csce750 — Analysis of Algorithms
Fall 2019 — Lecture Notes: Lower Bounds

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

1 Can we sort by comparisons faster than $\Theta(n \log n)$?

CLRS 8.1

We have seen two sorting algorithms that run in $\Theta(n \log n)$ time in the worst case. Can we do any better?

If the algorithm is based on **comparisons** between array elements, then the answer is: No.

(Chapter 8 describes a few sorting algorithms *not* based on comparisons that can be faster.)

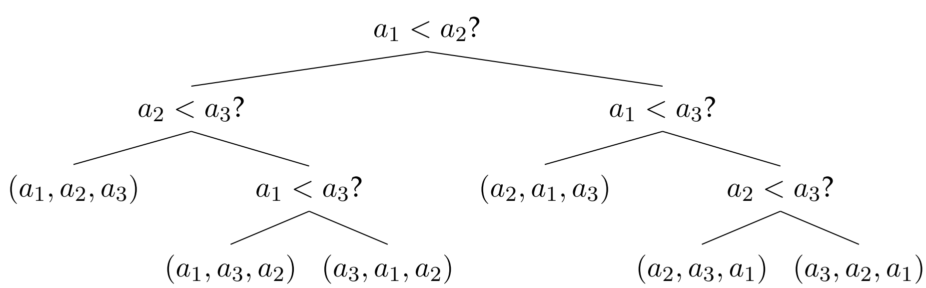
2 Decision trees

Given a comparison-based algorithm and an input size n, we can build a **decision tree**.

- **Internal nodes** are labeled with comparisons.
- **Edges** show the algorithm's progress based on the results of each comparison.
- **Leaves** represent final permutations of the elements.
- Other algorithm details are not shown in the decision tree.

Key idea: The height of the tree gives a lower bound on the run time of the algorithm.

3 Example: Sorting for $n = 3$



4 How short can a sorting decision tree be?

In any decision tree that correctly sorts n distinct elements:

- There must be at least $n!$ leaves, one for each permutation of the n elements.
- A binary tree with height h has at most 2^h leaves.

Therefore $n! \leq 2^h$, and we have

$$
\begin{aligned}
T(n) &\geq h \geq \log n! = \log\left(\prod_{i=1}^{n} i\right) \\
&= \sum_{i=1}^{n} \log i \geq \sum_{i=\lceil n/2 \rceil}^{n} \log i \\
&\geq \sum_{i=\lceil n/2 \rceil}^{n} \log \lceil n/2 \rceil \geq (n/2)\log(n/2) \\
&= \Omega(n \log n)
\end{aligned}
$$

Therefore: Any correct comparison-based sorting algorithm takes $\Omega(n \log n)$ time.