csce750 — Analysis of Algorithms
Fall 2019 — Lecture Notes: Solving Recurrences — Master Theorem

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

1 Master theorem: Simple version

Theorem: Consider the recurrence

$$T(n) = aT(n/b) + \Theta(n^d).$$

If $a > b^d$ then $T(n) = \Theta(n^{\log_b a})$.
If $a = b^d$ then $T(n) = \Theta(n^d \log n)$.
If $a < b^d$ then $T(n) = \Theta(n^d)$.

For this simple version, the final added part must be a polynomial.

2 Master theorem: Real version

Theorem: Consider the recurrence

$$T(n) = aT(n/b) + f(n).$$

1. If there exists $\epsilon > 0$, for which $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If there exists $\epsilon > 0$, for which $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $af(n/b) \leq cf(n)$ for some constant c and sufficiently large n, then $T(n) = \Theta(f(n))$.

3 Example 1

$$T(n) = 9T(n/3) + n$$

We have $a = 9$, $b = 3$, and $f(n) = n$.

Compare $n^{\log_3 9} = n^2$ to n. Observe that $n = O(n^{2-\epsilon})$, with $\epsilon = 1$.

Therefore, the first case applies, and $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$.

4 Example 2

$$T(n) = T(2n/3) + 1$$

We have $a = 1$, $b = 3/2$, and $f(n) = 1$.

Compare $n^{\log_{3/2} 1} = 1$ to 1. Observe that $1 = \Theta(1)$.

Therefore, the second case applies, and $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(\log n)$.

5 *Example 3*

$$T(n) = 3T(n/4) + n \log n$$

We have $a = 3$, $b = 4$, and $f(n) = n \log n$.

Compare $n^{\log_4 3}$ to $n \log n$. Observe that $n \log n = \Omega(n^{\log_4 3+\epsilon})$, as long as $\log_4 3 + \epsilon \leq 1$. (For example, choose $\epsilon = 0.2$.)

The "regularity condition" also holds.

Therefore, the third case applies, and $T(n) = \Theta(n \log n)$.

6 *Example 4*

$$T(n) = 2T(n/2) + n \log n$$

We have $a = 2$, $b = 2$, and $f(n) = n \log n$.

Compare $n^{\log_2 2} = n$ to $n \log n$. Observe that, although $n \log n = \Omega(n)$, for any $\epsilon > 0$, $n \log n \neq \Omega(n^{1+\epsilon})$.

(Intuition: Even a very tiny polynomial term n^{ϵ} will eventually grow faster than $\log n$.)

Therefore, the Master theorem does not apply.