1 Introduction

A hash table is a specific kind of data structure implementing these operations:

- INSERT(k)
- SEARCH(k)
- DELETE(k)

The keys are drawn from a universe \( \mathcal{U} \):

\[
\mathcal{U} = \{0, 1, \ldots, |\mathcal{U}| - 1\}
\]

(We are skipping CLRS 10, on “Elementary Data Structures.”)

2 A simple solution: Direct address tables

We could use an array \( T \) of size \(|\mathcal{U}|\), and store the element with key \( k \) at \( T[k] \).

**Advantage:** Fast. All operations \( \Theta(1) \) time. Very simple.

**Disadvantage:** Requires lots of memory if \( \mathcal{U} \) is large.

3 Hashing

**Key idea:** Instead of storing \( k \) at \( T[k] \), choose an efficiently computable hash function:

\[
h : \mathcal{U} \to \{0, \ldots, m - 1\},
\]

with \( m < |\mathcal{U}| \).

Store \( k \) at \( T[h(k)] \).

(Intuition: The hash function must be deterministic, and should be “random-looking.”)
4 Building a hash function

Creating hash functions is not a precise science, but there are some useful patterns.

- **Division method:**
  \[ h(k) = k \mod m \]

- **Multiplication method:**
  \[ h(k) = \lfloor m(kA - \lfloor kA \rfloor) \rfloor \]

With good choices for \( m \) and \( A \), both of these tend to produce good approximations of the simple uniform hashing assumption:

For each \( k \) we insert and each \( j = 0, \ldots, m - 1 \),

\[ Pr(h(k) = j) = \frac{1}{m} \]

This leads to an average case analysis, because it starts with an assumption about the inputs we’ll get.

5 Example: Python hash function for string keys

```python
static long string_hash(PyStringObject *a) {
    register Py_ssize_t len;
    register unsigned char *p;
    register long x;

    if (a->ob_shash != -1)
        return a->ob_shash;
    len = Py_SIZE(a);
    p = (unsigned char *) a->ob_sval;
    x = *p << 7;
    while (--len >= 0)
        x = (1000003*x) ^ *p++;
    x ^= Py_SIZE(a);
    if (x == -1)
        x = -2;
    a->ob_shash = x;
    return x;
}
```

6 Collisions

Because \( m < |U| \), we must be prepared for collisions, which occur when we insert two distinct keys \( k_1 \) and \( k_2 \), with \( h(k_1) = h(k_2) \).

There are two primary choices for resolving collisions:

- **Chaining** — Use a secondary data structure (linked list, hash table, etc.) for each element of \( T \).
- **Probing** — If \( h(k) \) is occupied, try somewhere else.
7 Analysis: Hashing with linked-list chaining

Searching, worst case: Collision every time. $\Theta(n)$ time.

Note: Insert is faster: $\Theta(1)$.

8 Analysis: Hashing with linked-list chaining

Searching, under simple uniform hashing assumption: For an unsuccessful search:

$$A(n) = \Theta(1) + \sum_{i=0}^{m-1} \frac{1}{m} \text{length}(T[i])$$

$$= \Theta(1) + \frac{1}{m} \sum_{i=0}^{m-1} \text{length}(T[i])$$

$$= \Theta(1) + \frac{n}{m}$$

$$= \Theta(1 + \frac{n}{m})$$

For a successful search: Similar, but messier. (CLRS 259)

$\Theta \left(1 + \frac{n}{m}\right)$

If $n = O(m)$ — table size proportional to number of keys — then this is $O(1 + cm/m) = O(1)$ time.

9 A word about probing

In the probing method, the table itself stores everything, without any other data structures.

Simple example: If $h(k)$ is occupied, try $h(k) + 1$, then $h(k) + 2$, then ... . (For better choices, see CLRS 272.)

Advantage: Can be more memory efficient, because all of the available memory is devoted to the table. No pointers.

Disadvantage: Deleting is harder. (Why?)