An Illustrated Proof of The Front-Door Adjustment Theorem

Mohammad Ali Javidian\textsuperscript{1}  Marco Valtorta\textsuperscript{1}

\textsuperscript{1}Department of Computer Science
University of South Carolina

June, 2018
<table>
<thead>
<tr>
<th>Outline</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The Back-Door Criterion</td>
</tr>
<tr>
<td>• Definition 1. (Back-Door)</td>
</tr>
<tr>
<td>• Theorem 1. (Back-Door Adjustment)</td>
</tr>
<tr>
<td>2. The Front-Door Criterion</td>
</tr>
<tr>
<td>• Definition 2. (Front-Door)</td>
</tr>
<tr>
<td>• Theorem 2. (Front-Door Adjustment)</td>
</tr>
<tr>
<td>3. do Calculus</td>
</tr>
<tr>
<td>4. Proof of Theorem 2.</td>
</tr>
</tbody>
</table>
Definition 1. (Back-Door)

A set of variables $Z$ satisfies the *back-door criterion* relative to an ordered pair of variables $(X_i, X_j)$ in a DAG $G$ if:

(i) no node in $Z$ is a descendant of $X_i$; and
(ii) $Z$ blocks every path between $X_i$ and $X_j$ that contains an arrow into $X_i$.

Figure: $S_1 = \{X_3, X_4\}$ and $S_2 = \{X_4, X_5\}$ would qualify under the back-door criterion, but $S_3 = \{X_4\}$ would not because $X_4$ does not $d$-separate $X_i$ from $X_j$ along the path $(X_i, X_3, X_1, X_4, X_2, X_5, X_j)$.
Definition 1. (Back-Door)

**Back-Door Criterion**

A set of variables $Z$ satisfies the *back-door criterion* relative to an ordered pair of variables $(X_i, X_j)$ in a DAG $G$ if:

(i) no node in $Z$ is a descendant of $X_i$; and

(ii) $Z$ blocks every path between $X_i$ and $X_j$ that contains an arrow into $X_i$.
Definition 1. (Back-Door)

A set of variables \( Z \) satisfies the \textit{back-door criterion} relative to an ordered pair of variables \((X_i, X_j)\) in a DAG \( G \) if:

(i) no node in \( Z \) is a descendant of \( X_i \); and

(ii) \( Z \) blocks every path between \( X_i \) and \( X_j \) that contains an arrow into \( X_i \).
Definition 1. (Back-Door)

Back-Door Criterion

A set of variables $Z$ satisfies the back-door criterion relative to an ordered pair of variables $(X_i, X_j)$ in a DAG $G$ if:

(i) no node in $Z$ is a descendant of $X_i$; and

(ii) $Z$ blocks every path between $X_i$ and $X_j$ that contains an arrow into $X_i$.

Figure: $S_1 = \{X_3, X_4\}$ and $S_2 = \{X_4, X_5\}$ would qualify under the back-door criterion, but $S_3 = \{X_4\}$ would not because $X_4$ does not $d$-separate $X_i$ from $X_j$ along the path $(X_i, X_3, X_1, X_4, X_2, X_5, X_j)$.
<table>
<thead>
<tr>
<th>1</th>
<th>The Back-Door Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Definition 1. (Back-Door)</td>
</tr>
<tr>
<td></td>
<td>Theorem 1. (Back-Door Adjustment)</td>
</tr>
<tr>
<td>2</td>
<td>The Front-Door Criterion</td>
</tr>
<tr>
<td></td>
<td>Definition 2. (Front-Door)</td>
</tr>
<tr>
<td></td>
<td>Theorem 2. (Front-Door Adjustment)</td>
</tr>
<tr>
<td>3</td>
<td>\textit{do} Calculus</td>
</tr>
<tr>
<td>4</td>
<td>Proof of Theorem 2.</td>
</tr>
</tbody>
</table>
### Back-Door Criterion

**Back-Door Adjustment Theorem**

If a set of variables $Z$ satisfies the back-door criterion relative to $(X, Y)$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y | \hat{x}) = \sum_z P(y | x, z) P(z). \tag{1}$$
If a set of variables $Z$ satisfies the back-door criterion relative to $(X, Y)$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y|\hat{x}) = \sum_z P(y|x, z)P(z).$$  (1)
Outline

1. The Back-Door Criterion
   - Definition 1. (Back-Door)
   - Theorem 1. (Back-Door Adjustment)

2. The Front-Door Criterion
   - Definition 2. (Front-Door)
   - Theorem 2. (Front-Door Adjustment)

3. *do* Calculus

4. Proof of Theorem 2.
A set of variables $Z$ satisfies the *front-door criterion* relative to an ordered pair of variables $(X, Y)$ in a DAG $G$ if:

(i) $Z$ intercepts all directed paths from $X$ to $Y$;
(ii) there is no unblocked back-door path from $X$ to $Z$; and
(iii) all back-door paths from $Z$ to $Y$ are blocked by $X$. 

Mohammad Ali Javidian, Marco Valtorta
An Illustrated Proof of The Front-Door Adjustment Theorem
A set of variables $Z$ satisfies the *front-door criterion* relative to an ordered pair of variables $(X, Y)$ in a DAG $G$ if:

(i) $Z$ intercepts all directed paths from $X$ to $Y$;
A set of variables $Z$ satisfies the *front-door criterion* relative to an ordered pair of variables $(X, Y)$ in a DAG $G$ if:

(i) $Z$ intercepts all directed paths from $X$ to $Y$;

(ii) there is no unblocked back-door path from $X$ to $Z$; and
A set of variables $Z$ satisfies the front-door criterion relative to an ordered pair of variables $(X, Y)$ in a DAG $G$ if:

(i) $Z$ intercepts all directed paths from $X$ to $Y$;

(ii) there is no unblocked back-door path from $X$ to $Z$; and

(iii) all back-door paths from $Z$ to $Y$ are blocked by $X$. 
A set of variables $Z$ satisfies the \textit{front-door criterion} relative to an ordered pair of variables $(X, Y)$ in a DAG $G$ if:

(i) $Z$ intercepts all directed paths from $X$ to $Y$;

(ii) there is no unblocked back-door path from $X$ to $Z$; and

(iii) all back-door paths from $Z$ to $Y$ are blocked by $X$.

\textbf{Figure:} A diagram representing the front-door criterion.
The Back-Door Criterion

Definition 1. (Back-Door)

Theorem 1. (Back-Door Adjustment)

The Front-Door Criterion

Definition 2. (Front-Door)

Theorem 2. (Front-Door Adjustment)

do Calculus

Proof of Theorem 2.

Outline

1. The Back-Door Criterion
   - Definition 1. (Back-Door)
   - Theorem 1. (Back-Door Adjustment)

2. The Front-Door Criterion
   - Definition 2. (Front-Door)
   - Theorem 2. (Front-Door Adjustment)

3. do Calculus

4. Proof of Theorem 2.
Theorem 2. (Front-Door Adjustment)

Front-Door Criterion

If a set of variables $Z$ satisfies the front-door criterion relative to $(X, Y)$ and if $P(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y | \hat{x}) = \sum_{z} P(z | x) \sum_{x'} P(y | x', z) P(x').$$
Theorem 2. (Front-Door Adjustment)

Front-Door Adjustment Theorem

If a set of variables $Z$ satisfies the front-door criterion relative to $(X, Y)$ and if $P(x, z) > 0$, then the causal effect of $X$ on $Y$ is identifiable and is given by the formula

$$P(y|\hat{x}) = \sum_z P(z|x) \sum_{x'} P(y|x', z) P(x').$$
Rules of *do* Calculus

Preliminary Notation

![Diagram](https://via.placeholder.com/150)

**Figure:** Subgraphs of $G$ used in the derivation of causal effects.
Inference Rules

Rules of do Calculus

**Rule 1** (Insertion/deletion of observations):

\[ P(y|\hat{x}, z, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_X}. \]
Inference Rules

Rules of do Calculus

**Rule 1** (Insertion/deletion of observations):

\[ P(y|\hat{x}, z, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \!\!\!\!\!\perp Z|X, W)_{G_X}. \]

**Rule 2** (Action/observation exchange):

\[ P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp \!\!\!\!\!\perp Z|X, W)_{G_{X\hat{Z}}}. \]

where \( Z(W) \) is the set of Z-nodes that are not ancestors of any W-node in \( G_X \).
Inference Rules

Rules of \textit{do} Calculus

\textbf{Rule 1} (Insertion/deletion of observations):

\[ P(y|\hat{x}, z, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_X}. \]

\textbf{Rule 2} (Action/observation exchange):

\[ P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{\overline{X}Z}}. \]

\textbf{Rule 3} (Insertion/deletion of actions):

\[ P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{\overline{XZ}(W)}}. \]

where \( Z(W) \) is the set of \( Z \)-nodes that are not ancestors of any \( W \)-node in \( G_X \).
Proof of Front-Door Adjustment Theorem

Step 1: Compute $P(z|\hat{x})$

- $X \perp \perp Z$ in $G_X$ because there is no outgoing edge from $X$ in $G_X$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from $X$ to $Z$ are blocked.
Proof of Front-Door Adjustment Theorem

Step 1: Compute $P(z | \hat{x})$

- $X \perp Z$ in $G_X$ because there is no outgoing edge from $X$ in $G_X$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from $X$ to $Z$ are blocked.
Proof of Front-Door Adjustment Theorem

Step 1: Compute $P(z|\hat{x})$

- $X \perp\!\!\!\!\perp Z$ in $G_X$ because there is no outgoing edge from $X$ in $G_X$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from $X$ to $Z$ are blocked.

- $G$ satisfies the applicability condition for Rule 2:
  \[ P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp\!\!\!\!\perp Z|X, W)_{G} \]

Mohammad Ali Javidian, Marco Valtorta
An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 1: Compute $P(z|\hat{x})$

- $X \perp Z$ in $G_X$ because there is no outgoing edge from $X$ in $G_X$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from $X$ to $Z$ are blocked.

- $G$ satisfies the applicability condition for Rule 2:

\[
P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp Z|X, W)_{G_X Z}.
\]

- In Rule 2, set $y = z, x = \emptyset, z = x, w = \emptyset$: $P(z|\hat{x}) = P(z|x)$ because $(Z \perp X)_{G_X}$. 

Mohammad Ali Javidian, Marco Valtorta
University of South Carolina
An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 1: Compute $P(z | \hat{x})$

- $X \perp \perp Z$ in $G_X$ because there is no outgoing edge from $X$ in $G_X$, and also by condition (ii) of the definition of the front-door criterion, all back-door paths from $X$ to $Z$ are blocked.

- $G$ satisfies the applicability condition for Rule 2:
  $$P(y | \hat{x}, \hat{z}, w) = P(y | \hat{x}, z, w) \quad \text{if} \quad (Y \perp \perp Z | X, W)_{G_{XZ}}.$$

- In Rule 2, set $y = z, x = \emptyset, z = x, w = \emptyset$:
  $$P(z | \hat{x}) = P(z | x) \quad \text{because} \quad (Z \perp \perp X)_{G_X}.$$
Proof of Front-Door Adjustment Theorem

Step 2: Compute $P(y | \hat{z})$

$$P(y | \hat{z}) = \sum_x P(y | x, \hat{z}) P(x | \hat{z}).$$
Proof of Front-Door Adjustment Theorem

Step 2: Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
- $X \perp \perp Z$ in $G_Z$ because there is no incoming edge to $Z$ in $G_Z$, and also all paths from $X$ to $Z$ either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.

Mohammad Ali Javidian, Marco Valtorta

University of South Carolina

An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 2: Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
- $X \perp \perp Z$ in $G_Z$ because there is no incoming edge to $Z$ in $G_Z$, and also all paths from $X$ to $Z$ either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.
Proof of Front-Door Adjustment Theorem

Step 2: Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
- $X \perp \perp Z$ in $G_Z$ because there is no incoming edge to $Z$ in $G_Z$, and also all paths from $X$ to $Z$ either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.

$G$ satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_X, Z(w)}.$$
Proof of Front-Door Adjustment Theorem

Step 2: Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
- $X \perp\perp Z$ in $G_Z$ because there is no incoming edge to $Z$ in $G_Z$, and also all paths from $X$ to $Z$ either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.

$G$ satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp\perp Z|X, W)_{G_{X,Z}(W)}.$$

- In Rule 3, set $y = x, x = \emptyset, z = z, w = \emptyset$: 

Mohammad Ali Javidian, Marco Valtorta

An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 2: Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z})$.
- $X \perp \perp Z$ in $G_Z$ because there is no incoming edge to $Z$ in $G_Z$, and also all paths from $X$ to $Z$ either by condition (ii) of the definition of the front-door criterion (blue-type paths), or because of existence of a collider node on the path (green-type paths) are blocked.

- $G$ satisfies the applicability condition for Rule 3:
  $$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{X,Z}(W)}.$$

- In Rule 3, set $y = x, x = \emptyset, z = z, w = \emptyset$:
  $$P(x|\hat{z}) = P(x) \quad \text{because} \quad (Z \perp \perp X)_{G_Z}.$$
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute $P(y|\bar{z})$

- $(Z \perp \perp Y|X)_{G_Z}$ because there is no outgoing edge from $Z$ in $G_Z$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$. 
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute $P(y|\hat{x})$

- $(Z \perp\!\!\!\!\!\perp Y|X)_{G_Z}$ because there is no outgoing edge from $Z$ in $G_Z$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$. 

Mohammad Ali Javidian, Marco Valtorta
An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute $P(y|\hat{z})$

- $(Z \perp\!
\perp Y|X)_{G_Z}$ because there is no outgoing edge from $Z$ in $G_Z$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$.

- $G$ satisfies the applicability condition for Rule 2: $P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w)$ if $(Y \perp\!
\perp Z|X, W)_{G_{XZ}}$.
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute \( P(y|\hat{z}) \)

- \( (Z \perp \perp Y|X)_{G_Z} \) because there is no outgoing edge from \( Z \) in \( G_Z \), and also by condition (iii) of the definition of the front-door criterion, all back-door paths from \( Z \) to \( Y \) are blocked by \( X \).

- \( G \) satisfies the applicability condition for Rule 2:  
  \[ P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{XZ}}. \]

- In Rule 2, set \( y = y, x = \emptyset, z = z, w = x \):
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute \( P(y|\hat{z}) \)

- \((Z \perp \perp Y|X)_{G_Z}\) because there is no outgoing edge from \(Z\) in \(G_Z\), and also by condition (iii) of the definition of the front-door criterion, all back-door paths from \(Z\) to \(Y\) are blocked by \(X\).

- \(G\) satisfies the applicability condition for Rule 2:
  \[
P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{XZ}}.
  \]

In Rule 2, set \(y = y, x = \emptyset, z = z, w = x\):

\[
P(y|x, \hat{z}) = P(y|x, z) \quad \text{because} \quad (Z \perp \perp Y|X)_{G_Z}.
\]
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z}) = \sum_x P(y|x, z)P(x)$. 

Mohammad Ali Javidian, Marco Valtorta
An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 2 (continued): Compute $P(y|\hat{z})$

- $P(y|\hat{z}) = \sum_x P(y|x, \hat{z})P(x|\hat{z}) = \sum_x P(y|x, z)P(x)$.

- This formula is a special case of the back-door formula in Theorem 1.
Proof of Front-Door Adjustment Theorem

Step 3: Compute $P(y|\hat{x})$

$$P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x}).$$
Proof of Front-Door Adjustment Theorem

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.
- $(Y \perp \perp Z|X)_{G_{XZ}}$ because there is no outgoing edge from $Z$ in $G_{XZ}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$. 
Proof of Front-Door Adjustment Theorem

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.
- $(Y \perp \perp Z|X)_{G_{\overline{X}Z}}$ because there is no outgoing edge from $Z$ in $G_{\overline{X}Z}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$. 

Mohammad Ali Javidian, Marco Valtorta
An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.
- $(Y \perp \perp Z|X)_{G_{XZ}}$ because there is no outgoing edge from $Z$ in $G_{XZ}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$.

- $G$ satisfies the applicability condition for Rule 2: $P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w)$ if $(Y \perp \perp Z|X, W)_{G_{XZ}}$. 
Proof of Front-Door Adjustment Theorem

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.
- $(Y \perp \perp Z|X)_{G_{XZ}}$ because there is no outgoing edge from $Z$ in $G_{XZ}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$.

- $G$ satisfies the applicability condition for Rule 2: $P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w)$ if $(Y \perp \perp Z|X, W)_{G_{XZ}}$.
- In Rule 2, set $y = y, x = x, z = z, w = \emptyset$.
Proof of Front-Door Adjustment Theorem

Step 3: Compute $P(y|\hat{x})$

- $P(y|\hat{x}) = \sum_z P(y|z, \hat{x})P(z|\hat{x})$.
- $(Y \perp \perp Z|X)_{G_{XZ}}$ because there is no outgoing edge from $Z$ in $G_{XZ}$, and also by condition (iii) of the definition of the front-door criterion, all back-door paths from $Z$ to $Y$ are blocked by $X$.

$G$ satisfies the applicability condition for Rule 2:

$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, z, w)$ if $(Y \perp \perp Z|X, W)_{G_{XZ}}$.

In Rule 2, set $y = y, x = x, z = z, w = \emptyset$:

$P(y|z, \hat{x}) = P(y|\hat{z}, \hat{x})$ because $(Y \perp \perp Z|X)_{G_{XZ}}$. 

Mohammad Ali Javidian, Marco Valtorta
An Illustrated Proof of The Front-Door Adjustment Theorem
Proof of Front-Door Adjustment Theorem
Step 3 (continued): Compute $P(y|\hat{x})$

$(Y \perp X|Z)_{G_{XZ}}$ because there is no incoming edge to $X$ in $G_{XZ}$, and also all paths from $X$ to $Y$ are blocked either because of condition (i) of the definition of the front-door criterion (blue-type paths) [directed paths from $X$ to $Y$], or because of the existence of a collider on the path (green-type paths) (note that the case $T \in Z$ cannot happen because there is no incoming edge to $Z$ in $G_{XZ}$).
Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

$(Y \perp X|Z)_{G_{XZ}}$ because there is no incoming edge to $X$ in $G_{XZ}$, and also all paths from $X$ to $Y$ are blocked either because of condition (i) of the definition of the front-door criterion (blue-type paths)[directed paths from $X$ to $Y$], or because of the existence of a collider on the path (green-type paths) (note that the case $T \in Z$ cannot happen because there is no incoming edge to $Z$ in $G_{XZ}$).
Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

- $G$ satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \!\!\!\!\perp Z|X, W)_G^{X, Z(W)}.$$
Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

- $G$ satisfies the applicability condition for Rule 3:
  \[ P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{X,Z(W)}}. \]

- In Rule 3, set $y = y, x = z, z = x, w = \emptyset$: 
  \[ P(y|\hat{x}) = \sum_z P(y|z, \hat{x}) P(z|\hat{x}) = \sum_x \sum_{x'} P(y|x', \hat{x}) P(x'). \]
Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

- $G$ satisfies the applicability condition for Rule 3:

$$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \text{ if } (Y \perp \!\!\!\!\!\!\perp Z|X, W)_{G_X, Z(W)}.$$

- In Rule 3, set $y = y$, $x = z$, $z = x$, $w = \emptyset$:

$$P(y|\hat{z}, \hat{x}) = P(y|\hat{z}) \text{ because } (Y \perp \!\!\!\!\!\!\perp Z|X)_{G_X Z}.$$
Proof of Front-Door Adjustment Theorem

Step 3 (continued): Compute $P(y|\hat{x})$

- $G$ satisfies the applicability condition for Rule 3:
  $$P(y|\hat{x}, \hat{z}, w) = P(y|\hat{x}, w) \quad \text{if} \quad (Y \perp \perp Z|X, W)_{G_{\overline{x},\overline{z}(w)}}.$$

- In Rule 3, set $y = y$, $x = z$, $z = x$, $w = \emptyset$:
  $$P(y|\hat{z}, \hat{x}) = P(y|\hat{x}) \quad \text{because} \quad (Y \perp \perp Z|X)_{G_{\overline{x},\overline{z}}}. $$

- $P(y|\hat{x}) = \sum_{z} P(y|z, \hat{x})P(z|\hat{x}) = \sum_{z} P(z|x) \sum_{x'} P(y|x', z)P(x').$
Reference For Further Reading

J. Pearl.

*Causality: Models, reasoning, and inference.*