csce750 — Analysis of Algorithms
Fall 2019 — Lecture Notes: Shortest Paths on Graphs

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

1 Introduction

CLRS 24

Single-source shortest path problem:

- Input: A weighted directed graph G with no negative weights, stored as adjacency lists, and a **start vertex** s in G.
- Output: For each vertex v in G, a path in G from s to v that minimizes the total weight of edges crossed.

Key idea: Subpaths of shortest paths are also shortest paths.

- If $v_i \rightsquigarrow v_k \rightsquigarrow v_j$ is a shortest path from v_i to v_j,
- then $v_i \rightsquigarrow v_k$ is a shortest path from v_i to v_k,
- and $v_k \rightsquigarrow v_j$ is a shortest path from v_k to v_j.

2 Dijkstra's algorithm

For each vertex v, keep track of:

- $v.d$: the length of the shortest known path from s to v
- $v.\pi$: a **predecessor** of vertex v on that path

Use a priority queue Q of vertices, keyed by their d values.

- Start with all nodes in Q. Start with each $v.d = \infty$, except at the start node.
- For the node v with the lowest d, consider each edge $v \to u$.
- If $v.d + w(v, u) < u.d$, update $u.d$ and $u.\pi$, then DECREASEKEY on u.

3 Dijkstra's analysis

With a simple array for the priority queue:

- Initialization: $O(V)$
- V EXTRACTMIN operations: $O(V^2)$
- E DECREASEKEY operations: $O(E)$

- Total: $T(n) = O(V) + O(V^2) + O(E) = O(V^2)$

With a binary heap:

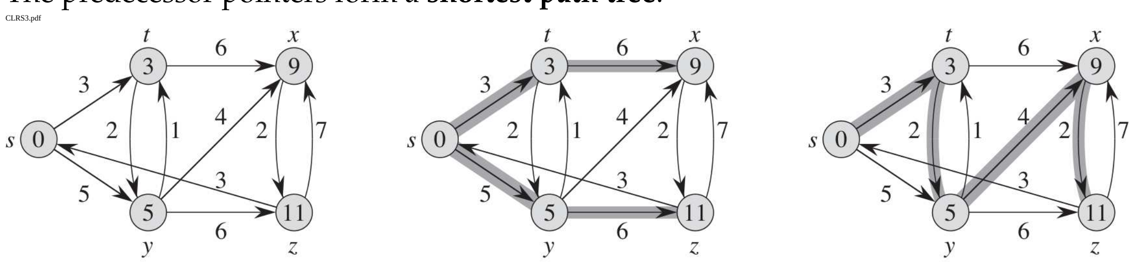
- Initialization: (BUILDMINHEAP): $O(V)$
- V EXTRACTMIN operations: $O(V \log V)$
- E DECREASEKEY operations: $O(E \log V)$
- Total: $T(n) = O(V) + O(V \log V) + O(E \log V) = O(E \log V)$

With a Fibonacci heap:

- Initialization (V INSERT operations): $O(V)$
- V EXTRACTMIN operations: $O(V \log V)$
- E DECREASEKEY operations: $O(E)$
- Total: $T(n) = O(V) + O(V \log V) + O(E) = O(E + V \log V)$

4 Shortest path trees

The predecessor pointers form a **shortest path tree**.



(a) (b) (c)

5 Be careful about negative-weight edges!

Recall that we assumed that no edges have negative weights. What happens if this assumption is violated?

- Dijkstra's algorithm may give incorrect results. When?
- The shortest path may not even be well-defined. When?

6 Bellman-Ford algorithm

For negative weights, use the Bellman-Ford algorithm instead.

BELLMANFORD(G, w, s)

for $v \in G.V$ **do**
  $v.d = \infty$
  $v.\pi = \text{NIL}$
end for
for $i = 1, \ldots, |G.V| - 1$ **do**
  for each edge (u, v) in $G.E$ **do**
    if $u.d + w(u, v) < v.d$ **then**
      $v.d = u.d + w(u, v)$
      $v.\pi = u$
    end if
  end for
end for
for each edge (u, v) in $G.E$ **do**
  if $u.d + w(u, v) < v.d$ **then**
    return "Negative weight cycle found."
  end if
end for