ with productions

\[
U \Rightarrow 1 \text{ and } Z \Rightarrow 0
\]

Rewrite the $G_3$ production replacing all $0$s with $Z$'s and $1$'s with $U$'s

Then most productions are ok except for

\[
S \Rightarrow ZAZ \mid UBU
\]
\[
A \Rightarrow ZAZ \mid UBU
\]
\[
B \Rightarrow ZAZ \mid UBU
\]
$S \rightarrow ZAZ$

Add $N_1$

$S \rightarrow ZN_1, \ N_1 \rightarrow AZ$

Similarly replace add $N_2$

$S \rightarrow UN_2, \ N_2 \rightarrow BU$

We also for the $A$ production, we need to add $N_3$ and $N_4$

$A \rightarrow ZN_3, \ N_3 \rightarrow AZ$

$A \rightarrow UN_4, \ N_4 \rightarrow BU$

And similarly for the $B$ production we need to add $N_5$ and $N_6$

$B \rightarrow ZN_5, \ N_5 \rightarrow AZ$

$B \rightarrow UN_6, \ N_6 \rightarrow BU$

And we are done.
\[ S \rightarrow A A A | B \]
\[ A \rightarrow a A | B \]
\[ B \rightarrow c \]

a) Remove \( c \) production

Note: 
\[ A \rightarrow B \Rightarrow c \] so \( A \) and \( B \) are nullable
\[ S \rightarrow B \Rightarrow c \] so \( S \) is nullable

\[ S \rightarrow A | AA | AAA | B \]
\[ A \rightarrow a | a A | B \]

b) Unit productions

\((S,S)\), \((A,A)\) and \((B,B)\) are unit pairs

\[ S \Rightarrow A \quad (S,A) \text{ is a unit pair} \]
\[ S \Rightarrow B \quad (S,B) \]
\[ A \Rightarrow B \quad (A,B) \]

\((S,S)\) adds \( S \Rightarrow AA | AAA \)
\((A,A)\) adds \( A \Rightarrow a | a A \)
\((S,A)\) adds \( S \Rightarrow a | a A \)
\((S,B)\) adds nothing
\((A,B)\) adds nothing

\[ G_2 \quad S \Rightarrow AA | AAA | a | a A \]
\[ A \Rightarrow a | a A \]

\( \boxed{c} \) Useless symbols "\( c \)" were removed in last step.

Both \( S \) and \( A \) are generating \( S \Rightarrow a, A \Rightarrow a \)
Add nonterminal \( N \) and \( N \rightarrow a \)

Replace \( S \rightarrow AAA \) with
\( S \rightarrow AR \quad R \rightarrow AA \)

Yielding

\[
\begin{align*}
S & \rightarrow AA \mid a \mid NA \\
A & \rightarrow a \mid NA \\
N & \rightarrow a \\
R & \rightarrow AA
\end{align*}
\]