csce750 — Analysis of Algorithms
Fall 2019 — Lecture Notes: Summations

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

There are a couple of goals for this section.

- *To remind you of the basics of expressing the run time of an iterative algorithm using a summation.*
- *To remind you of some of the most commonly-used identities for simplifying summations.*
- *To demonstrate a few 'tricks' that can be used to solve many summations that occur in the analysis of algorithms.*

1 Definition

CLRS A

A common tool for analyzing iterative algorithms is the **summation**:

$$\sum_{i=\ell}^{u} a_i = a_\ell + a_{\ell+1} + \cdots + a_{u-1} + a_u$$

If the upper limit is infinite, we interpret this as an implicit limit:

$$\sum_{i=\ell}^{\infty} a_i = \lim_{u\to\infty} \sum_{i=\ell}^{u} a_i$$

2 Two examples

Do these two algorithms have the same asymptotic running time?

```
int AlgA(int n) {
  int sum = 0;
  for(int i=0; i<n; i++) {
    for(int j=0; j<i; j++) {
      sum++;
    }
  }
  return sum;
}
```

```
int AlgB(int n) {
  int sum = 0;
  for(int i=0; i<n; i++) {
    for(int j=0; j<n; j++) {
      sum++;
    }
  }
  return sum;
}
```

3 Same asymptotic run times

Yes, the run times are both $\Theta(n^2)$.

$$A(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} i = \frac{(n-1)n}{2} = \Theta(n^2)$$

$$B(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1 = \sum_{i=0}^{n-1} n = n^2 = \Theta(n^2)$$

4 Two more examples

Do these two algorithms have the same asymptotic running time?

```
int AlgC(int n) {
  int sum = 0;
  for(int i=1; i<n; i*=2) {
    for(int j=0; j<i; j++) {
      sum++;
    }
  }
  return sum;
}
```

```
int AlgD(int n) {
  int sum = 0;
  for(int i=1; i<n; i*=2) {
    for(int j=0; j<n; j++) {
      sum++;
    }
  }
  return sum;
}
```

5 Different asymptotic run times

No, the asymptotic run times are different.
Observe that i is always a power of 2. Let $i = 2^k$, so that $k = \lg i$.

$$\begin{aligned} C(n) &= \sum_{k=0}^{\lceil \lg n \rceil - 1} \sum_{j=0}^{2^k - 1} 1 = \sum_{k=0}^{\lceil \lg n \rceil - 1} 2^k \\ &= 2^{\lceil \lg n \rceil} - 1 = \Theta(n) \end{aligned}$$

$$\begin{aligned} D(n) &= \sum_{k=0}^{\lceil \lg n \rceil - 1} \sum_{j=0}^{n-1} 1 = \sum_{k=0}^{\lceil \lg n \rceil - 1} n \\ &= \lceil \lg n \rceil \, n = \Theta(n \log n) \end{aligned}$$

6 Some common summations

The arithmetic series:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

Sums of squares, cubes, and higher powers:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \Theta(n^4)$$

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

7 Proof for $\sum i^k$

To show that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$, we need both upper and lower bounds.

Upper bound:

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}$$

Lower bound:

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil n/2 \rceil}^{n} i^k \geq \sum_{i=\lceil n/2 \rceil}^{n} \left(\frac{n}{2}\right)^k = \frac{n}{2} \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} n^{k+1}$$

Therefore, we can use $c_1 = \frac{1}{2^{k+1}}$ and $c_2 = 1$ as the constants in the Θ definition.

The trick we're using here for the upper bound is sometimes called 'bounding the terms'. The idea is to simplify the expression by replacing all of the terms with something larger, which makes the value of entire summation larger. For the lower bound, we are also bounding the terms, but we are also using another trick that we might call 'discarding terms'. If we simply eliminate some non-negative terms, we make the entire summation smaller.

8 Finite geometric series

For constants r and n, the **finite geometric series** is:

$$\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r - 1}$$

9 Deriving the finite geometric series formula

To prove this closed form is correct, let S denote the sum. We have:

$$\begin{aligned}
S - rS &= \sum_{i=0}^{n-1} r^i - r \sum_{i=0}^{n-1} r^i \\
&= \sum_{i=0}^{n-1} r^i - \sum_{i=0}^{n-1} r^{i+1} \\
&= \sum_{i=0}^{n-1} r^i - \sum_{i=1}^{n} r^i \\
&= 1 - r^n
\end{aligned}$$

Then solve for S:

$$S = \frac{r^n - 1}{r - 1}$$

10 Infinite geometric series

For a constant $r < 1$, the **infinite geometric series** is:

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

11 Deriving the infinite geometric series formula

We can use the finite version, along with the definition of an infinite summation, to evaluate this:

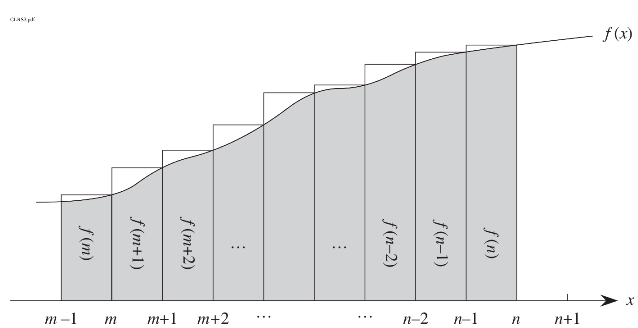
$$\begin{aligned}
\sum_{i=0}^{\infty} r^i &= \lim_{n \to \infty} \sum_{i=0}^{n-1} r^i \\
&= \lim_{n \to \infty} \frac{1 - r^n}{1 - r} \\
&= \frac{1}{1-r} - \frac{1}{1-r} \lim_{n \to \infty} r^n \\
&= \frac{1}{1-r}
\end{aligned}$$

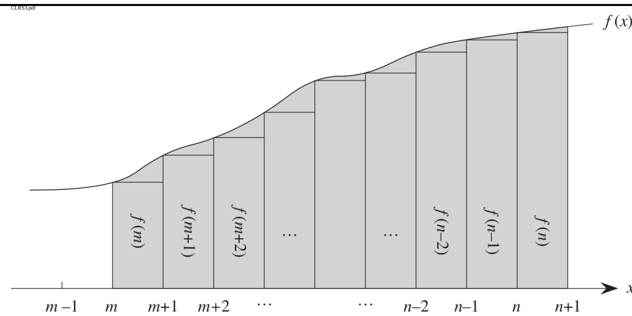
12 Bounding with integrals

Sometimes, an integral may be easier to evaluate than a sum.

If $f(n)$ is monotone increasing:

$$\int_{m-1}^{n} f(x)\,\mathrm{d}x \le \sum_{i=m}^{n} f(i) \le \int_{m}^{n+1} f(x)\,\mathrm{d}x$$





If $f(n)$ is monotone decreasing:

$$\int_{m}^{n+1} f(x)\,\mathrm{d}x \le \sum_{i=m}^{n} f(i) \le \int_{m-1}^{n} f(x)\,\mathrm{d}x$$