1 Recursion trees

We can solve many recurrences by drawing a recursion tree.

- **Nodes**: Label with the contribution to the total for that ‘recursive call’.
  
  … *not* counting what happens inside children.

- **Children**: One for each appearance of a recurrent term.

After drawing such a tree, we can solve the recurrence:

1. Count (or bound) the depth of the leaves.
2. Compute (or bound) the sum for each level.
3. Compute (or bound) the sum across all levels.

2 Example: Mergesort recurrence

\[
T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
2T\left(\frac{n}{2}\right) + \Theta(n) & \text{otherwise}
\end{cases}
\]

\[
\begin{array}{c}
\text{cn} \\
\downarrow \\
\text{cn/2} \\
\downarrow \\
\text{cn/4} \quad \text{cn/4} \\
\downarrow \\
\text{cn/4} \quad \text{cn/4} \\
\downarrow \\
\vdots \\
\Theta(1) \quad \ldots \quad \Theta(1)
\end{array}
\]

3 Example: Mergesort recurrence

- Depth of the leaves: \( \lg n \)
- Sum for each level: \( cn \)
- Sum across all levels: \( cn \lg n + \Theta(n) = \Theta(n \log n) \).
4 Example: Another divide-and-conquer recurrence

\[ T(n) = \begin{cases} 
\Theta(1) & \text{if } n \leq 1 \\
3T\left(\frac{n}{4}\right) + \Theta(n^2) & \text{otherwise} 
\end{cases} \]

5 Example continued

- Depth of the leaves: \( \log_4 n \)
- Sum for each level: \( 3^i c(n/4^i)^2 = (3/16)^i cn^2 \).
- Sum across all levels:

\[
T(n) = \sum_{i=0}^{\log_4 n} \left(\frac{3}{16}\right)^i cn^2 \\
\leq \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 \\
= \frac{1}{1 - (3/16)} cn^2 \\
= \frac{16}{13} cn^2 \\
= O(n^2)
\]

Note also: \( T(n) = \Omega(n^2) \). (Why?)

At depth \( i \), the ‘problem size’ is \( n/2^i \). To get down to the base case, we need this value to be 1 or less.

6 A lopsided tree

\[ T(n) = T(n/3) + T(2n/3) + O(n) \]
Depth of the (deepest) leaves: $\log_{3/2} n$

- Sum for each level: $\leq cn$

- Sum across all levels: $cn \lg n = O(n \log n)$.

7 Some branches terminate before others

Note that this recurrence does not produce a complete tree!

For $n = 6$ (assuming $\lfloor n/3 \rfloor$ and $\lfloor 2n/3 \rfloor$):

Therefore, the sum from the previous slide gives an upper bound. We could also get a lower bound by truncating the tree at the level of its shallowest leaves.