csce750 — Analysis of Algorithms
Fall 2019 — Lecture Notes: Amortized Analysis

This document contains slides from the lecture, formatted to be suitable for printing or individual reading, and with some supplemental explanations added. It is intended as a supplement to, rather than a replacement for, the lectures themselves — you should not expect the notes to be self-contained or complete on their own.

1 Introduction

CLRS 17

Amortized analysis is a technique for measuring the time needed to perform a *sequence* of operations on a data structure.

Your textbook describes three overlapping methods of amortized analysis:

- **Aggregate method**: Sum the total work across any sequence of n operations, and divide by n.
- **Accounting method**: Add extra costs to early, less expensive operations, to "prepay" for later, more expensive operations.
- **Potential method**: Define a "potential function" on the complete data structure, and sum the actual cost with the change in potential.

We will focus only on the **potential method**, which is more powerful than the other two.

Key idea: Amortized analysis is intended to capture the idea that "expensive" operations are rare enough to be acceptable, by analyzing *sequences* rather than individual operations.

2 Example data structure: Multipop Stack

Consider a stack-like data structure with the following operations:

- PUSH(x)
- POP()
- MULTIPOP(k) – try to pop k times, but stop if stack is empty.

Suppose we implement a data structure with these operations using a linked list.

How long does each operation take?

How long can a sequence of n operations take?

3 Goal of amortized analysis

We want to assign an **amortized cost** to each operation.

Notation:

- Actual cost of operation i: c_i
- Amortized cost of operation i: $\widehat{c_i}$

We need to guarantee that, for **any** sequence of n operations,

$$\sum_{i=1}^{n} \widehat{c_i} \geq \sum_{i=1}^{n} c_i.$$

4 Potential method

Let $D_i \in \mathcal{D}$ denote a 'snapshot' of the data structure *after* operation i.

1. **Define** a **potential function** Φ that maps data structure snapshots to real numbers.

$$\Phi : \mathcal{D} \to [0, \infty)$$

Intuition: The potential should represent the amount of "prepayment" that has been done.

- Inexpensive, common operations generally increase the potential.
- Expensive but infrequent operations generally decrease the potential.

5 Valid potential functions

2. **Verify** that that potential function has these two properties:

- The initial data structure has zero potential:

$$\Phi(D_0) = 0$$

- The potential is never negative:

$$\Phi(D_i) \geq 0 \text{ for all } i$$

6 Computing amortized costs

3. **Compute** the amortized cost of operation i as the actual cost plus the change in potential:

$$\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

7 Why the potential method works

This process is useful because the sum telescopes:

$$\begin{aligned}
\sum_i \widehat{c_i} &= \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) \\
&= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0) \\
&= \sum_{i=1}^{n} c_i + \Phi(D_n) \\
&\geq \sum_{i=1}^{n} c_i
\end{aligned}$$

For any sequence of operations, the actual cost is less than or equal to the amortized cost.

8 Multipop Stack: Potential method

1. Choose a potential function:

$$\Phi(S) = \text{ number of items in stack } S$$

2. Verify that the potential function is valid:

   - Do we have $\Phi(D_0) = 0$?
   - Do we have $\Phi(D_i) \geq 0$ for all i?

3. Compute amortized costs:

   - PUSH: $\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2 = \Theta(1)$
   - POP: $\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0 = \Theta(1)$
   - MULTIPOP:

$$\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = \min(k, s) - \min(k, s) = 0 = \Theta(1)$$

9 Example data structure: Dynamic tables

Consider an array-like data structure with these operations:

- INSERTATEND(k)
- LOOKUP(i)

Implement using arrays, and reallocating a new bigger array when needed.

TABLEINSERT(x)
if T.size $= 0$ **then**
  allocate T.table with 1 slot
  T.size $= 1$
else if T.num $= T$.size **then**
  allocate N with $2T$.size slots
  insert all items from T.table into N
  free T.table
  T.table $= N$
  T.size $= 2T$.size
end if
insert x into T.table
T.num $= T$.num $+ 1$

10 Dynamic tables: Potential method

1. Choose a potential function:
$$\Phi(T) = 2T.\text{num} - T.\text{size}$$
2. Verify that the potential function is valid:
   - Do we have $\Phi(D_0) = 0$?
   - Do we have $\Phi(D_i) \geq 0$ for all i?
3. Compute amortized costs:
   - INSERT (elementary): $c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3 = \Theta(1)$.
   - INSERT (reallocation):
$$\begin{aligned}\widehat{c_i} &= c_i + \Phi(D_i) - \Phi(D_{i-1}) \\ &= n_i + (2n_i - s_i) - (2n_{i-1} - s_{i-1}) \\ &= n_i + 2n_i - 2n_{i-1} - 2n_{i-1} + n_{i-1} \\ &= 3n_i - 3n_{i-1} = 3 = \Theta(1)\end{aligned}$$
   - LOOKUP: $c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 0 = 0 = \Theta(1)$.

One way to understand this potential function is that we want something

- *equal to the table size when the table is full, and*
- *zero right after the table is reallocated.*

That captures the idea that simple insert operations should increase the potential to 'save up' for the expensive reallocate step in the future.
When computing the amortized costs, note that $s_{i-1} = n_{i-1}$ (since the table was full), and $n_i - n_{i-1} = 1$ (since we've inserted a single item).