CHAPTER

FIVE

STATE ESTIMATION AND STOCHASTIC CONTROL

5.1 INTRODUCTION

Up to this point we have considered only the control algorithms for process control without much discussion of the problems of process measurement and random process disturbances. However, in most industrial processes the total state vector can seldom be measured and the number of outputs is much less than the number of states. In addition, the process measurements are often corrupted by significant experimental error, and the process itself is subject to random, unmodeled upsets. Thus without some consideration of these problems in the total control system design, the measurements used for feedback control will often be inadequate for acceptable control system performance. In this chapter we shall begin by discussing on-line *state estimation* techniques which may be used to provide acceptable estimates of all the state variables (even those not directly measured) in the face of measurement error and process disturbances. Following this, a brief introduction to *stochastic feedback control*, which is explicitly designed for systems with random disturbances and measurement errors, will be provided along with some illustrative applications. First we shall treat systems described by ordinary differential equations and then discuss methods for distributed parameter systems.

A full, rigorous treatment of these topics requires a thorough background in the theory of stochastic processes. However, very few process control engineers have this preparation, so the approach to be used here will be *operational*, the goal being to provide general, useful results for the engineer through formal plausible derivations of key results. For the reader who wishes a fuller treatment of the theory and alternative approaches, Refs. [1–7] are recommended.

State Estimation

In this chapter we are interested in state estimates which can be used with real-time control schemes; thus we shall consider only *sequential state estimation* algorithms in any detail. Nonsequential methods, which require a complete data base before computation begins, shall be mentioned only in passing. By *sequential estimation* we mean that initial *a priori* estimates of the process states are continually updated and the best current estimates used in the control application. State estimation algorithms may be classified into three categories as shown in Fig. 5.1:

1. *Smoothing*, in which estimates at time t are made from data taken both before and after time t. Thus, smoothing does not provide current estimates at time t, but only estimates of the state at some time $t_1 - t$ in the past. This is

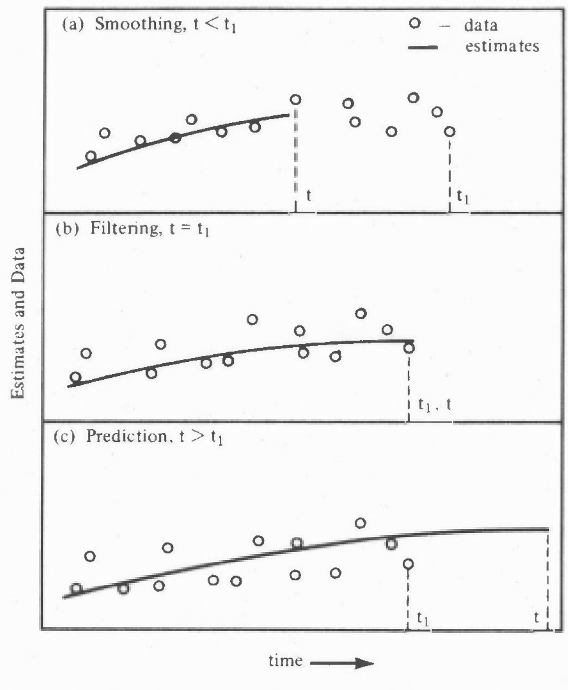


Figure 5.1 Three types of state estimation: (*a*) smoothing, (*b*) filtering, (*c*) prediction.

illustrated in Fig. 5.1*a*. Smoothing is basically a nonsequential technique and will not be discussed to any great extent here.

2. *Filtering*, in which estimates at time t are made from data up to time t, but not beyond. Thus, estimates are at $t = t_1$, the current time, as shown in Fig. 5.1*b*. Filtering is the most common estimation technique employed with feedback controllers because the most up-to-date state estimates are provided in a sequential fashion.
3. *Prediction*, in which estimates at time t are made from data up to time t_1, where $t > t_1$. This type of estimation, shown in Fig. 5.1*c*, is employed when one must extrapolate ahead of data measurements. This situation might arise when there are analysis delays in measurements of outputs such as concentration (e.g., in chromatographic analysis) or when the states themselves have time delays in them (e.g., as in flow-through piping).

In what follows we shall concentrate mainly on *filtering* and *prediction*.

Fundamentally, state estimation is the problem of determining the values of the state variables from only a knowledge of the outputs (data) and the inputs (controls, disturbances). Clearly, for this to be successful, this input-output information must provide a *unique state estimate*, which implies system *observability* (see Sec. 5.2 for a broader discussion of observability). For sequential estimation (filtering or prediction), the structure of the problem is shown in Fig. 5.2. One has available a process model corrupted by a noise process $\xi(t)$ due to either unknown disturbances or model error. In addition one has a corrupted

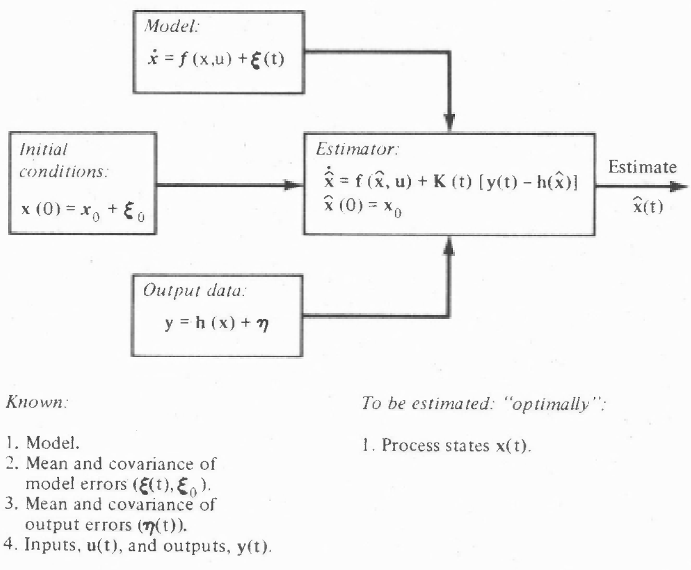


Figure 5.2 The structure of a sequential estimation device.

estimate of the initial conditions. Finally, output data, which are some combination of the state variables $\mathbf{h}(\mathbf{x})$, are measured with some error $\eta(t)$. Thus, $\mathbf{x}$, $\mathbf{y}$ are random variables evolving in time. It is assumed that the statistics of the noise processes $\xi(t)$, $\eta(t)$ and the initial error ξ_0 are known (i.e., the form of the error distribution plus distribution moments are known). Given the statistics of ξ_0, $\xi(t)$, it is possible to envision a distribution of possible process states $\mathbf{p}(\mathbf{x}, t)$ at each time resulting from the stochastic model. This distribution will evolve in time as shown in Fig. 5.3 for a single state variable. Notice that there are no data taken, and thus the predicted probability distribution widens considerably with time. If the measurements $\mathbf{y}(t)$ are available, then it is possible to consider the *conditional probability distribution* $p(x(t)|Y)$, which is the probability distribution of the state given the set of data $y(t')$, $0 \leq t' \leq t$ (denoted by Y). This distribution, plotted in Fig. 5.4, can be narrowed with time as shown so that the estimates improve with time due to the measured information. The degree of narrowing depends on the statistics of the process noise and measurement errors. The choice of the best estimate for the process state depends on the criterion used. If one wishes to minimize the *square of the deviation* between the estimate $\hat{x}(t)$ and the true state $x(t)$ (which is unknown), then the best estimate will be the *mean* of the distribution $p(x|Y)$, [i.e., $\hat{x}(t) = \bar{x}(t)$]. If, on the other hand, one wishes to maximize the *likelihood* that the estimate is the true state, then the peak in the distribution function $p(x|Y)$ would be the estimate [i.e., $x(t) = x_m(t)$]. In general these two criteria yield different estimates, though for some distributions (such as the Gaussian) they are identical.

The form of the estimation equations for a least squares criterion is shown in Fig. 5.2. The estimate can be determined by solving the differential equation for the mean of the state probability distribution. Notice that the evolution equation has two parts, one arising directly from the model and a second "feedback" term correcting the estimate for discrepancies between the actual

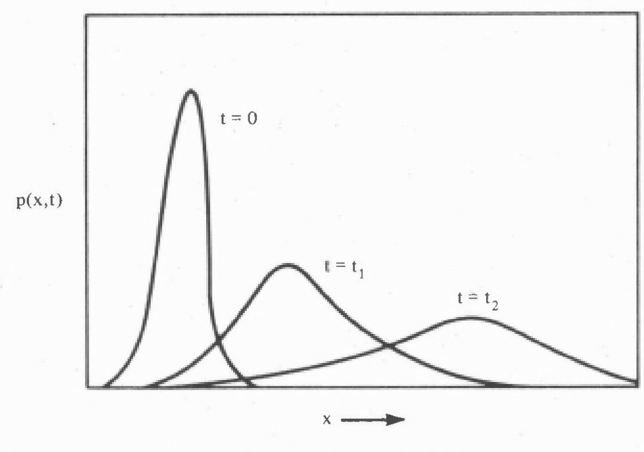


Figure 5.3 Evolution of the state probability distribution for a stochastic process without data.

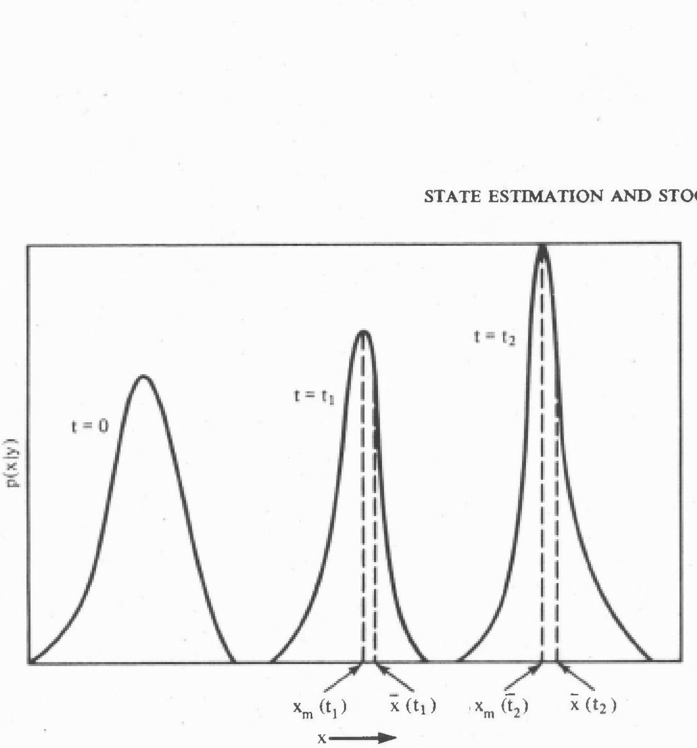


Figure 5.4 Evolution of probability distributions of system state conditional on the data. $\bar{x}(t)$ is the mean of the distribution, and $x_m(t)$ the mode or maximum likelihood value.

output data $\mathbf{y}(t)$ and the theoretical value of $\mathbf{y}(t)$ if the state estimates were correct, $\mathbf{h}(\hat{\mathbf{x}}(t))$. The magnitude of this feedback correction is controlled by a gain matrix $\mathbf{K}(t)$ which depends on the error statistics of the model and the output data. Generally speaking, $\mathbf{K}(t)$ will be large when the errors in the model are relatively larger than the errors in the data. Conversely, when the output data have a large relative error, the value of $\mathbf{K}(t)$ will be small. Thus the estimator naturally relies heaviest on the most precise information available to it. These very general concepts will be fleshed out with practical details in the ensuing sections.

Stochastic Feedback Control

Stochastic control is concerned with controller design when the measured outputs are random variables due to measurement errors and process noise. Secs. 5.4 and 5.7 below shall treat controller design for stochastic processes.

5.2 STATE ESTIMATION FOR LINEAR SYSTEMS DESCRIBED BY ORDINARY DIFFERENTIAL EQUATIONS

As in the case of feedback control, for state estimation a large body of powerful results are available for linear systems. Fortunately, a great many practical estimation problems are linear or nearly linear, so that linear state estimation techniques have both theoretical and practical importance. Although this class of system is now routinely discussed in standard references [1–7], one should note that Kalman and Bucy [8, 9] were the first to define the structure of the theory which follows.

Let us consider the system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \boldsymbol{\xi}(t) \tag{5.2.1}$$

$$\mathbf{y} = \mathbf{C}(t)\mathbf{x} + \boldsymbol{\eta}(t) \tag{5.2.2}$$

$$\mathbf{x}(0) = \mathbf{x}_0 + \boldsymbol{\xi}_0 \tag{5.2.3}$$

where $\mathbf{x}$ is an n vector of states, $\mathbf{y}$ is an l vector of continuous time outputs, $\boldsymbol{\xi}(t)$ is an n vector of random process noise, $\boldsymbol{\eta}(t)$ is an l vector of random measurement error, $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are $n \times n$ and $l \times n$ time-varying matrices, $\mathbf{x}_0$ is an estimate of the initial state, and $\boldsymbol{\xi}_0$ is its random error. The variables $\mathbf{x}(t)$, $\mathbf{y}(t)$ are stochastic random variables having some probability distribution at any instant of time, and Eq. (5.2.1) is a stochastic differential equation. A mathematically rigorous analysis of such a system requires more sophisticated mathematics than is assumed here, and the reader is urged to consult [4, 5] for these details. In our treatment here, the results shall be developed through more formal means.

Observability

Let us now discuss the observability property in some detail. Roughly speaking, a system is *observable* if it is possible to determine all the state variables at some time t_0 based on a knowledge of the system output $\mathbf{y}(t)$ and control $\mathbf{u}(t)$ over some finite time interval. To be more precise, we can state that:

If every initial system state $\mathbf{x}(t_0)$ *can be determined through knowledge of the system control* $\mathbf{u}(t)$ *and system output* $\mathbf{y}(t)$ *over some finite time interval* $t_0 \le t \le t_1$, *then the system is completely observable.* Clearly it is also possible to have systems which are partially observable, i.e., in which only a subset of the state variables are observable.

Note that observability is independent of the noise processes $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$, $\boldsymbol{\xi}_0$ and is only a property of the deterministic model equations.

Conditions for observability have been derived for a number of classes of systems [1–11]. For example, for the linear system of Eqs. (5.2.1) and (5.2.2) having constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, it can be shown that the system is *completely observable* if and only if the rank of an $n \times nl$ "observability matrix" L_0 is n, where

$$\mathbf{L}_0 \equiv \left[\mathbf{C}^T \,\vdots\, \mathbf{A}^T\mathbf{C}^T \,\vdots\, (\mathbf{A}^T)^2\mathbf{C}^T \,\vdots\, \cdots \,\vdots\, (\mathbf{A}^T)^{n-1}\mathbf{C}^T\right] \tag{5.2.4}$$

In order to see how this condition arises, let us consider the system of Eq. (5.2.1) with zero control action and zero process noise. The solution to Eq. (5.2.1) in that case is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \left(c_0\mathbf{I} + c_1 t\mathbf{A} + \cdots + c_{n-1}t^{n-1}\mathbf{A}^{n-1}\right)\mathbf{x}_0 \tag{5.2.5}$$

Now from Eq. (5.2.2),

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x} = \left(c_0\mathbf{C} + c_1 t\mathbf{C}\mathbf{A} + \cdots + c_{n-1}t^{n-1}\mathbf{C}\mathbf{A}^{n-1}\right)\mathbf{x}_0 \tag{5.2.6}$$

Now for the system to be observable, one must be able to identify the initial conditions $\mathbf{x}_0$ from the data $\mathbf{y}(t)$. [For if $\mathbf{x}_0$ is known, then the model can provide all future estimates $\mathbf{x}(t)$.] Thus we must be able to construct a *pseudo-inverse* between $\mathbf{x}_0$ and $\mathbf{y}(t)$ in Eq. (5.2.6). One can show that by multiplying both sides of Eq. (5.2.6) by $\exp(\mathbf{A}t)^T$ and integrating from zero to t_f, one obtains

$$\begin{aligned}\mathbf{x}_0 = \Bigg[\int_0^{t_f} &(c_0\mathbf{C} + c_1 t\mathbf{C}\mathbf{A} + \cdots + c_{n-1}t^{n-1}\mathbf{C}\mathbf{A}^{n-1})^T \\ &\times (c_0\mathbf{C} + \cdots + c_{n-1}t^{n-1}\mathbf{C}\mathbf{A}^{n-1})\,dt\Bigg]^{-1} \\ &\times \int_0^{t_f} (c_0\mathbf{C} + c_1 t\mathbf{C}\mathbf{A} + \cdots + c_{n-1}t^{n-1}\mathbf{C}\mathbf{A}^{n-1})^T \mathbf{y}(t)\,dt \qquad (5.2.7)\end{aligned}$$

and that this is the desired pseudo-inverse. Now for the pseudo-inverse to exist, it is required that the $n \times n$ matrix

$$\begin{aligned}\mathbf{M} = \int_0^{t_f} &(c_0\mathbf{C}^T + c_1 t\mathbf{A}^T\mathbf{C}^T + \cdots + c_{n-1}t^{n-1}\mathbf{A}^{T^{n-1}}\mathbf{C}^T) \\ &\times (c_0\mathbf{C} + c_1 t\mathbf{C}\mathbf{A} + \cdots + c_{n-1}t^{n-1}\mathbf{C}\mathbf{A}^{n-1})\,dt \qquad (5.2.8)\end{aligned}$$

be nonsingular (i.e., have rank n). $\mathbf{M}$ may be written

$$\mathbf{M} = \int_0^{t_f} [\mathbf{C}^T \,\vdots\, \mathbf{A}^T\mathbf{C}^T \cdots \,\vdots\, \mathbf{A}^{T^{n-1}}\mathbf{C}^T] \begin{bmatrix} c_0\mathbf{I} \\ c_1 t\mathbf{I} \\ \vdots \\ c_{n-1}t^{n-1}\mathbf{I} \end{bmatrix} \times [c_0\mathbf{I} \,\vdots\, c_1 t\mathbf{I} \cdots \,\vdots\, c_{n-1}t^{n-1}\mathbf{I}] \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} dt$$

where $\mathbf{I}$ is an $l \times l$ identity matrix. This may be reduced to

$$M = [C^T A^T C^T \cdots \mathbf{A}^{T^{n-1}}\mathbf{C}^T] \int_0^{t_f} \mathbf{T}\,dt \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \qquad (5.2.9)$$

where the $nl \times nl$ matrix $\mathbf{T}$ has $l \times l$ blocks of diagonal elements $[c_k c_j t^k t^j]$, $k, j = 0, 1, \ldots, n-1$. By application of the fundamental laws of linear algebra, it is possible to show that $\mathbf{M}$ will have rank n if and only if the $n \times nl$ observability matrix given by Eq. (5.2.4) has rank n.

To generalize these results, Kalman [10] showed that the system of Eqs. (5.2.1) to (5.2.3) with time-varying $\mathbf{A}$, $\mathbf{C}$ matrices will be completely observable at time $t_f > t_0$ if and only if

$$\mathbf{M}(t_0, t_f) = \int_{t_0}^{t_f} \mathbf{\Phi}(t, t_0)^T \mathbf{C}^T(t)\mathbf{C}(t)\mathbf{\Phi}(t, t_0)\, dt \tag{5.2.10}$$

is positive definite. Here $\mathbf{\Phi}(t, t_0)$ is the *fundamental matrix solution* defined by

$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0) \qquad \mathbf{\Phi}(t_0, t_0) = \mathbf{I} \tag{3.3.12}$$

The proof of this result follows the same arguments as that for constant coefficient systems. The output is related to the initial conditions by

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 \tag{5.2.11}$$

so that by multiplying both sides by $\mathbf{\Phi}^T(t, t_0)\mathbf{C}^T(t)$ and integrating one obtains the inverse relation

$$\mathbf{x}_0 = \mathbf{M}(t_0, t_f)^{-1} \int_{t_0}^{t_f} \mathbf{\Phi}^T(t, t_0)\mathbf{C}^T(t)\mathbf{y}(t)\, dt \tag{5.2.12}$$

Now it is easy to see [2, 7] that the symmetric matrix $\mathbf{M}(t_0, t_f)$ must be nonsingular (and hence positive definite) for the inversion Eq. (5.2.12) to exist.

In the case where measurements are taken at discrete intervals of time t_i, so that Eq. (5.2.3) becomes

$$\mathbf{y}(t_i) = \mathbf{C}(t_i)\mathbf{x}(t_i) + \boldsymbol{\eta}(t_i) \qquad i = 1, 2, \ldots \tag{5.2.13}$$

the observability condition for h sampling points $t_1, t_2, \ldots, t_h$ is that the *discrete data observability matrix* $\mathbf{L}_{0d}$

$$\mathbf{L}_{0d} = \left[\mathbf{\Phi}^T(t_1, t_0)\mathbf{C}^T(t_1), \mathbf{\Phi}^T(t_2, t_0)\mathbf{C}^T(t_2), \ldots, \mathbf{\Phi}^T(t_h, t_0)\mathbf{C}^T(t_h)\right] \tag{5.2.14}$$

have rank n [11]. The proof of this result follows from the same approach as used above.

A weaker property than observability is *detectability*. *Detectability is the property that all unstable modes of the process are observable.*

Clearly any *observable* system is also *detectable*. The property of detectability is important for control because one may successfully design a control system for an unobservable but detectable system so as to estimate and control the unstable modes.

Let us now illustrate these results with some examples.

Example 5.2.1 Let us consider the CSTR system of Example 3.2.14 described by

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 + u_1 \tag{3.2.123}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + u_2 \tag{3.2.124}$$

where it is only possible to measure the concentration of species A:

$$y_1 = x_1 \tag{5.2.15}$$

In this case, the matrix $\mathbf{A}$ is given by Eq. (3.2.128) and

$$\mathbf{C} = [1 \quad 0]$$

so that the observability matrix is

$$\mathbf{L}_0 = [\mathbf{C}^T \,\vdots\, \mathbf{A}^T\mathbf{C}^T] = \begin{bmatrix} 1 & \vdots & -(1 + \mathrm{Da}_1) \\ 0 & \vdots & 0 \end{bmatrix}$$

which has only rank 1. Thus the system is *not* observable. However, because all the eigenvalues of $\mathbf{A}$ are negative, there are no unstable modes and the system is *detectable*.

Example 5.2.2 If we change the output equation in Example 5.2.1 to

$$y_1 = x_2 \tag{5.2.16}$$

so that one is measuring species B, then

$$\mathbf{C} = [0 \quad 1]$$

and the observability matrix is

$$\mathbf{L}_0 = \begin{bmatrix} 0 & \mathrm{Da}_1 \\ 1 & -(1 + \mathrm{Da}_3) \end{bmatrix}$$

which has rank 2. Thus the system of Eqs. (3.2.123), (3.2.124), and (5.2.16) is *completely observable.*

There are important practical implications to the observability property which are illustrated for these problems. For the output $y_1 = x_1$ in Example 5.2.1, there is not sufficient information available to uniquely determine x_2, and thus *it is impossible to design an estimation scheme to estimate both x_1 and x_2* from x_1 measurements alone. Conversely, when $y_1 = x_2$ is chosen as an output, as in Example 5.2.2, there is enough information to determine x_1 uniquely, and one could devise an estimation scheme to do this. The physical reasons for this can be seen by noting that x_2 depends on x_1 and x_2, while x_1 is independent of x_2; therefore by measuring $x_2(t)$ and knowing $\mathbf{u}(t)$ in Eq. (3.2.124), $x_1(t)$ can be determined. Conversely, if we measure $x_1(t)$ and know $\mathbf{u}(t)$, $x_2(t)$ can take on any values and still satisfy the information constraints.

Example 5.2.3 To illustrate the observability conditions for nonautonomous linear systems, let us again consider the CSTR system from Example 3.2.14. The equations in the absence of control are

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 \tag{5.2.17}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 \tag{5.2.18}$$

Suppose we are interested in the system observability over the time interval during startup $0 < t < t_f$. Further, let us suppose that the temperature is increasing with time, causing k_1 and k_2 to increase with time. Thus for this problem

$$\mathbf{A}(t) = \begin{bmatrix} -[1 + \mathrm{Da}_1(t)] & 0 \\ \mathrm{Da}_1(t) & -[1 + \mathrm{Da}_3(t)] \end{bmatrix} = \begin{bmatrix} a_{11}(t) & 0 \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \tag{5.2.19}$$

Let us further assume that our measuring device can measure x_1 or x_2 but not both simultaneously. If we assume a measuring device of the form

$$y(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{5.2.20}$$

where $\mathbf{C} = [c_1, c_2]$ and

$$c_1(t) = \begin{cases} 1 & 0 \le t \le \dfrac{t_f}{2} \\ 0 & \dfrac{t_f}{2} < t \le t_f \end{cases}$$

$$c_2(t) = \begin{cases} 0 & 0 \le t \le \dfrac{t_f}{2} \\ 1 & \dfrac{t_f}{2} \le t \le t_f \end{cases} \tag{5.2.21}$$

then we sample $x_1(t)$ for the first half of the startup and $x_2(t)$ for the second half.

Applying the observability conditions, Eq. (5.2.8),

$$\mathbf{M}(0, t_f) = \int_0^{t_f/2} \mathbf{\Phi}(t, 0)^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{\Phi}(t, 0)\, dt + \int_{t_f/2}^{t_f} \mathbf{\Phi}\left(t, \frac{t_f}{2}\right)^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Phi}\left(t, \frac{t_f}{2}\right) dt \tag{5.2.22}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

is given by

$$\begin{aligned} \dot{\phi}_{11}(t, 0) &= a_{11}(t)\phi_{11}(t, 0) & \phi_{11}(0, 0) &= 1 \\ \dot{\phi}_{12}(t, 0) &= a_{11}(t)\phi_{12}(t, 0) & \phi_{12}(0, 0) &= 0 \\ \dot{\phi}_{21}(t, 0) &= a_{21}(t)\phi_{11}(t, 0) + a_{22}(t)\phi_{21}(t, 0) & \phi_{21}(0, 0) &= 0 \\ \dot{\phi}_{22}(t, 0) &= a_{21}(t)\phi_{12}(t, 0) + a_{22}(t)\phi_{22}(t, 0) & \phi_{22}(0, 0) &= 1 \end{aligned} \tag{5.2.23}$$

which immediately implies $\phi_{12}(t, 0) \equiv 0$. Thus

$$\mathbf{M}(0, t_f) = \int_0^{t_f/2} \begin{bmatrix} \phi_{11}(t, 0) & \phi_{21}(t, 0) \\ 0 & \phi_{22}(t, 0) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11}(t, 0) & 0 \\ \phi_{21}(t, 0) & \phi_{22}(t, 0) \end{bmatrix} dt$$

$$+ \int_{t_f/2}^{t_f} \begin{bmatrix} \phi_{11}\left(t, \frac{t_f}{2}\right) & \phi_{21}\left(t, \frac{t_f}{2}\right) \\ 0 & \phi_{22}\left(t, \frac{t_f}{2}\right) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11}\left(t, \frac{t_f}{2}\right) & 0 \\ \phi_{21}\left(t, \frac{t_f}{2}\right) & \phi_{22}\left(t, \frac{t_f}{2}\right) \end{bmatrix} dt \tag{5.2.24}$$

or

$$\mathbf{M}(0, t_f) = \int_0^{t_f/2} \begin{bmatrix} \phi_{11}(t, 0)^2 & 0 \\ 0 & 0 \end{bmatrix} dt + \int_{t_f/2}^{t_f} \begin{bmatrix} \phi_{21}\left(t, \frac{t_f}{2}\right)^2 & \phi_{21}\phi_{22} \\ \phi_{21}\phi_{22} & \phi_{22}\left(t, \frac{t_f}{2}\right)^2 \end{bmatrix} dt \tag{5.2.25}$$

Now clearly the process is *not observable* for $0 \le t \le t_f/2$ because the first term is *not* positive definite. However, when the sampling device is changed to x_2 at $t = t_f/2$, the system can be observable, as shown by the second term of Eq. (5.2.25), which can be positive definite. The final conclusion has to be based on the specific values of $a_{11}(t)$, $a_{21}(t)$, $a_{22}(t)$.

Note that the second term in Eq. (5.2.25) has a singular matrix for an integrand, but the integral

$$\begin{bmatrix} \int_{t_f/2}^{t_f} \phi_{21}\left(t, \frac{t_f}{2}\right)^2 dt & \int_{t_f/2}^{t_f} \phi_{21}\left(t, \frac{t_f}{2}\right)\phi_{22}\left(t, \frac{t_f}{2}\right) dt \\ \int_{t_f/2}^{t_f} \phi_{21}\left(t, \frac{t_f}{2}\right)\phi_{22}\left(t, \frac{t_f}{2}\right) dt & \int_{t_f/2}^{t_f} \phi_{22}^2\left(t, \frac{t_f}{2}\right) dt \end{bmatrix}$$

is in general *not singular* and could be positive definite.

Optimal State Estimation

Let us now discuss the concept of *optimal estimation* and objective functionals for estimation. There are many possible objectives one could use in defining optimal state estimates (e.g., minimal least squares, maximum likelihood, minimum maximum error); however, in our discussion here we shall deal with the minimum least squares objective and shall phrase our presentation so that it may be most easily extended to nonlinear problems. The weighted least squares objective to be considered along with the system equations (5.2.1) to (5.2.3) is

$$I = \tfrac{1}{2}[\mathbf{x}(0) - \mathbf{x}_0]^T \mathbf{P}_0^{-1}[\mathbf{x}(0) - \mathbf{x}_0] + \frac{1}{2}\int_0^{t_f}\{(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x})^T \mathbf{R}(t)(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x})$$
$$+ [\mathbf{y}(t) - \mathbf{C}(t)\mathbf{x}(t)]^T \mathbf{Q}(t)[\mathbf{y}(t) - \mathbf{C}(t)\mathbf{x}(t)]\}\, dt \qquad (5.2.26)$$

where the first term minimizes the squared error of initial condition estimates, the second term minimizes the integral squared modeling error, and the third term minimizes the integral squared measurement error. The weighting factors $\mathbf{P}_0^{-1}$, $\mathbf{R}(t)$, $\mathbf{Q}(t)$ are chosen based on the statistics of the problem (see the discussion to follow). Let us assume the noise processes $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$ in Eqs. (5.2.1) to (5.2.3) to be Gaussian and uncorrelated in time (i.e., *white* noise) as well as uncorrelated with the initial state. Also, we assume that the expected value relations

$$\mathcal{E}(\boldsymbol{\xi}(t)) = \mathbf{0} \qquad \mathcal{E}(\boldsymbol{\eta}(t)) = \mathbf{0} \qquad \mathcal{E}(\boldsymbol{\xi}(t)\boldsymbol{\xi}(\tau)^T) = \mathbf{R}^{-1}(t)\delta(t-\tau)$$
$$\mathcal{E}(\boldsymbol{\eta}(t)\mathbf{x}^T(0)) = \mathbf{0} \qquad \mathcal{E}(\boldsymbol{\xi}(t)\mathbf{x}^T(0)) = \mathbf{0} \qquad \mathcal{E}(\boldsymbol{\xi}(t)\boldsymbol{\eta}^T(\tau)) = \mathbf{0}$$
$$\mathcal{E}(\mathbf{x}(0)) = \mathbf{x}_0 \qquad \mathcal{E}([\mathbf{x}_0 - \mathbf{x}(0)][\mathbf{x}_0 - \mathbf{x}(0)]^T) = \mathbf{P}_0$$
$$\mathcal{E}(\boldsymbol{\eta}(t)\boldsymbol{\eta}(\tau)^T) = \mathbf{Q}^{-1}(t)\delta(t-\tau) \qquad (5.2.27)$$

hold, where $\mathbf{P}_0$ is the covariance of the initial state errors, $\mathbf{R}^{-1}(t)$ is the covariance of the process noise, and $\mathbf{Q}^{-1}$ is the covariance of the measurement errors. Under these conditions one can show [1–7] that the stochastic process $\mathbf{x}(t)$ described by Eq. (5.2.1) has a Gaussian distribution at each point in time and the measurements $\mathbf{y}(t)$ given by Eq. (5.2.2) will also be Gaussianly distributed. Thus, the conditional probability distribution $p(x|Y)$ will also be Gaussian and the least squares estimate, the maximum likelihood estimate, and the minimum maximum deviation estimates are all the same.

There is a very elegant and powerful theory which applies to linear system state estimation. However, the scope is enormous and requires mathematical background not expected of the reader; thus Refs. [1–7] should be consulted for more theoretical details. We shall present key results in a formal way and concentrate on the applications of the theory.

We shall now proceed to derive state estimation algorithms which estimate $\mathbf{x}(t)$ such that the objective, Eq. (5.2.26), is minimized. This problem may be reformulated by defining $\mathbf{u}(t) = \dot{\mathbf{x}} - \mathbf{A}\mathbf{x}$, and rewriting the objective

$$I = \tfrac{1}{2}[\mathbf{x}(0) - \mathbf{x}_0]^T \mathbf{P}_0^{-1}[\mathbf{x}(0) - \mathbf{x}_0] + \frac{1}{2}\int_0^{t_f}\{\mathbf{u}(t)^T \mathbf{R}(t)\mathbf{u}(t) + [\mathbf{y}(t) - \mathbf{C}(t)\mathbf{x}(t)]$$
$$\times \mathbf{Q}(t)[\mathbf{y}(t) - \mathbf{C}(t)\mathbf{x}(t)]\}\, dt \qquad (5.2.28)$$

Thus the estimation problem can be posed as a deterministic *optimal control problem*; i.e., select the control $\mathbf{u}(t)$ such that I in Eq. (5.2.28) is minimized subject to the constraints

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t) \qquad (5.2.29)$$
$$\mathbf{x}(0) \text{ unspecified} \qquad (5.2.30)$$

Note that after the optimal "control" $\mathbf{u}(t)$ is found, Eqs. (5.2.29) and (5.2.30) can be used to generate the optimal state estimates.

Applying the maximum principle (see Chap. 3) to this problem, one obtains

$$H = \tfrac{1}{2}\left[\mathbf{u}^T\mathbf{R}\mathbf{u} + (\mathbf{y} - \mathbf{C}\mathbf{x})^T\mathbf{Q}(\mathbf{y} - \mathbf{C}\mathbf{x})\right] + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} + \mathbf{u}) \tag{5.2.31}$$

and the condition $\partial H/\partial \mathbf{u} = \mathbf{0}$ yields

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\boldsymbol{\lambda}(t) \tag{5.2.32}$$

where

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \mathbf{x}} = \left[\mathbf{C}^T\mathbf{Q}(\mathbf{y} - \mathbf{C}\mathbf{x}) - \mathbf{A}^T\boldsymbol{\lambda}\right]^T \tag{5.2.33}$$

or

$$\dot{\boldsymbol{\lambda}} = -\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{x} - \mathbf{A}^T\boldsymbol{\lambda} + \mathbf{C}^T\mathbf{Q}\mathbf{y} \tag{5.2.34}$$

Because both $\mathbf{x}(0)$, $\mathbf{x}(t_f)$ are free, then there are two boundary conditions on $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda}(t_f) = \mathbf{0} \tag{5.2.35}$$

and

$$\mathbf{x}(0) = \mathbf{x}_0 - \mathbf{P}_0\,\boldsymbol{\lambda}(0) \tag{5.2.36}$$

By substituting Eq. (5.2.32) into Eq. (5.2.29), one obtains

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{R}^{-1}\boldsymbol{\lambda} \tag{5.2.37}$$

Now Eqs. (5.2.34) to (5.2.37) form a two-point boundary value problem which can be solved for $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$ and thus produce the optimal estimates. To make things more explicit, we shall denote $\hat{\mathbf{x}}(t|t_f)$, $\hat{\mathbf{u}}(t|t_f)$ as the optimal estimates and controls at time t, with data $\mathbf{y}(t)$ up to time t_f. Thus $\hat{\mathbf{x}}(t|t_f)$ is the estimate found from the two-point boundary value problem of Eqs. (5.2.34) to (5.2.37). We shall now make a transformation

$$\hat{\mathbf{x}}(t|t_f) = \mathbf{w}(t) - \mathbf{P}(t)\boldsymbol{\lambda}(t) \tag{5.2.38}$$

where the n vector $\mathbf{w}(t)$ and $n \times n$ matrix $\mathbf{P}(t)$ are to be determined. If we substitute Eq. (5.2.38) into Eq. (5.2.37), then one obtains for each side of the equation

$$\text{RHS} = \mathbf{A}(\mathbf{w} - \mathbf{P}\boldsymbol{\lambda}) - \mathbf{R}^{-1}\boldsymbol{\lambda} \tag{5.2.39}$$

$$\begin{aligned}\text{LHS} &= \dot{\mathbf{w}} - \dot{\mathbf{P}}\boldsymbol{\lambda} - \mathbf{P}\dot{\boldsymbol{\lambda}} \\ &= \dot{\mathbf{w}} - \dot{\mathbf{P}}\boldsymbol{\lambda} + \mathbf{P}\left[\mathbf{C}^T\mathbf{Q}\mathbf{C}(\mathbf{w} - \mathbf{P}\boldsymbol{\lambda}) + \mathbf{A}^T\boldsymbol{\lambda} - \mathbf{C}^T\mathbf{Q}\mathbf{y}\right]\end{aligned} \tag{5.2.40}$$

Collecting terms, one obtains

$$\dot{\mathbf{w}} - \mathbf{P}\mathbf{C}^T\mathbf{Q}(\mathbf{y} - \mathbf{C}\mathbf{w}) - \mathbf{A}\mathbf{w} = (\dot{\mathbf{P}} - \mathbf{P}\mathbf{A}^T - \mathbf{A}\mathbf{P} - \mathbf{R}^{-1} + \mathbf{P}\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{P})\boldsymbol{\lambda} \tag{5.2.41}$$

Now we can choose to define $\mathbf{w}(t)$, $\mathbf{P}(t)$ such that the coefficients in Eq. (5.2.41) vanish and choose the boundary conditions to satisfy Eqs. (5.2.35) and (5.2.36).

Thus we have

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{P}\mathbf{C}^T\mathbf{Q}(\mathbf{y} - \mathbf{C}\mathbf{w}) \qquad \mathbf{w}(0) = \mathbf{x}_0 \tag{5.2.42}$$

$$\dot{\mathbf{P}} = \mathbf{P}\mathbf{A}^T + \mathbf{A}\mathbf{P}^T + \mathbf{R}^{-1} - \mathbf{P}\mathbf{C}^T\mathbf{Q}\mathbf{C}\mathbf{P} \qquad \mathbf{P}(0) = \mathbf{P}_0 \tag{5.2.43}$$

Note that the state estimates may be found by first solving Eqs. (5.2.42) and (5.2.43) forward in time to produce $\mathbf{w}(t)$, $\mathbf{P}(t)$, then solving backward in time using (5.2.35), (5.2.37), and (5.2.38) to find the optimal estimates $\hat{\mathbf{x}}(t|t_f)$. This estimate $\hat{\mathbf{x}}(t|t_f)$ is the minimal least squares estimate at t, conditional on all the data in the interval $0 \leq t' \leq t_f$. Thus, this is a *smoothed* estimate, *nonsequential*, and usually impossible to obtain in real time.

The *filtering estimate*, which is needed for online control, is $\hat{\mathbf{x}}(t_f|t_f)$, the minimal least squares estimate at t_f conditional on all the data up to time t_f. From Eq. (5.2.35) one sees that at the end of a data period $t = t_f$, $\boldsymbol{\lambda}(t_f)$ always vanishes. Thus for any t_f, Eq. (5.2.38) yields the result

$$\hat{\mathbf{x}}(t_f|t_f) \equiv \mathbf{w}(t_f) \tag{5.2.44}$$

Thus the filtered estimates are determined from the sequential real-time equation (5.2.42) where t is always the current time.

$$\dot{\hat{\mathbf{x}}}(t|t) = \mathbf{A}(t)\hat{\mathbf{x}}(t|t) + \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{Q}(t)[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t|t)] \qquad \hat{\mathbf{x}}(0|0) = \mathbf{x}_0 \tag{5.2.45}$$

The $n \times n$ matrix function $\mathbf{P}(t)$ is given by Eq. (5.2.43).

Let us now investigate the statistics of these filtered estimates. The error between the true state $\mathbf{x}(t)$ and the filter estimate is given by the stochastic variable

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t|t) \tag{5.2.46}$$

Making use of Eqs. (5.2.1) and (5.2.45), one obtains the stochastic error process

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{Q}(t)\mathbf{C}(t)]\mathbf{e}(t) + \boldsymbol{\xi}(t) - \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{Q}(t)\boldsymbol{\eta}(t) \tag{5.2.47}$$

with the initial condition taken from Eq. (5.2.3) as

$$\mathbf{e}(0) = \boldsymbol{\xi}_0 \tag{5.2.48}$$

The expected value of the error process

$$\bar{\mathbf{e}}(t) = \mathcal{E}(\mathbf{e}(t)) \tag{5.2.49}$$

can be found by taking expectations of Eqs. (5.2.47) and (5.2.48)

$$\dot{\bar{\mathbf{e}}}(t) = (\mathbf{A} - \mathbf{P}\mathbf{C}^T\mathbf{Q}\mathbf{C})\bar{e}(t) \qquad \bar{\mathbf{e}}(0) = \mathbf{0} \tag{5.2.50}$$

The solution to this equation yields $\bar{\mathbf{e}}(t)$ identically zero, so the estimate has no bias. The covariance of the estimation error can be found from

$$\mathbf{Cov}(t, \tau) = \mathcal{E}(\mathbf{e}(t)\mathbf{e}^T(\tau)) \tag{5.2.51}$$

However, because of the statistical assumptions we have made, the estimates are uncorrelated in time and the covariance depends only on one time, i.e.,

$$\mathbf{Cov}(t, \tau) = \mathbf{Cov}(t) = \mathcal{E}(\mathbf{e}(t)\mathbf{e}^T(\tau)\delta(t-\tau)) = \mathcal{E}(\mathbf{e}(t)\mathbf{e}^T(t)) \quad (5.2.52)$$

Postmultiplying Eq. (5.2.47) by $\mathbf{e}^T(t)$ and taking expectations, one obtains

$$\frac{d\,\mathbf{Cov}(t)}{dt} = \mathcal{E}\left[\mathbf{e}(t)\dot{\mathbf{e}}^T + \dot{\mathbf{e}}(t)\mathbf{e}^T\right] = \mathcal{E}\left\{\mathbf{e}(t)\left[\mathbf{e}^T(\mathbf{A}^T - \mathbf{C}^T\mathbf{QCP}) + \boldsymbol{\xi}^T - \boldsymbol{\eta}^T\mathbf{QCP}\right]\right.$$
$$\left. + \left[(\mathbf{A} - \mathbf{PC}^T\mathbf{QC})\mathbf{e} + \boldsymbol{\xi} - \mathbf{PC}^T\mathbf{Q}\boldsymbol{\eta}\right]\mathbf{e}^T\right\} \quad (5.2.53)$$

Now formally it is possible to write down the solution to Eq. (5.2.47) as

$$\mathbf{e}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{\xi}_0 + \int_0^t \boldsymbol{\Phi}(t, \tau)\left[\boldsymbol{\xi}(\tau) - \mathbf{P}(\tau)\mathbf{C}^T(\tau)\mathbf{Q}(\tau)\boldsymbol{\eta}(\tau)\right] d\tau \quad (5.2.54)$$

where $\boldsymbol{\Phi}(t, \tau)$ is the transition matrix of the system. Because the noise is uncorrelated in time and $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are uncorrelated, postmultiplying Eq. (5.2.54) by $\boldsymbol{\xi}^T(t)$ or $\boldsymbol{\eta}^T(t)$ gives

$$\mathcal{E}(\mathbf{e}(t)\boldsymbol{\xi}^T(t)) = \mathcal{E}\left(\int_0^t \boldsymbol{\Phi}(t, \tau)\boldsymbol{\xi}(\tau)\boldsymbol{\xi}^T(t)\, d\tau\right)$$
$$= \int_0^t \boldsymbol{\Phi}(t, \tau)\mathbf{R}^{-1}(t)\delta(t-\tau)\, d\tau = \tfrac{1}{2}\mathbf{R}^{-1}(t) \quad (5.2.55)$$

$$\mathcal{E}(\mathbf{e}(t)\boldsymbol{\eta}^T(t)) = \mathcal{E}\left(-\int_0^t \boldsymbol{\Phi}(t, \tau)\mathbf{P}(\tau)\mathbf{C}^T(\tau)\mathbf{Q}(\tau)\boldsymbol{\eta}(\tau)\boldsymbol{\eta}^T(t)\, d\tau\right)$$
$$= -\int_0^t \boldsymbol{\Phi}(t, \tau)\mathbf{P}(\tau)\mathbf{C}^T(\tau)\mathbf{Q}(\tau)\mathbf{Q}^{-1}(t)\delta(t-\tau)\, d\tau = -\tfrac{1}{2}\mathbf{P}(t)\mathbf{C}^T(t)$$
$$(5.2.56)$$

where the factor $\frac{1}{2}$ arises in Eqs. (5.2.55) and (5.2.56) because the delta function takes nonzero values at the upper limit of integration.

Substituting into Eq. (5.2.53), one obtains

$$\frac{d\,\mathbf{Cov}(t)}{dt} = \mathbf{Cov}(t)(\mathbf{A}^T - \mathbf{C}^T\mathbf{QCP}) + \tfrac{1}{2}\mathbf{R}^{-1} + \tfrac{1}{2}\mathbf{PC}^T\mathbf{QCP}$$
$$+ (\mathbf{A} - \mathbf{PC}^T\mathbf{QC})\mathbf{Cov}(t) + \tfrac{1}{2}\mathbf{R}^{-1} + \tfrac{1}{2}\mathbf{PC}^T\mathbf{QCP} \quad (5.2.57)$$

where $\mathbf{Cov}(0) = \mathbf{P}_0$. It is useful to notice that Eqs. (5.2.57) and (5.2.43) are identical if we let

$$\mathbf{Cov}(t) = \mathbf{P}(t) \quad (5.2.58)$$

Thus we see that $\mathbf{P}(t)$ is the filter *estimate covariance* and can be precomputed because it does not depend on the estimate $\hat{\mathbf{x}}(t|t)$ or the data $\mathbf{y}(t)$. To summarize our optimal estimator for linear systems, one has the following results:

1. *Smoothed estimates* $\hat{\mathbf{x}}(t|t_f)$ for the system of Eqs. (5.2.1) to (5.2.3) and (5.2.26) can be found by solving Eqs. (5.2.45) and (5.2.53) for the filtered estimates $\hat{\mathbf{x}}(t|t) = \mathbf{w}(t)$ and filter covariance $\mathbf{P}(t)$, then solving Eqs. (5.2.37) backwards,

which takes the form

$$\dot{\hat{\mathbf{x}}}(t|t_f) = \mathbf{A}(t)\hat{\mathbf{x}}(t|t_f) + \mathbf{R}^{-1}\mathbf{P}^{-1}[\hat{\mathbf{x}}(t|t_f) - \hat{\mathbf{x}}(t|t)] \tag{5.2.59}$$

where $\hat{\mathbf{x}}(t_f|t_f)$ is known from the filtering result.

By subtracting the true state $\mathbf{x}(t)$ from both sides of Eq. (5.2.38) it is possible to show that the smoothed estimate covariance

$$\mathbf{P}(t|t_f) \equiv \mathcal{E}\{[\hat{\mathbf{x}}(t|t_f) - \mathbf{x}(t)][\hat{\mathbf{x}}(t|t_f) - \mathbf{x}(t)]^T\} \tag{5.2.60}$$

can be related to the filtered covariance by

$$\mathbf{P}(t|t_f) = \mathbf{P}(t) - \mathbf{P}(t)\mathbf{\Lambda}(t)\mathbf{P}(t) \tag{5.2.61}$$

where

$$\mathbf{\Lambda}(t) = \mathcal{E}(\mathbf{P}^{-1}\mathbf{e}\boldsymbol{\lambda}^T + \boldsymbol{\lambda}\mathbf{e}^T\mathbf{P}^{-1} - \boldsymbol{\lambda}\boldsymbol{\lambda}^T) \tag{5.2.62}$$

If we note that Eq. (5.2.34) can be written

$$\dot{\boldsymbol{\lambda}} = -(\mathbf{A}^T - \mathbf{C}^T\mathbf{QCP})\boldsymbol{\lambda} + \mathbf{C}^T\mathbf{Q}\boldsymbol{\eta} + \mathbf{C}^T\mathbf{QCe} \qquad \boldsymbol{\lambda}(t_f) = \mathbf{0} \tag{5.2.63}$$

then substituting Eqs. (5.2.43), (5.2.47), and (5.2.63) into Eq. (5.2.62) yields

$$\dot{\mathbf{\Lambda}}(t) = -(\mathbf{A}^T - \mathbf{C}^T\mathbf{QCP})\mathbf{\Lambda} - \mathbf{\Lambda}(\mathbf{A} - \mathbf{PC}^T\mathbf{QC}) + \mathbf{C}^T\mathbf{QC} \qquad \mathbf{\Lambda}(t_f) = \mathbf{0} \tag{5.2.64}$$

Because $\mathbf{\Lambda}$ is positive definite, the covariance for the smoothed estimate is always as good as or better than the filtered estimate. An alternative equation for $\mathbf{P}(t|t_f)$ is [1]

$$\begin{aligned}\dot{\mathbf{P}}(t|t_f) = {} & (\mathbf{A} + \mathbf{R}^{-1}\mathbf{P}^{-1})\mathbf{P}(t|t_f) + \mathbf{P}(t|t_f)(\mathbf{A}^T + \mathbf{P}^{-1}\mathbf{R}^{-1}) \\ & -\mathbf{R}^{-1} - \mathbf{PC}^T\mathbf{QCP} \qquad \mathbf{P}(t_f|t_f) = \mathbf{P}(t_f)\end{aligned} \tag{5.2.65}$$

2. *Filtering estimates* $\hat{\mathbf{x}}(t|t)$ for the system of Eqs. (5.2.1) to (5.2.3) and (5.2.26) can be found by solving Eq. (5.2.45). The estimate covariance $\mathbf{P}(t)$ is computed from Eq. (5.2.43). Notice that the filter estimates may have covariances which increase or decrease with time depending on the noise covariances $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$.
3. *Prediction estimates* $\hat{\mathbf{x}}(t|t_0)$ for the system of Eqs. (5.2.1) to (5.2.3) and (5.2.26) are estimates at time $t > t_0$ for a system having no data after time t_0. These arise directly from the filtering equations if we let $\mathbf{Q}(t) \to 0$ for $t > t_0$ in Eqs. (5.2.43) and (5.2.45). Thus the prediction equations are

$$\dot{\hat{\mathbf{x}}}(t|t_0) = \mathbf{A}\hat{\mathbf{x}}(t|t_0) \tag{5.2.66}$$

where $\hat{\mathbf{x}}(t_0|t_0)$ is the filter estimate at time t_0. The prediction covariances $\mathbf{P}(t|t_0)$ are given by

$$\begin{aligned}\dot{\mathbf{P}}(t|t_0) &= \mathbf{P}(t|t_0)\mathbf{A}^T(t) + \mathbf{A}(t)\mathbf{P}(t|t_0) + \mathbf{R}^{-1} \\ \mathbf{P}(t_0|t_0) &= \mathbf{P}(t_0)\end{aligned} \tag{5.2.67}$$

Note that the prediction equations yield estimates *not conditional on data*, and $\mathbf{P}(t|t_0)$ generally tends to increase with time.

Let us now review some important properties of the estimation equations. First, all covariance matrices are *symmetric* and *positive semidefinite*. Thus the calculation of the covariance matrix having n^2 elements

$$\mathbf{P}(t) = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ P_{n1} & \cdots & \cdots & P_{nn} \end{bmatrix}$$

requires only the solution of $(n^2 + n)/2$ ODE equations because $P_{ij} = P_{ji}$. Second, observe that the filtering equations can be written in the form

$$\dot{\hat{\mathbf{x}}}(t|t) = \mathbf{A}\hat{\mathbf{x}} + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)] \tag{5.2.68}$$

where

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{Q}(t) \tag{5.2.69}$$

so that the filter behaves like the process model except for the feedback correction terms coming from the data. Note how similar this is to the optimal linear-quadratic feedback control law seen in Chap. 3. The covariance equations are Riccati equations very similar to those for $\mathbf{S}(t)$ in feedback control problems. In fact there is a very precise *duality* relationship between these two problems (see [8, 9]).

There are a limited number of more general results available for linear systems. Estimation equations when the process and sampling noise are correlated, or when the errors are correlated in time (so-called *colored noise*), for example, have been developed. The reader is urged to consult [1–7] as well as the recent literature for a discussion of these cases. The situation when the noise is non-Gaussian is particularly difficult to analyze because higher moments of the process state must be treated. Thus it is usually desirable to assume Gaussian distributions for the noise because these are completely described by the first two moments. Practically speaking, a wide range of naturally occurring noise processes can be well approximated by Gaussian distributions due to the central limit theorem of statistics.

Before continuing our discussion of theoretical results, let us illustrate the application of the filter with an example problem.

Example 5.2.4 Let us consider the CSTR problem discussed in Example 5.2.1. The system is described by

$$\dot{x}_1 = -(1 + \mathrm{Da}_1)x_1 + \xi_1(t) \qquad x_1(0) = x_{10} + \xi_{10} \tag{5.2.70}$$

$$\dot{x}_2 = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + \xi(t) \qquad x_2(0) = x_{20} + \xi_{20} \tag{5.2.71}$$

Recall that we know from Example 5.2.2 that the deterministic system is observable if we measure $x_2(t)$; thus let us choose our measuring device as

$$y(t) = x_2(t) + \eta(t) \tag{5.2.72}$$

Now let us assume that the process noise processes $\xi_1(t)$, $\xi_2(t)$ (possibly due to flow variations, temperature variations, or other unmeasured process disturbances) are white, Gaussian at every t with mean zero and covariance $R^{-1}(t)$, i.e.,

$$\mathcal{E}(\xi(t)\xi^T(\tau)) = \mathbf{R}^{-1}(t)\delta(t-\tau) \tag{5.2.73}$$

where

$$\mathbf{R}^{-1} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{E}(\xi_1(t)^2) & \mathcal{E}(\xi_1(t)\xi_2(t)) \\ \mathcal{E}(\xi_2(t)\xi_1(t)) & \mathcal{E}(\xi_2(t)^2) \end{bmatrix} \tag{5.2.74}$$

Similarly we assume that $\eta(t)$ and ξ_0 are white Gaussian with zero mean and covariances

$$\mathcal{E}(\eta(t)\eta(\tau)) = Q^{-1}(t)\delta(t-\tau) \tag{5.2.75}$$

$$\mathcal{E}(\xi_0\,\xi_0^T) = \mathbf{P}_0 \tag{5.2.76}$$

where

$$\mathbf{P}_0 = \begin{bmatrix} P_{110} & P_{120} \\ P_{210} & P_{220} \end{bmatrix} = \begin{bmatrix} \mathcal{E}(\xi_{10}^2) & \mathcal{E}(\xi_{10}\xi_{20}) \\ \mathcal{E}(\xi_{20}\xi_{10}) & \mathcal{E}(\xi_{20}^2) \end{bmatrix} \tag{5.2.77}$$

Applying Eq. (5.2.45), the filter equations become

$$\dot{\hat{x}}_1 = -(1+\mathrm{Da}_1)\hat{x}_1 + P_{12}(t)Q(t)[y - \hat{x}_2(t)] \tag{5.2.78}$$

$$\dot{\hat{x}}_2 = \mathrm{Da}_1\hat{x}_1 - (1+\mathrm{Da}_3)\hat{x}_2 + P_{22}(t)Q(t)[y - \hat{x}_2(t)] \tag{5.2.79}$$

where the precomputable covariance equations arise from Eq. (5.2.43) and take the form

$$\dot{P}_{11}(t) = -2(1+\mathrm{Da}_1)P_{11} + \rho_{11} - Q(P_{12})^2$$

$$\dot{P}_{12}(t) = \mathrm{Da}_1 P_{11} - (2+\mathrm{Da}_1+\mathrm{Da}_3)P_{12} + \rho_{12} - Q\,P_{12}P_{22}$$

$$\dot{P}_{22}(t) = 2\mathrm{Da}_1 P_{12} - 2(1+\mathrm{Da}_3)P_{22} + \rho_{22} - Q(P_{22})^2 \tag{5.2.80}$$

where symmetry requires $P_{12}(t) = P_{21}(t)$.

To implement this state estimator, one must compute $\mathbf{P}(t)$ off-line and store it in the process control computer. Then $y(t)$ is fed to the estimator equations (5.2.78) and (5.2.79), which are numerically integrated in real time to yield current estimates of x_1 and x_2.

To demonstrate the performance of this estimation scheme, numerical computations were carried out for the parameter values

$$\mathrm{Da}_1 = 3.0 \qquad \mathrm{Da}_3 = 1.0 \qquad t_f = 2.0 \qquad x_{10} = 1.0 \qquad x_{20} = 0.0$$

$$\mathbf{P}(0) = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.01 \end{bmatrix} \qquad Q^{-1} = 0.0025 \qquad \mathbf{R}^{-1} = \mathbf{0}$$

and the filter initial guesses were

$$\hat{x}_1(0) = 0.85 \qquad \hat{x}_2(0) = 0.15$$

Measurement errors having a zero mean and variance $\sigma = 0.05$ were simulated by a random number generator and added to the x_2 values to produce the sensor signals $y(t)$. The estimate covariances $P_{ij}(t)$ are plotted in Fig. 5.5. Because $\mathbf{R}^{-1} = \mathbf{0}$, all the P_{ij} decline to zero as $t \to \infty$. The "true" states x_1, x_2, the filter estimates $\hat{x}_1$, $\hat{x}_2$, and the measurement noise $\eta(t)$ are shown in Fig. 5.6. Note how the estimates converge toward the "true" values even with high measurement error and poor initial estimates.

This example problem serves to illustrate a shortcoming in the theoretical results we have presented so far. Our results require that the measuring device yield data continuously in time. However, for this example problem we are measuring a concentration in the reactor, and many composition measurements can only be made by sampling at discrete intervals in time (e.g., through on-line chromatography). Thus we need to extend our estimation results to include discrete time sampling.

Estimation with Discrete Time Data

For *samples discrete in time*, the output device should be modeled by

$$\mathbf{y}(t_k) = \mathbf{C}(t_k)\mathbf{x}(t_k) + \boldsymbol{\eta}(t_k) \qquad k = 1, 2, \ldots \tag{5.2.81}$$

where $\boldsymbol{\eta}(t_k)$ is Gaussian white (uncorrelated in time) noise with zero mean and covariance $\mathbf{Q}_k^{-1}$. It can be shown [12] that the proper estimation equations for

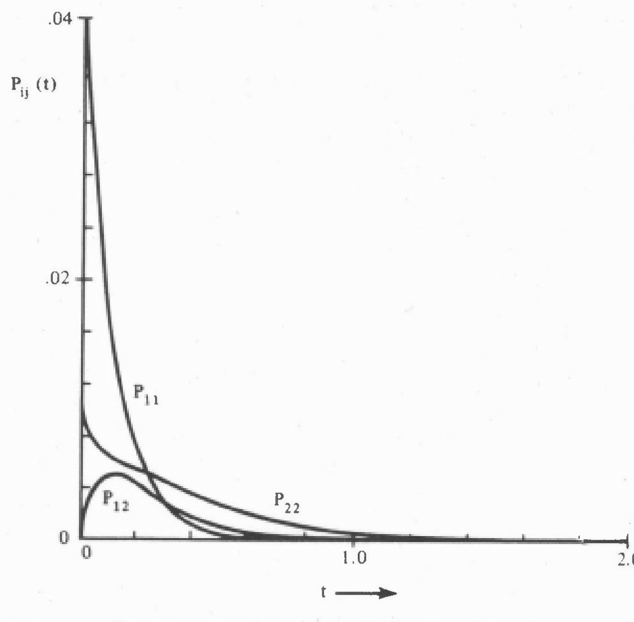


Figure 5.5 Estimate covariances for CSTR example with continuous data.

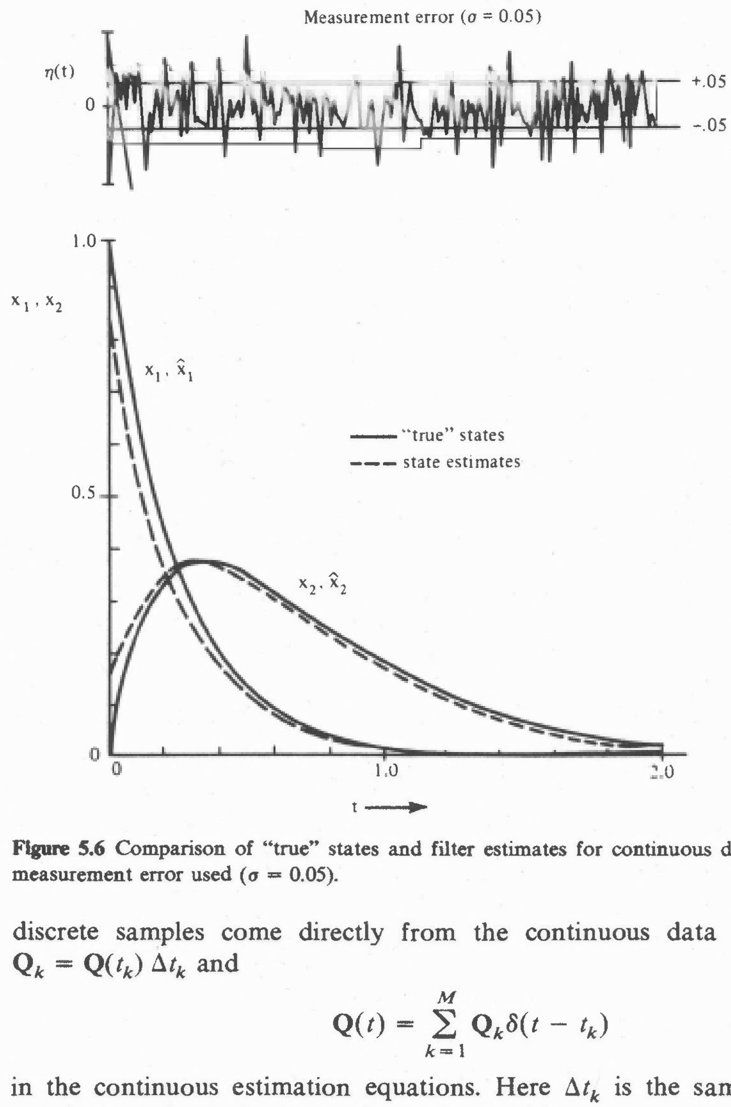


Figure 5.6 Comparison of "true" states and filter estimates for continuous data. $\eta(t)$ is the actual measurement error used ($\sigma = 0.05$).

discrete samples come directly from the continuous data results if we let $\mathbf{Q}_k = \mathbf{Q}(t_k)\,\Delta t_k$ and

$$\mathbf{Q}(t) = \sum_{k=1}^{M} \mathbf{Q}_k \delta(t - t_k) \tag{5.2.82}$$

in the continuous estimation equations. Here Δt_k is the sampling interval at sample k, and any time t_f can be represented by

$$t_f = \sum_{k=1}^{M} \Delta t_k$$

Substituting Eq. (5.2.82) into Eqs. (5.2.43) and (5.2.45), one obtains the following discrete time *filtering equations*. Between samples, $t_{k-1} < t < t_k$, one must use the prediction equations

$$\dot{\hat{\mathbf{x}}}(t|t_{k-1}) = \mathbf{A}(t)\hat{\mathbf{x}}(t|t_{k-1}) \tag{5.2.83}$$

$$\dot{\mathbf{P}}(t|t_{k-1}) = \mathbf{P}(t|t_{k-1})\mathbf{A}^T(t) + \mathbf{A}(t)\mathbf{P}(t|t_{k-1}) + \mathbf{R}^{-1}(t) \tag{5.2.84}$$

while at sampling points the updating equations

$$\hat{x}(t_k|t_k) = \hat{x}(t_k|t_{k-1}) + \mathbf{K}(t_k)[\mathbf{y}(t_k) - \mathbf{C}(t_k)\hat{\mathbf{x}}(t_k|t_{k-1})] \tag{5.2.85}$$

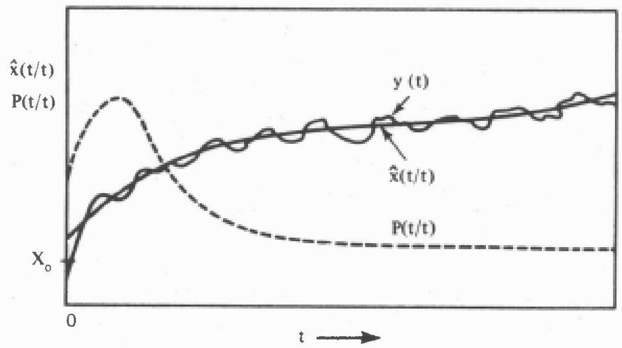
$$\mathbf{P}(t_k|t_k) = \mathbf{P}(t_k|t_{k-1}) - \mathbf{K}(t_k)\mathbf{C}(t_k)\mathbf{P}(t_k|t_{k-1}) \tag{5.2.86}$$

must be used. Here

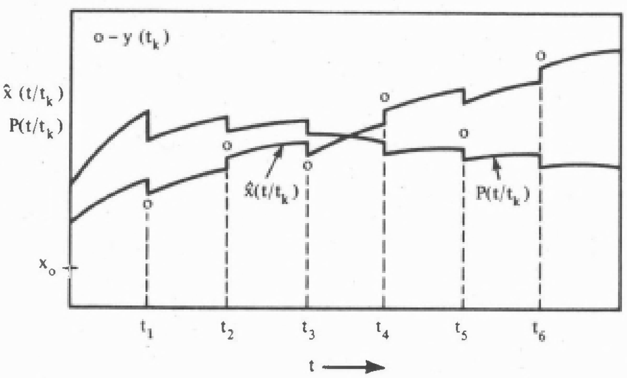
$$\mathbf{K}(t_k) = \mathbf{P}(t_k|t_{k-1})\mathbf{C}^T(t_k)[\mathbf{C}(t_k)\mathbf{P}(t_k|t_{k-1})\mathbf{C}^T(t_k) + \mathbf{Q}_k^{-1}]^{-1} \tag{5.2.87}$$

Thus both the filter estimates and the covariance matrix have discontinuities at sampling points t_k. Example trajectories of the filter estimates and covariance for both continuous and discrete data are shown in Fig. 5.7. Note that in the continuous case the covariance tends to a positive steady-state value determined by the sampling and process noise levels. For the discrete case, the covariance tends to increase between samples and be reduced at sampling points.

We are now ready to summarize our results for discrete data estimation.



(a) Continuous data



(b) Discrete data

Figure 5.7 Typical filtering trajectories for (a) continuous data, and for (b) discrete data.

1. *For smoothing estimates*, one solves the filtering equations (5.2.83) to (5.2.87) forward in time and then uses Eq. (5.2.59) to generate the smoothed estimates $\hat{\mathbf{x}}(t|t_f)$. The covariance equation for the smoothed estimates can be found by substituting Eqs. (5.2.84) and (5.2.86) into Eq. (5.2.65) and integrating backward. Note that although the filtered estimates are discontinuous at sampling points, the smoothed estimates are continuous.
2. *For filtering estimates*, $\hat{\mathbf{x}}(t|t_{k-1})$, one solves the prediction equations (5.2.83) and (5.2.84) between samples and the updating equations (5.2.85) to (5.2.87) at the sampling points.
3. Since *prediction estimates* involve no data, the prediction equations are independent of the sampling device.

Note that just as in the continuous data case, the covariance equations can be precomputed off-line, so that only the filter equations need be solved in real time.

Let us illustrate the discrete data filter with an example problem.

Example 5.2.5 Let us again consider the CSTR filtering problem discussed in Example 5.2.4, with the exception that here the sampling device is discrete in time:

$$y(t_k) = x_2(t_k) + \eta(t_k) \tag{5.2.88}$$

The filtering equations between samples become

$$\dot{\hat{x}}_1 = -(1 + \mathrm{Da}_1)\hat{x}_1 \qquad t_{k-1} < t < t_k \tag{5.2.89}$$

$$\dot{\hat{x}}_2 = \mathrm{Da}_1\hat{x}_1 - (1 + \mathrm{Da}_3)\hat{x}_2 \tag{5.2.90}$$

and the updating equations at samples take the form

$$\hat{x}_1(t_k|t_k) = \hat{x}_1(t_k|t_{k-1}) + \frac{P_{12}(t_k|t_{k-1})}{P_{22}(t_k|t_{k-1}) + Q_k^{-1}} \times [y(t_k) - \hat{x}_2(t_k|t_{k-1})] \tag{5.2.91}$$

$$\hat{x}_2(t_k|t_k) = \hat{x}_2(t_k|t_{k-1}) + \frac{P_{22}(t_k|t_{k-1})}{P_{22}(t_k|t_{k-1}) + Q_k^{-1}} \times [y(t_k) - \hat{x}_2(t_k|t_{k-1})] \tag{5.2.92}$$

The covariance equations between samples, $t_{k-1} < t < t_k$, are

$$\dot{P}_{11}(t|t_{k-1}) = -2(1 + \mathrm{Da}_1)P_{11}(t|t_{k-1}) + \rho_{11} \tag{5.2.93}$$

$$\dot{P}_{12}(t|t_{k-1}) = \mathrm{Da}_1 P_{11}(t|t_{k-1}) - (2 + \mathrm{Da}_1 + \mathrm{Da}_3)P_{12}(t|t_{k-1}) + \rho_{12} \tag{5.2.94}$$

$$\dot{P}_{22}(t|t_{k-1}) = 2\mathrm{Da}_1 P_{12}(t|t_{k-1}) - 2(1 + \mathrm{Da}_3)P_{22}(t|t_{k-1}) + \rho_{22} \tag{5.2.95}$$

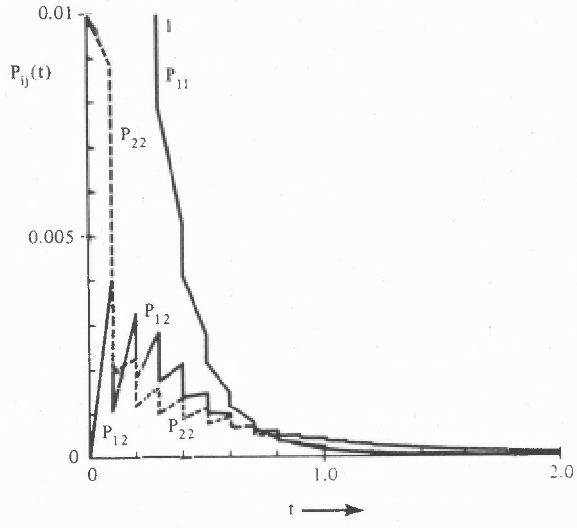


Figure 5.8 Estimate covariances for CSTR example with discrete data.

and the updating equations at samples are

$$P_{11}(t_k|t_k) = P_{11}(t_k|t_{k-1}) - \frac{[P_{12}(t_k|t_{k-1})]^2}{P_{22}(t_k|t_{k-1}) + Q_k^{-1}} \tag{5.2.96}$$

$$P_{12}(t_k|t_k) = P_{12}(t_k|t_{k-1}) - \frac{P_{12}(t_k|t_{k-1})P_{22}(t_k|t_{k-1})}{P_{22}(t_k|t_{k-1}) + Q_k^{-1}} \tag{5.2.97}$$

$$P_{22}(t_k|t_k) = P_{22}(t_k|t_{k-1}) - \frac{P_{22}(t_k|t_{k-1})^2}{P_{22}(t_k|t_{k-1}) + Q_k^{-1}} \tag{5.2.98}$$

In order to illustrate the estimator behavior, numerical calculations were carried out for the same parameters and conditions as in Example 5.2.4. However, in this case the data are obtained only at discrete times $t_n = n\,\Delta t$ where $\Delta t = 0.05$. The estimate covariances are plotted in Fig. 5.8, where one should note how the P_{ij} usually increase between samples and make a discontinuous improvement after each measurement point. The state estimates, which have the same type of discontinuous behavior, are shown in Fig. 5.9 when there is no measurement error. Even with intermittant measurements, the filter estimates converge rapidly to the true state.

Observers

Observers are estimators for the state variables of a deterministic system, i.e., a system without any significant process noise or measurement errors. Like filtering, observers are used to reconstruct the full-state vector for the system $\mathbf{x}(t)$ from the available outputs $\mathbf{y}(t)$. In this sense, the optimal estimators just

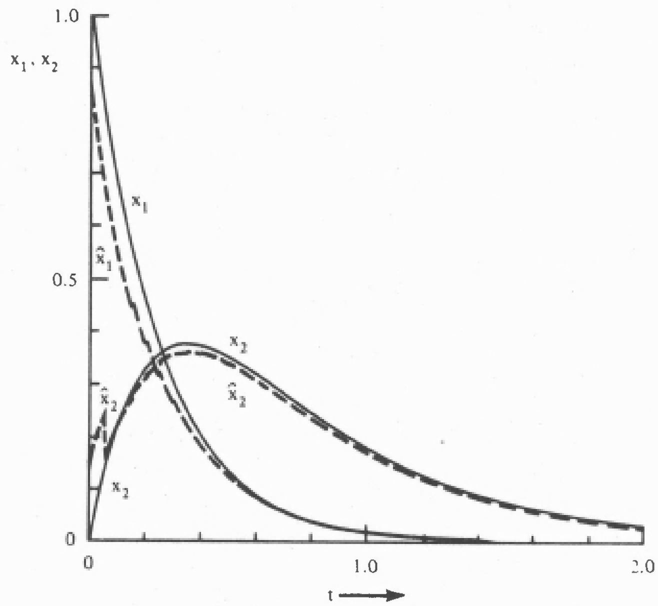


Figure 5.9 Comparison of "true" states and filter estimates for discrete data.

discussed may be considered as a type of optimal, stochastic observer. Observer theory [6, 13, 14] has been highly developed for linear systems. To illustrate the properties of observers, let us consider the linear deterministic system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{5.2.99}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x} \tag{5.2.100}$$

which has n states and l outputs. If the system is observable, then one can construct a deterministic observer in the following way. Let us define a new variable, $\mathbf{z}(t)$ of dimension $n - l$, which may be related to the state by the relation

$$\mathbf{z}(t) = \mathbf{T}(t)\mathbf{x}(t) \tag{5.2.101}$$

where the $(n - l) \times n$ matrix $\mathbf{T}(t)$ is chosen so that the $n \times n$ matrix

$$\begin{bmatrix} \mathbf{T}(t) \\ \hdashline \mathbf{C}(t) \end{bmatrix}$$

is nonsingular. Then the full state vector may be expressed in terms of $\mathbf{z}$, $\mathbf{y}$ as

$$\mathbf{x} = \begin{bmatrix} \mathbf{T}(t) \\ \hdashline \mathbf{C}(t) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z}(t) \\ \hdashline \mathbf{y}(t) \end{bmatrix} \tag{5.2.102}$$

Now if $\begin{bmatrix} \mathbf{T}(t) \\ \hdashline \mathbf{C}(t) \end{bmatrix}^{-1}$ is partitioned as

$$\begin{bmatrix} \mathbf{T}(t) \\ \hdashline \mathbf{C}(t) \end{bmatrix}^{-1} \equiv [\boldsymbol{\Omega}_1 \,\vdots\, \boldsymbol{\Omega}_2] \tag{5.2.103}$$

where $\boldsymbol{\Omega}_1$ is an $n \times (n-l)$ matrix and $\boldsymbol{\Omega}_2$ is an $n \times l$ matrix, then

$$\mathbf{x}(t) = \boldsymbol{\Omega}_1\mathbf{z}(t) + \boldsymbol{\Omega}_2\mathbf{y}(t) \tag{5.2.104}$$

It now remains to determine $\mathbf{z}(t)$. Let us differentiate Eq. (5.2.101) to yield

$$\dot{\mathbf{z}} = \dot{\mathbf{T}}\mathbf{x} + \mathbf{T}\dot{\mathbf{x}} = \dot{\mathbf{T}}[\boldsymbol{\Omega}_1\mathbf{z}(t) + \boldsymbol{\Omega}_2\mathbf{y}(t)] + \mathbf{TA}[\boldsymbol{\Omega}_1\mathbf{z}(t) + \boldsymbol{\Omega}_2\mathbf{y}(t)] \tag{5.2.105}$$

or

$$\dot{\mathbf{z}} = (\mathbf{TA}\boldsymbol{\Omega}_1 + \dot{\mathbf{T}}\boldsymbol{\Omega}_1)\mathbf{z}(t) + (\mathbf{TA}\boldsymbol{\Omega}_2 + \dot{\mathbf{T}}\boldsymbol{\Omega}_2)\mathbf{y}(t) \tag{5.2.106}$$

where we note that from Eq. (5.2.103)

$$\boldsymbol{\Omega}_1\mathbf{T} + \boldsymbol{\Omega}_2\mathbf{C} = \mathbf{I} \tag{5.2.107}$$

and

$$\begin{bmatrix} \mathbf{T} \\ \hdashline \mathbf{C}(t) \end{bmatrix} [\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2] = \left[\begin{array}{c:c} \mathbf{T}\boldsymbol{\Omega}_1 & \mathbf{T}\boldsymbol{\Omega}_2 \\ \hdashline \mathbf{C}\boldsymbol{\Omega}_1 & \mathbf{C}\boldsymbol{\Omega}_2 \end{array}\right] = \mathbf{I} \tag{5.2.108}$$

Thus

$$\mathbf{T}\boldsymbol{\Omega}_1 = \mathbf{I} \qquad \mathbf{T}\boldsymbol{\Omega}_2 = \mathbf{0} \qquad \mathbf{C}\boldsymbol{\Omega}_1 = \mathbf{0} \qquad \mathbf{C}\boldsymbol{\Omega}_2 = \mathbf{I} \tag{5.2.109}$$

From Eq. (5.2.107) one sees that

$$(\dot{\boldsymbol{\Omega}}_1\mathbf{T}) + (\dot{\boldsymbol{\Omega}}_2\mathbf{C}) = \mathbf{0}$$

and from Eq. (5.2.109),

$$\begin{aligned} \dot{\mathbf{T}}\boldsymbol{\Omega}_1 &= -\mathbf{T}\dot{\boldsymbol{\Omega}}_1 \\ \dot{\mathbf{T}}\boldsymbol{\Omega}_2 &= -\mathbf{T}\dot{\boldsymbol{\Omega}}_2 \end{aligned} \tag{5.2.110}$$

Thus Eq. (5.2.106) becomes

$$\dot{\mathbf{z}}(t) = (\mathbf{TA}\boldsymbol{\Omega}_1 - \mathbf{T}\dot{\boldsymbol{\Omega}}_1)\mathbf{z}(t) + (\mathbf{TA}\boldsymbol{\Omega}_2 - \mathbf{T}\dot{\boldsymbol{\Omega}}_2)\mathbf{y}(t) \tag{5.2.111}$$

and our observer takes the form

$$\dot{\hat{\mathbf{z}}} = (\mathbf{TA}\boldsymbol{\Omega}_1 - \mathbf{T}\dot{\boldsymbol{\Omega}}_1)\hat{\mathbf{z}} + (\mathbf{TA}\boldsymbol{\Omega}_2 - \mathbf{T}\dot{\boldsymbol{\Omega}}_2)\mathbf{y}(t) \tag{5.2.112}$$

$$\hat{\mathbf{x}}(t) = \boldsymbol{\Omega}_1\hat{\mathbf{z}} + \boldsymbol{\Omega}_2\mathbf{y}(t) \tag{5.2.113}$$

The estimate error may be found by subtracting the actual process, Eqs. (5.2.111) and (5.2.104), from the estimator equations (5.2.112) and (5.2.113) and noting that in general $\hat{\mathbf{z}}(0) - \mathbf{z}(0) \neq \mathbf{0}$ because we don't know the initial conditions. Thus the errors in $\mathbf{z}$, $\mathbf{x}$ are given by

$$\mathbf{e}_z = \hat{\mathbf{z}} - \mathbf{z}$$

$$\mathbf{e}_x = \hat{\mathbf{x}} - \mathbf{x}$$

and

$$\dot{\mathbf{e}}_z = \left(\mathbf{T}\mathbf{A}\boldsymbol{\Omega}_1 - \mathbf{T}\dot{\boldsymbol{\Omega}}_1\right)\mathbf{e}_z \tag{5.2.114}$$

$$\mathbf{e}_x = \boldsymbol{\Omega}_1\mathbf{e}_z \qquad \mathbf{e}_z = \mathbf{T}\mathbf{e}_x \qquad \dot{\mathbf{e}}_x = \dot{\boldsymbol{\Omega}}_1\mathbf{e}_z + \boldsymbol{\Omega}\dot{\mathbf{e}}_z \tag{5.2.115}$$

or*

$$\dot{\mathbf{e}}_x = \left[\boldsymbol{\Omega}_1\mathbf{T}\mathbf{A} - \boldsymbol{\Omega}_1\mathbf{T}\dot{\boldsymbol{\Omega}}_1\mathbf{T} + \dot{\boldsymbol{\Omega}}_1\mathbf{T}\right]\mathbf{e}_x \tag{5.2.116}$$

and the observer has an exponential rate of convergence with rate given by the eigenvalues of the matrix in square brackets.

By employing Eq. (5.2.110) it is possible to show that

$$\dot{\mathbf{e}}_x = \left(\boldsymbol{\Omega}_1\mathbf{T}\mathbf{A} + \boldsymbol{\Omega}_1\dot{\mathbf{T}}\boldsymbol{\Omega}_1\mathbf{T} + \dot{\boldsymbol{\Omega}}_1\mathbf{T}\right)\mathbf{e}_x \tag{5.2.117}$$

but applying $\boldsymbol{\Omega}_1\mathbf{T}\mathbf{e}_x = \mathbf{e}_x$ and Eq. (5.2.107) yields

$$\dot{\mathbf{e}}_x = \left[(\mathbf{I} - \boldsymbol{\Omega}_2\mathbf{C})\mathbf{A} + \boldsymbol{\Omega}_1\dot{\mathbf{T}} + \dot{\boldsymbol{\Omega}}_1\mathbf{T}\right]\mathbf{e}_x \tag{5.2.118}$$

or

$$\dot{\mathbf{e}}_x = \left(\mathbf{A} - \boldsymbol{\Omega}_2\mathbf{C}\mathbf{A} - \boldsymbol{\Omega}_2\dot{\mathbf{C}} - \dot{\boldsymbol{\Omega}}_2\mathbf{C}\right)\mathbf{e}_x \qquad \mathbf{e}_x(0) = \mathbf{e}_{x_0} \tag{5.2.119}$$

where $\mathbf{e}_{x_0}$ is the initial error in the state estimates. Note that one may control the rate of convergence of the observer by adjusting $\boldsymbol{\Omega}_2$ in Eq. (5.2.119).

One should realize that there are infinitely many transformations $\mathbf{T}$, $\boldsymbol{\Omega}_1$, $\boldsymbol{\Omega}_2$ which satisfy Eqs. (5.2.107) and (5.2.108); thus one has great freedom of choice. For systems having constant matrices $\mathbf{A}$, $\mathbf{C}$, one possible observer design procedure would be

1. Choose $\boldsymbol{\Omega}_2$ such that $\mathbf{A} - \boldsymbol{\Omega}_2\mathbf{C}\mathbf{A}$ has the desired eigenvalues for convergence.
2. Choose some $\boldsymbol{\Omega}_1$,$\mathbf{T}$ such that Eq. (5.2.107) is satisfied.

Observers have also been developed for stochastic systems (e.g., [6, 15–18]), where there are fewer estimation equations to solve than for optimal state estimation because only $n - l$ states need estimation. However, the available experience suggests that observers are much less robust in the face of measurement and process noise than the optimal estimators. Thus observers should only be considered for implementation in control systems with small measurement and process noise.

Let us illustrate the application of an observer with an example problem.

* Note that Eq. (5.2.115) implies $\boldsymbol{\Omega}_1\mathbf{T}\mathbf{e}_x \equiv \mathbf{e}_x$ but *not* $\boldsymbol{\Omega}_1\mathbf{T} = \mathbf{I}$.

Example 5.2.6 Consider the state estimation problem for the CSTR discussed in Example 5.2.4 and apply an observer to the problem. If we measure x_2, then the equations are

$$\dot{x}_1 = -(1 + \mathrm{Da}_1)x_1$$
$$\dot{x}_2 = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2$$
$$y = x_2$$

Recall that we have already proved that this example is observable. Let us choose

$$z = \mathbf{Tx} = T_1 x_1 + T_2 x_2 \tag{5.2.120}$$

Thus

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ 0 & 1 \end{bmatrix} \mathbf{x} \tag{5.2.121}$$

and by inversion,

$$\mathbf{x} = \begin{bmatrix} \dfrac{1}{T_1} & -\dfrac{T_2}{T_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{1}{T_1} \\ 0 \end{bmatrix} z + \begin{bmatrix} \dfrac{-T_2}{T_1} \\ 1 \end{bmatrix} y \tag{5.2.122}$$

so that

$$\Omega_1 = \begin{bmatrix} \dfrac{1}{T_1} \\ 0 \end{bmatrix} \qquad \Omega_2 = \begin{bmatrix} -\dfrac{T_2}{T_1} \\ 1 \end{bmatrix} \tag{5.2.123}$$

Now the observer error as given by Eq. (5.2.119) is

$$\dot{\mathbf{e}}_x = \begin{bmatrix} -(1 + \mathrm{Da}_1) + \dfrac{T_2}{T_1}\mathrm{Da}_1 & \dfrac{-T_2}{T_1}(1 + \mathrm{Da}_3) \\ 0 & 0 \end{bmatrix} \mathbf{e}_x \qquad \mathbf{e}_x(0) = \begin{bmatrix} e_{x_{10}} \\ 0 \end{bmatrix} \tag{5.2.124}$$

Note that there is no error in the estimates of x_2 because they are directly measured. Also, the eigenvalue of the x_1 estimate is $-(1 + \mathrm{Da}_1) + (T_2/T_1)\mathrm{Da}_1$. Thus by choosing T_2/T_1 large and negative we can get rapid convergence of the observer.

The observer equations (5.2.112) then become

$$\dot{\hat{z}} = [-T_1(1 + \mathrm{Da}_1) + T_2\mathrm{Da}_1, \; -T_2(1 + \mathrm{Da}_3)] \begin{bmatrix} \dfrac{1}{T_1} \\ 0 \end{bmatrix} \hat{z}$$
$$+ [-T_1(1 + \mathrm{Da}_1) + T_2\mathrm{Da}_1, \; -T_2(1 + \mathrm{Da}_3)] \begin{bmatrix} \dfrac{-T_2}{T_1} \\ 1 \end{bmatrix} y$$

or

$$\dot{\hat{z}} = \left[-(1 + \mathrm{Da}_1) + \frac{T_2}{T_1}\mathrm{Da}_1 \right]\hat{z}$$

$$+ \left[T_2(1 + \mathrm{Da}_1) - \frac{T_2^2}{T_1}\mathrm{Da}_1 - T_2(1 + \mathrm{Da}_3) \right] y \qquad (5.2.125)$$

$$\hat{x}_1 = \frac{1}{T_1}\hat{z} - \frac{T_2}{T_1}y \qquad (5.2.126)$$

$$\hat{x}_2 = y \qquad (5.2.127)$$

In order to illustrate the performance of the observer for the conditions given in Example 5.2.4, one must choose T_1 and T_2. To aid this choice, note that the eigenvalue associated with the error in x_1 may be determined from Eq. (5.2.124) as

$$\lambda = \left[-(1 + \mathrm{Da}_1) + \frac{T_2}{T_1}\mathrm{Da}_1 \right]$$

which for $\mathrm{Da}_1 = 3.0$ is

$$\lambda = -4 + 3\frac{T_2}{T_1}$$

Thus the rate of convergence can be made as fast as desired (in the absence of measurement and process errors) by adjusting T_2/T_1. As specific examples, Fig. 5.10 shows the rate of convergence when $T_1 = 1$, $T_2 = -4$,

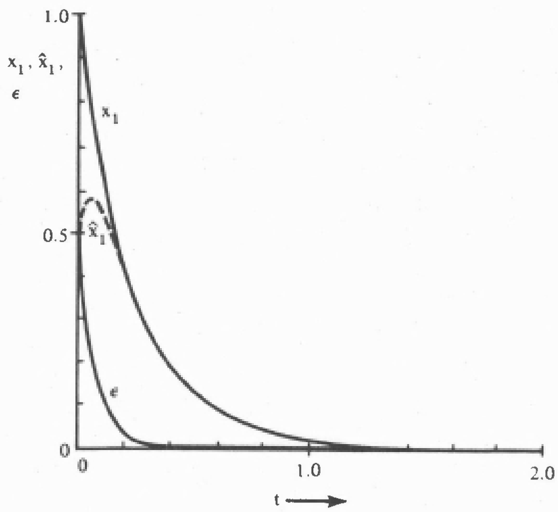


Figure 5.10 Observer performance for $T_1 = 1$, $T_2 = -4$, $\lambda = -16$, $\epsilon = x_1 - \hat{x}_1$.

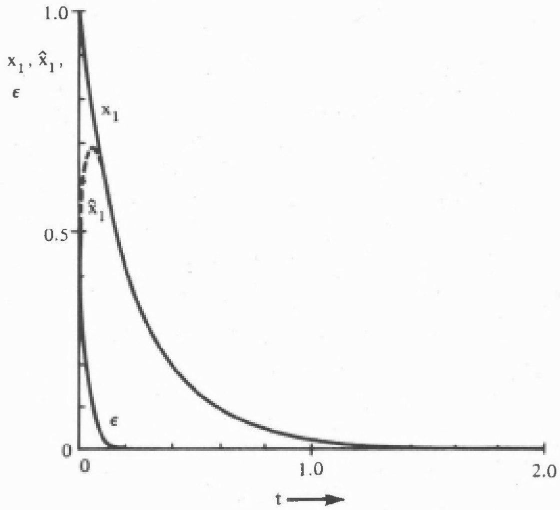


Figure 5.11 Observer performance for $T_1 = 1$, $T_2 = -8$, $\lambda = -28$, $\epsilon = x_1 - \hat{x}_1$.

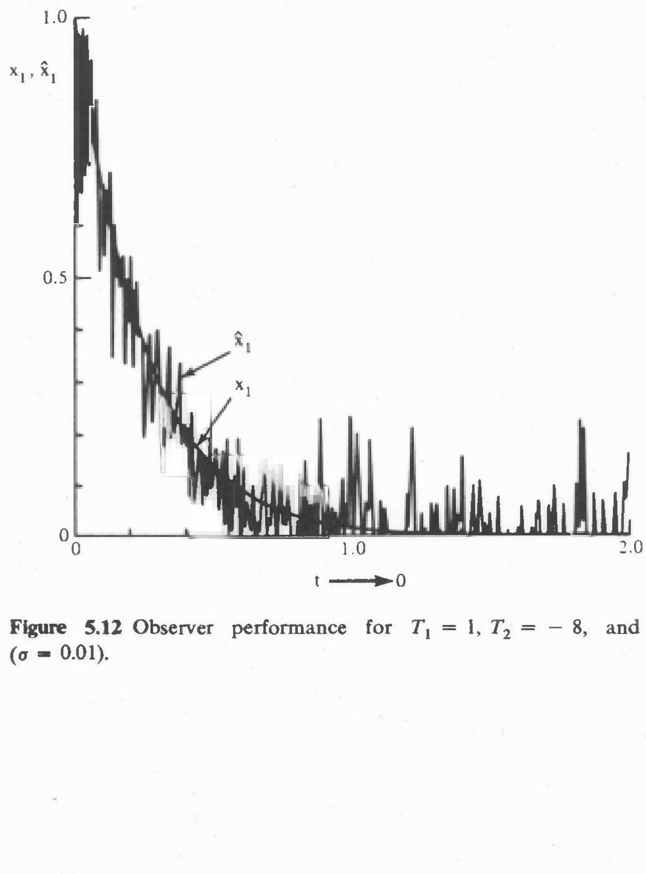


Figure 5.12 Observer performance for $T_1 = 1$, $T_2 = -8$, and random measurement error ($\sigma = 0.01$).

$\lambda = -16$, while Fig. 5.11 gives the observer performance when $T_1 = 1$, $T_2 = -8$, $\lambda = -28$. In both these cases the estimated initial condition for x_1 was $\hat{x}_1(0) = 0.5$, far from the true value of 1.0, and yet the observer converged quickly.

The difficulty observers have with measurement or process noise is illustrated in Fig. 5.12, where the random noise with standard deviation $\sigma = 0.01$ (five times smaller than the noise levels easily handled by the filter in Example 5.2.4) has been added to the x_2 measurements. The observer estimates are so bad as to be useless for control purposes. Thus observers are not recommended for estimation where noise is significant.

5.3 STATE ESTIMATION FOR NONLINEAR SYSTEMS DESCRIBED BY ORDINARY DIFFERENTIAL EQUATIONS

In this section we shall extend our treatment of state estimation to a broad class of nonlinear systems modeled *formally* by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) + \boldsymbol{\xi}(t) \tag{5.3.1}$$

$$\mathbf{x}(0) = \mathbf{x}_0 + \boldsymbol{\xi}_0 \tag{5.3.2}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}, t) + \boldsymbol{\eta}(t) \tag{5.3.3}$$

where Eqs. (5.3.1) and (5.3.2) represent the nonlinear system state equations, and Eq. (5.3.3) the nonlinear measuring device. The noise processes $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$ as well as the initial error $\boldsymbol{\xi}_0$ are assumed to have zero mean and unspecified distributions.

Unfortunately, for nonlinear systems the conditional probability distributions for the state evolving in time are not Gaussian even when $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$, and $\boldsymbol{\xi}_0$ are assumed to have Gaussian distributions. This means that an infinite number of moments are required to determine the distribution and that the moments are coupled in increasing order; i.e., the first moment depends on the second, the second on the third, etc. This structure of the statistics means that approximations must be made in order to obtain a computationally feasible filter.

Furthermore, a new type of differential calculus (e.g., the Ito calculus) must be used to rigorously treat these nonlinear time-varying probability distributions (see [4]). The Ito calculus would require, for example, that Eq. (5.3.1) be rewritten in differential form

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\, dt + d\boldsymbol{\beta} \tag{5.3.4}$$

where $\boldsymbol{\xi}(t)$ is the formal definition of $d\boldsymbol{\beta}/dt$. These considerations carry us into deep mathematical waters and too far afield from our discussion of applications. Thus we urge the reader to pursue these developments independently [4]. In this chapter we shall proceed with formal derivations of nonlinear state estimators and where possible compare them with rigorously derived results.

Observability

In any discussion of estimation, observability must be a primary question. Unfortunately, the results available for nonlinear systems are more difficult to apply than the results for linear systems. Basically the approach that has been used is to linearize the nonlinear equations about some nominal trajectory $\bar{\mathbf{x}}(t)$ and apply the known results for linear, nonautonomous systems. Hwang and Seinfeld [19] have a good discussion of this approach.

To establish conditions for observability, let us linearize Eqs. (5.3.1) to (5.3.3) about a nominal trajectory $\bar{\mathbf{x}}(t)$ satisfying Eq. (5.3.1) and beginning at $\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0$, and define

$$\delta\mathbf{x}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}(t) \qquad \delta\mathbf{y}(t) = \mathbf{y}(t) - \bar{\mathbf{y}}(t)$$

$$\mathbf{A}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}(t)} \qquad \mathbf{C}(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}(t)} \tag{5.3.5}$$

where we neglect the noise processes $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$, $\boldsymbol{\xi}_0$ because observability is a property of the deterministic system. Then the linearized system is

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}(t)\delta\mathbf{x}(t) \qquad \delta\mathbf{x}(0) = \delta\mathbf{x}_0 \tag{5.3.6}$$

$$\delta\mathbf{y}(t) = \mathbf{C}(t)\delta\mathbf{x}(t) \tag{5.3.7}$$

We can now apply the results of the last section to provide the result: *The nonlinear system of Eqs.* (5.3.1) *to* (5.3.3) *is observable for all initial conditions in some neighborhood of* $\mathbf{x}_0$ *provided the matrix*

$$\mathbf{M}(0, t_f) = \int_0^{t_f} \mathbf{\Phi}(t, 0)^T \mathbf{C}^T(t)\mathbf{C}(t)\mathbf{\Phi}(t, 0)\, dt \tag{5.3.8}$$

is positive definite for $t_f > 0$, where, of course, $\Phi(t, \tau)$ is the fundamental matrix solution associated with the solution to Eq. (5.3.6). This result can be extended to the entire $\mathbf{x}$ domain by selecting a grid of initial conditions covering the entire domain.

In principle the application of this criterion could involve extensive calculation to map the entire domain of initial conditions in marginal cases. However, in practice one often finds that observability is determined by the problem structure and not dependent on $\mathbf{x}_0$ in a complex manner. Hence simple linearized observability tests are usually adequate for nonlinear problems. We shall illustrate these results with an example problem.

Example 5.3.1 Let us consider an adiabatic batch reactor in which the reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ is taking place. The modeling equations for the reactor take the form

$$\frac{dc_A}{dt'} = -k_1 c_A \qquad c_A(0) = c_{A_0} \tag{5.3.9}$$

$$\frac{dc_B}{dt'} = k_1 c_A - k_2 c_B \qquad c_B(0) = c_{B_0} \tag{5.3.10}$$

$$\frac{dT}{dt'} = J_1 k_1 c_A + J_2 k_2 c_B = -J_1 \frac{dc_A}{dt'} - J_2\left(\frac{dc_A}{dt'} + \frac{dc_B}{dt'}\right) \qquad T(0) = T_0 \tag{5.3.11}$$

where

$$J_i = -\Delta H_i/\rho C_p \qquad i = 1, 2$$

Now integrating the heat balance one obtains

$$T(t) = T_0 + (J_1 + J_2)(c_{A_0} - c_A) + J_2(c_{B_0} - c_B) \tag{5.3.12}$$

Thus if we know c_A, c_B, we can calculate T from Eq. (5.3.12). The velocity constants k_1, k_2 are Arrhenius functions of temperature, so that

$$k_i = A_{i0}e^{-E_i/RT} \qquad i = 1, 2 \tag{5.3.13}$$

and this provides the strongest nonlinearity in the problem. Now the question is, if one measures $c_B(t)$, will the nonlinear system be observable?

Let us put the equations in reduced form by defining

$$x_1 = c_A \qquad x_2 = c_B \qquad t = t'A_{10} \qquad p = \frac{E_2}{E_1}$$

$$\alpha_0 = \frac{R}{E_1}\left[T_0 + (J_1 + J_2)c_{A_0} + J_2c_{B_0}\right]$$

$$\alpha_1 = \frac{R}{E_1}(J_1 + J_2) \qquad \alpha_2 = \frac{J_2R}{E_1} \qquad \gamma = \frac{A_{20}}{A_{10}} \qquad \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} c_{A_0} \\ c_{B_0} \end{bmatrix} \tag{5.3.14}$$

then Eq. (5.3.12) is

$$\frac{RT}{E_1} = \alpha_0 - \alpha_1x_1 - \alpha_2x_2 \tag{5.3.15}$$

and the system becomes

$$\frac{dx_1}{dt} = -\exp\left(-\frac{1}{\alpha_0 - \alpha_1x_1 - \alpha_2x_2}\right)x_1 \qquad x_1(0) = x_{10} \tag{5.3.16}$$

$$\frac{dx_2}{dt} = \exp\left(-\frac{1}{\alpha_0 - \alpha_1x_1 - \alpha_2x_2}\right)x_1 - \gamma\exp\left(-\frac{p}{\alpha_0 - \alpha_1x_1 - \alpha_2x_2}\right)x_2$$

$$x_2(0) = x_{20} \tag{5.3.17}$$

Now let us see if the system is observable with the output*

$$y = x_2(t) \tag{5.3.18}$$

If we linearize about some nominal trajectory $\bar{x}$ with initial conditions $\bar{x}_{10}$, $\bar{x}_{20}$, then we get the linear perturbation equations:

$$\delta y(t) = \mathbf{C}\delta\mathbf{x}(t) \tag{5.3.19}$$

$$\delta\dot{\mathbf{x}} = \mathbf{A}(t)\delta\mathbf{x} \qquad \delta\mathbf{x}(0) = \delta\mathbf{x}_0 \tag{5.3.20}$$

* Note that the problem assumes a knowledge of α_0, which involves a linear combination of c_{A_0}, c_{B_0}. However, the value of α_0 could be known while c_{A_0} and c_{B_0} were individually unknown.

where

$$\mathbf{C} = [0 \quad 1] \tag{5.3.21}$$

and

$$\mathbf{A}(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$a_{11}(t) = -\exp\left(-\frac{1}{\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2}\right)\left[1 - \frac{\alpha_1\bar{x}_1}{(\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2)^2}\right] \tag{5.3.22}$$

$$a_{12}(t) = \frac{\alpha_2\bar{x}_1}{(\alpha_0 - \alpha_0\bar{x}_1 - \alpha_2\bar{x}_2)^2}\exp\left(-\frac{1}{\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2}\right) \tag{5.3.23}$$

$$a_{21}(t) = \exp\left(-\frac{1}{\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2}\right)\left[1 - \frac{\alpha_1\bar{x}_1}{(\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2)^2}\right] + \frac{\gamma p\alpha_1\bar{x}_2}{(\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2)^2}\exp\left(\frac{-p}{\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2}\right) \tag{5.3.24}$$

$$a_{22}(t) = \frac{-\alpha_2\bar{x}_1}{(\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2)^2}\exp\left(\frac{-1}{\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2}\right) - \gamma\exp\left(\frac{-p}{\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2}\right)\left[1 - \frac{p\alpha_2\bar{x}_2}{(\alpha_0 - \alpha_1\bar{x}_1 - \alpha_2\bar{x}_2)^2}\right] \tag{5.3.25}$$

Now since $\mathbf{C}^T\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we see that

$$\mathbf{M}(0, t_f) = \int_0^{t_f} \begin{bmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} dt \tag{5.3.26}$$

where the transition matrix $\mathbf{\Phi}(t, 0)$ is the solution of

$$\dot{\mathbf{\Phi}}(t, 0) = \mathbf{A}(t)\mathbf{\Phi}(t, 0) \qquad \mathbf{\Phi}(0, 0) = \mathbf{I} \tag{5.3.27}$$

Thus

$$\mathbf{M}(0, t_f) = \int_0^{t_f} \begin{bmatrix} \phi_{21}^2 & \phi_{21}\phi_{22} \\ \phi_{21}\phi_{22} & \phi_{22}^2 \end{bmatrix} dt \tag{5.3.28}$$

can be positive definite (the actual situation depends on the parameters and nominal trajectory) and the system could be observable.

A simpler test could be made if the nominal trajectory chosen were constant, in which case **A** would become a constant and the observability condition, Eq. (5.2.4), could be used. This requires that

$$\mathbf{L}_0 = \left[\mathbf{C}^T \,\vdots\, \mathbf{A}^T\mathbf{C}^T\right] = \begin{bmatrix} 0 & a_{21} \\ 1 & a_{22} \end{bmatrix}$$

have rank 2. For a_{21} given by Eq. (5.3.24) this condition is clearly satisfied.

Optimal Nonlinear State Estimation

There have been a large number of different estimation schemes developed for nonlinear systems (e.g., [4]), so that we shall only discuss a few here. We shall show how these results may be derived for the case of a minimum least squares objective:

$$\begin{aligned} I &= \tfrac{1}{2}\left[\mathbf{x}(0) - \mathbf{x}_0\right]^T \mathbf{P}_0^{-1}\left[\mathbf{x}(0) - \mathbf{x}_0\right] \\ &\quad + \tfrac{1}{2}\int_0^{t_f}\left\{\left[\dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, t)\right]^T \mathbf{R}(t)\left[\dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, t)\right]\right\} dt \\ &\quad + \tfrac{1}{2}\int_0^{t_f}\left\{\left[\mathbf{y}(t) - \mathbf{h}(\mathbf{x}, t)\right]^T \mathbf{Q}(t)\left[\mathbf{y}(t) - \mathbf{h}(\mathbf{x}, t)\right]\right\} dt \end{aligned} \tag{5.3.29}$$

where the weighting matrices $\mathbf{P}_0^{-1}$, $\mathbf{R}(t)$, and $\mathbf{Q}(t)$ can be chosen to reflect the errors in the initial estimate, the process model, and the measuring device.

To derive the state estimation equations for nonlinear systems, we can proceed in a formal way, as in the last section, to solve the optimal smoothing problem and then extend the results to filtering and prediction. Thus let us define

$$\mathbf{u}(t) = \dot{\mathbf{x}} - \mathbf{f}(\mathbf{x}, t) \tag{5.3.30}$$

so that the smoothing problem may be stated as determining the control $\mathbf{u}(t)$ such that the objective

$$\begin{aligned} I &= \tfrac{1}{2}\left[\mathbf{x}(0) - \mathbf{x}_0\right]^T \mathbf{P}_0^{-1}\left[\mathbf{x}(0) - \mathbf{x}_0\right] \\ &\quad + \tfrac{1}{2}\int_0^{t_f}\left\{\mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t) + \left[\mathbf{y}(t) - \mathbf{h}(\mathbf{x}, t)\right]^T \mathbf{Q}(t)\left[\mathbf{y}(t) - \mathbf{h}(\mathbf{x}, t)\right]\right\} dt \end{aligned} \tag{5.3.31}$$

is minimized subject to the constraints

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{u}(t) \qquad \mathbf{x}(0) \text{ unspecified} \tag{5.3.32}$$

Applying the maximum principle, the Hamiltonian is

$$H = \tfrac{1}{2}\left[\mathbf{u}^T\mathbf{R}\mathbf{u} + (\mathbf{y} - \mathbf{h})^T\mathbf{Q}(\mathbf{y} - \mathbf{h})\right] + \boldsymbol{\lambda}^T(\mathbf{f} + \mathbf{u}) \tag{5.3.33}$$

and the condition $\partial H/\partial \mathbf{u} = \mathbf{0}$ yields

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\boldsymbol{\lambda}(t) \tag{5.3.34}$$

where

$$\dot{\hat{\boldsymbol{\lambda}}}^T = -\frac{\partial H}{\partial \mathbf{x}} = \frac{\partial \hat{\mathbf{h}}^T}{\partial \mathbf{x}}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}}) - \frac{\partial \hat{\mathbf{f}}^T}{\partial \mathbf{x}}\hat{\boldsymbol{\lambda}} \tag{5.3.35}$$

subject to

$$\hat{\boldsymbol{\lambda}}(t_f) = \mathbf{0} \tag{5.3.36}$$

$$\hat{\mathbf{x}}(0) = \mathbf{x}_0 - \mathbf{P}_0\hat{\boldsymbol{\lambda}}(0) \tag{5.3.37}$$

Now upon substitution of Eq. (5.3.34) into Eq. (5.3.32) one obtains

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) - \mathbf{R}^{-1}(t)\hat{\boldsymbol{\lambda}}(t) \tag{5.3.38}$$

so that the nonlinear optimal smoothing problem can be found from the solution of Eqs. (5.3.35) and (5.3.36). Notice that this is an exact solution, with no approximations being necessary.

To produce the *filter* equations, let us again make use of the more explicit notation $\hat{\mathbf{x}}(t|t_f)$, $\hat{\boldsymbol{\lambda}}(t|t_f)$ denoting the optimal estimates and adjoint variables at time t, conditional on data up to time t_f. To derive the estimation equations we shall take advantage of the fact (see [20]) that there exists a decomposition of the estimation equations. If we recall that for *smoothing*, the estimates evolve by $\partial\hat{\mathbf{x}}(t|t_f)/\partial t_f$, the rate of change of the estimate at t with increasing data at time t_f, $t < t_f$, while for *prediction* we require $\partial\hat{\mathbf{x}}(t|t_f)/\partial t$, the rate of change of the estimate at time t with fixed data base, $t > t_f$, and for *filtering* the estimates are $d\hat{\mathbf{x}}(t_f|t_f)/dt_f$, the rate of change of the estimate at time t_f with both data base and estimate time changing together, it is straightforward to show that

$$\frac{d\hat{\mathbf{x}}(t_f|t_f)}{dt_f} = \left.\frac{\partial\hat{\mathbf{x}}(t|t_f)}{\partial t}\right|_{t=t_f} + \left.\frac{\partial\hat{\mathbf{x}}(t_f|T)}{\partial T}\right|_{T=t_f} \tag{5.3.39}$$

To show this, use the definition of a derivative

$$\begin{aligned}\frac{d\hat{\mathbf{x}}(t_f|t_f)}{dt_f} &= \lim_{\delta\to 0}\left[\frac{\hat{\mathbf{x}}(t_f+\delta|t_f+\delta) - \hat{\mathbf{x}}(t_f|t_f)}{\delta}\right]\\ &= \lim_{\delta\to 0}\left[\frac{\hat{\mathbf{x}}(t_f+\delta|t_f+\delta) - \hat{\mathbf{x}}(t_f|t_f+\delta)}{\delta} + \frac{\hat{\mathbf{x}}(t_f|t_f+\delta) - \hat{\mathbf{x}}(t_f|t_f)}{\delta}\right]\end{aligned} \tag{5.3.40}$$

which leads directly to Eq. (5.3.39).

We shall shortly make use of this decomposition property; however, first let us recognize that there exists a nonlinear transformation

$$\hat{\mathbf{x}}(t|t_f) = \hat{\mathbf{x}}[\boldsymbol{\lambda}(t|t_f)] \tag{5.3.41}$$

which is the solution to the two-point boundary value problem of Eqs. (5.3.35) to (5.3.38). Recall that for the linear problem, this transformation was linear and took the form of Eq. (5.2.38). However, for nonlinear systems a nonlinear transformation is required. By making use of the chain rule of calculus in Eq. (5.3.41), one obtains

$$\frac{\partial\hat{\mathbf{x}}(t|t_f)}{\partial t_f} = \frac{\partial\hat{\mathbf{x}}(t|t_f)}{\partial\hat{\boldsymbol{\lambda}}(t|t_f)}\frac{\partial\hat{\boldsymbol{\lambda}}(t|t_f)}{\partial t_f} \tag{5.3.42}$$

and one may define the matrix of "differential sensitivities"

$$\mathbf{P}(t|t_f) \equiv -\frac{\partial \hat{\mathbf{x}}(t|t_f)}{\partial \hat{\boldsymbol{\lambda}}(t|t_f)} \tag{5.3.43}$$

so that Eq. (5.3.42) becomes

$$\frac{\partial \hat{x}(t|t_f)}{\partial t_f} = -\mathbf{P}(t|t_f)\frac{\partial \hat{\boldsymbol{\lambda}}(t|t_f)}{\partial t_f} \tag{5.3.44}$$

Now making use of the decomposition property, Eq. (5.3.39), applied to $\hat{\boldsymbol{\lambda}}(t|t_f)$, one obtains

$$\frac{d\hat{\boldsymbol{\lambda}}(t_f|t_f)}{dt_f} = \left.\frac{\partial \hat{\boldsymbol{\lambda}}(t|t_f)}{\partial t}\right|_{t=t_f} + \left.\frac{\partial \hat{\boldsymbol{\lambda}}(t_f|T)}{\partial T}\right|_{T=t_f} \tag{5.3.45}$$

By considering the terminal condition, Eq. (5.3.36), on $\hat{\boldsymbol{\lambda}}(t_f|t_f)$,we see that the left-hand side of Eq. (5.3.45) must vanish. Let us now denote

$$\left.\frac{\partial \hat{\boldsymbol{\lambda}}(t|t_f)}{\partial t}\right|_{t=t_f} \equiv \hat{\boldsymbol{\lambda}}_t(t_f|t_f)$$

$$\left.\frac{\partial \hat{\boldsymbol{\lambda}}(t_f|T)}{\partial T}\right|_{T=t_f} \equiv \hat{\boldsymbol{\lambda}}_{t_f}(t_f|t_f) \tag{5.3.46}$$

with similar notation for $\hat{\mathbf{x}}(t|t_f)$ as well. Then the first term on the RHS of Eq. (5.3.45) is given by Eq. (5.3.35) with $t = t_f$. Thus

$$\hat{\boldsymbol{\lambda}}_{t_f}(t_f|t_f) = -\hat{\boldsymbol{\lambda}}_t(t_f|t_f) = -\mathbf{h}_x^T(\hat{\mathbf{x}}, t_f)\mathbf{Q}(t_f)[\mathbf{y}(t_f) - \mathbf{h}(\hat{\mathbf{x}}, t_f)] \tag{5.3.47}$$

If one substitutes Eq. (5.3.47) into Eq. (5.3.44) for $t = t_f$, one obtains

$$\hat{\mathbf{x}}_{t_f}(t_f|t_f) = -\mathbf{P}(t_f|t_f)\hat{\boldsymbol{\lambda}}_{t_f}(t_f|t_f) = \mathbf{P}(t_f|t_f)\hat{\mathbf{h}}_x^T\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}}) \tag{5.3.48}$$

and by evaluating Eq. (5.3.38) at $t = t_f$, the result

$$\hat{\mathbf{x}}_t(t_f|t_f) = \mathbf{f}(\hat{\mathbf{x}}(t_f|t_f), t_f) \tag{5.3.49}$$

is found. Now making use of the decomposition result, Eq. (5.3.39), one obtains the filter equations

$$\dot{\hat{\mathbf{x}}}(t_f|t_f) = \hat{\mathbf{x}}_t(t_f|t_f) + \hat{\mathbf{x}}_{t_f}(t_f|t_f)$$

or

$$\dot{\hat{\mathbf{x}}}(t_f|t_f) = \mathbf{f}(\hat{\mathbf{x}}, t_f) + \mathbf{P}(t_f|t_f)\mathbf{h}_x^T(\hat{\mathbf{x}}, t_f)\mathbf{Q}(t_f)[\mathbf{y}(t_f) - \mathbf{h}(\hat{\mathbf{x}}, t_f)] \tag{5.3.50}$$

Notice this has the same structure as the linear filtering equations—a predictive part coming from the modeling equations, and a "feedback" term incorporating the data. The decomposition result allows one to see these two contributions quite explicitly.

The initial condition for the filter arises by allowing $t_f \to 0$ and using Eq. (5.3.36) to see that

$$\hat{\boldsymbol{\lambda}}(0|0) = \mathbf{0} \tag{5.3.51}$$

so that Eq. (5.3.37) yields the initial filter estimate

$$\hat{\mathbf{x}}(0|0) = \mathbf{x}_0 \tag{5.3.52}$$

What remains now is to calculate the matrix of differential sensitivities $\mathbf{P}(t_f|t_f)$. If we differentiate Eq. (5.3.37) with respect to t_f, we obtain

$$\hat{\mathbf{x}}_{t_f}(0|t_f) = -\mathbf{P}_0\hat{\boldsymbol{\lambda}}_{t_f}(0|t_f) \tag{5.3.53}$$

Then letting t, $t_f \to 0$ in Eqs. (5.3.44) and (5.3.53), we see that

$$\mathbf{P}(0|0) = \mathbf{P}_0 \tag{5.3.54}$$

provides the initial value of the differential sensitivity.

To find the evolution equations for $\mathbf{P}(t_f|t_f)$, let us note that

$$\frac{\partial}{\partial t}\left[\hat{\mathbf{x}}_{t_f}(t|t_f)\right] \equiv \frac{\partial}{\partial t_f}\left[\hat{\mathbf{x}}_t(t|t_f)\right] \tag{5.3.55}$$

By evaluating the LHS using Eq. (5.3.44),

$$-\frac{\partial}{\partial t}\left[\mathbf{P}(t|t_f)\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\right] = -\mathbf{P}_t(t|t_f)\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f) - \mathbf{P}(t|t_f)\frac{\partial}{\partial t}\left[\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\right] \tag{5.3.56}$$

however,

$$\frac{\partial}{\partial t}\left[\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\right] = \frac{\partial}{\partial t_f}\left[\hat{\boldsymbol{\lambda}}_t(t|t_f)\right] = -\frac{\partial}{\partial t_f}\left[\hat{\mathbf{f}}_x^T\hat{\boldsymbol{\lambda}}(t|t_f) - \hat{\mathbf{h}}_x^T\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\right] \tag{5.3.57}$$

Now applying the chain rule of differentiation to Eq. (5.3.57),

$$\frac{\partial}{\partial t}\left[\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\right] = -\left\{\hat{\mathbf{f}}_x^T\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f) + \left[\hat{\mathbf{f}}_{xx}\hat{\boldsymbol{\lambda}}(t|t_f) + \hat{\mathbf{h}}_x^T\mathbf{Q}\hat{\mathbf{h}}_x^T - \hat{\mathbf{h}}_{xx}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\right]\hat{\mathbf{x}}_{t_f}(t|t_f)\right\} \tag{5.3.58}$$

or

$$\frac{\partial}{\partial t}\left[\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\right] = -\left\{\hat{\mathbf{f}}_x^T - \left[\hat{\mathbf{f}}_{xx}\hat{\boldsymbol{\lambda}}(t|t_f) + \hat{\mathbf{h}}_x^T\mathbf{Q}\hat{\mathbf{h}}_x - \hat{\mathbf{h}}_{xx}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\right]\mathbf{P}(t|t_f)\right\}\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f) \tag{5.3.59}$$

Thus the LHS of Eq. (5.3.55) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\left[\hat{\mathbf{x}}_{t_f}(t|t_f)\right] = -\Big(\mathbf{P}_t(t|t_f) - \mathbf{P}(t|t_f)\Big\{\hat{\mathbf{f}}_x^T - \Big[\hat{\mathbf{f}}_{xx}\hat{\boldsymbol{\lambda}}(t|t_f) + \hat{\mathbf{h}}_x^T\mathbf{Q}\hat{\mathbf{h}}_x \\ -\hat{\mathbf{h}}_{xx}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\Big]\mathbf{P}(t|t_f)\Big\}\Big)\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\end{aligned} \tag{5.3.60}$$

The right-hand side of Eq. (5.3.55) may be written

$$\begin{aligned}\frac{\partial}{\partial t_f}\left[\hat{\mathbf{x}}_t(t|t_f)\right] &= \frac{\partial}{\partial t_f}\left[\mathbf{f}(\hat{\mathbf{x}}, t) - \mathbf{R}^{-1}(t)\hat{\boldsymbol{\lambda}}(t|t_f)\right] = \hat{\mathbf{f}}_x\hat{\mathbf{x}}_{t_f}(t|t_f) - \mathbf{R}^{-1}(t)\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f) \\ &= -\left[\hat{\mathbf{f}}_x\mathbf{P}(t|t_f) + \mathbf{R}^{-1}(t)\right]\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)\end{aligned} \tag{5.3.61}$$

Upon comparing the RHS and LHS of Eq. (5.3.55), one sees that for the equation to hold for all $\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)$ the coefficients of $\hat{\boldsymbol{\lambda}}_{t_f}(t|t_f)$ must vanish, so that

$$\begin{aligned}\mathbf{P}_t(t|t_f) &= \mathbf{P}(t|t_f)\left[\hat{\mathbf{f}}_x^T - \hat{\mathbf{f}}_{xx}\hat{\boldsymbol{\lambda}}(t|t_f)\mathbf{P}(t|t_f)\right] \\ &\quad + \hat{\mathbf{f}}_x\mathbf{P}(t|t_f) - \mathbf{P}(t|t_f)\left[\hat{\mathbf{h}}_x^T\mathbf{Q}\hat{\mathbf{h}}_x - \mathbf{h}_{xx}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\right]\mathbf{P}(t|t_f) + \mathbf{R}^{-1}(t)\end{aligned} \tag{5.3.62}$$

and when $t \to t_f$, Eq. (5.3.62) becomes

$$\begin{aligned}\mathbf{P}_t(t_f|t_f) &= \mathbf{P}(t_f|t_f)\hat{\mathbf{f}}_x^T + \hat{\mathbf{f}}_x\mathbf{P}(t_f|t_f) - \mathbf{P}(t_f|t_f)\left[\hat{\mathbf{h}}_x^T\mathbf{Q}\hat{\mathbf{h}}_x - \hat{\mathbf{h}}_{xx}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\right]\mathbf{P}(t_f|t_f) \\ &\quad + \mathbf{R}^{-1}(t_f)\end{aligned} \tag{5.3.63}$$

However, remember that this is only part of the contribution because of the decomposition result:

$$\dot{\mathbf{P}}(t_f|t_f) = \mathbf{P}_t(t_f|t_f) + \mathbf{P}_{t_f}(t_f|t_f) \tag{5.3.64}$$

The first term on the RHS is given by Eq. (5.3.63), and the second term can be found by applying the chain rule to Eq. (5.3.43):

$$\mathbf{P}_{t_f}(t_f|t_f) = \frac{\partial \mathbf{P}(t_f|t_f)}{\partial \hat{\boldsymbol{\lambda}}(t_f|t_f)}\hat{\boldsymbol{\lambda}}_{t_f}(t_f|t_f) \tag{5.3.65}$$

If we define $\partial \mathbf{P}/\partial \hat{\boldsymbol{\lambda}}$ to be $\mathbf{P}'(t_f|t_f)$, then it will have the evolution equations

$$\dot{\mathbf{P}}'(t_f|t_f) = \mathbf{P}'_t(t_f|t_f) + \mathbf{P}'_{t_f}(t_f|t_f) \tag{5.3.66}$$

where

$$\mathbf{P}'_{t_f}(t_f|t_f) = \frac{\partial^2 \mathbf{P}(t_f|t_f)}{\partial \hat{\boldsymbol{\lambda}}^2(t_f|t_f)}\boldsymbol{\lambda}_{t_f}(t_f|t_f) \tag{5.3.67}$$

Thus the second terms in Eqs. (5.3.64) and (5.3.66) lead to an infinite set of upward-coupled moment equations which cannot be solved in closed form. This means that some approximation must be made to get a solution. The various nonlinear filters now in use differ only by the approximations made at this point.

The most commonly used nonlinear filter is a *first-order filter* which arises when one truncates the moments by setting $\mathbf{P}_{t_f}(t_f|t_f) = \mathbf{0}$. In this case the filter equations take the form

$$\dot{\hat{\mathbf{x}}}(t_f|t_f) = \hat{\mathbf{f}}(\hat{\mathbf{x}}, t_f) + \mathbf{P}(t_f|t_f)\hat{\mathbf{h}}_x^T\mathbf{Q}(t_f)\left[\mathbf{y}(t_f) - \mathbf{h}(\hat{\mathbf{x}}, t_f)\right], \qquad \hat{\mathbf{x}}(0|0) = \mathbf{x}_0 \tag{5.3.68}$$

and the differential sensitivities have the approximate solution

$$\begin{aligned}\dot{\mathbf{P}}(t_f|t_f) &= \mathbf{P}(t_f|t_f)\hat{\mathbf{f}}_x^T + \hat{\mathbf{f}}_x\mathbf{P}(t_f|t_f) \\ &\quad - \mathbf{P}(t_f|t_f)\left[\hat{\mathbf{h}}_x^T\mathbf{Q}\hat{\mathbf{h}}_x - \hat{\mathbf{h}}_{xx}\mathbf{Q}(\mathbf{y} - \hat{\mathbf{h}})\right]\mathbf{P}(t_f|t_f) + \mathbf{R}^{-1}(t_f) \\ &\qquad \mathbf{P}(0|0) = \mathbf{P}_0\end{aligned} \tag{5.3.69}$$

If in addition one assumes $\hat{\mathbf{h}}_{xx} = \mathbf{0}$, then Eqs. (5.3.68) and (5.3.69) represent the so-called *extended Kalman filter*.

Jazwinski [4] develops exact expressions for the estimates and covariance matrices for nonlinear problems having specific process noise and measurement error. However, these must be solved in some approximate manner as we have done here. The reader is referred to [4] for a variety of approximate filters.

One should note that, if desired, the present equations could be used to extend the solution to as high an order approximation as desired for $\mathbf{P}(t_f|t_f)$. The fact that first-order filters have been found to perform well in many applications means that these additional computations are often unnecessary.

As in the case of linear problems, the *prediction* equations only involve the properties of the model and take the form

$$\mathbf{x}_t(t|t_f) = \hat{f}(\hat{\mathbf{x}}(t|t_f), t) \qquad \begin{aligned} &\hat{\mathbf{x}}(0|0) = \mathbf{x}_0 \\ &t > t_f \end{aligned} \tag{5.3.70}$$

Let us now make some general comments about non-linear estimation.

1. The optimal smoothing and prediction problems may be solved *exactly* with no approximations.
2. The filtering problem usually involves approximations in the differential sensitivity equations to provide a reasonable solution.
3. In general, *both* the filter equations and the differential sensitivity equations must be solved on-line (in real time) because both involve the current state estimates and possibly even the data. However, further approximations will allow these differential sensitivities to be precomputed off-line. Let us illustrate the performance of a nonlinear filter with an example.

Example 5.3.2* Consider the continuous stirred tank reactor (CSTR) shown in Fig. 5.13. Only the reactor temperature can be measured; thus state estimation is used to determine the reactant concentration c_A. The modeling equations take the form

$$V\frac{dc_A}{dt'} = (c_{AF} - c_A)F - Vk_0 \exp(-E_a/RT)c_A$$

$$V\rho C_p\frac{dT}{dt'} = F\rho C_p(T_F - T) - (\Delta H)Vk_0 \exp(-E_a/RT)c_A - hA(T - T_c)$$

* This example resulted from a term project carried out by Barry Freehill as part of the graduate course in Advanced Process Control at the University of Wisconsin.

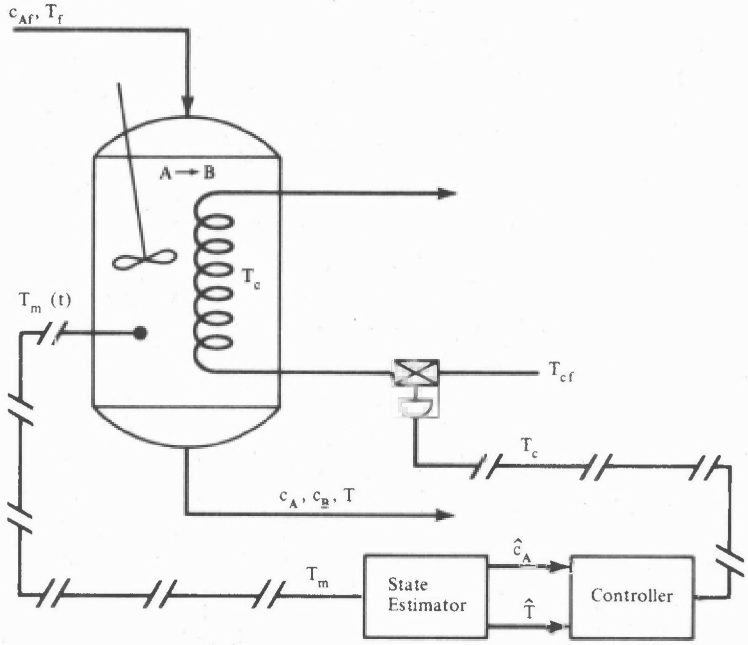


Figure 5.13 Nonlinear state estimator and controller for CSTR.

which may be put in dimensionless form by defining new variables

$$x_1 = \frac{c_{Af} - c_A}{c_{Af}} \qquad x_2 = \frac{T - T_f}{T_f}\gamma \qquad \beta = \frac{hA}{F\rho C_p} \qquad t = t'\frac{F}{V}$$

$$\gamma = \frac{E_a}{RT_f} \qquad \mathrm{Da} = \frac{k_0 e^{-\gamma} V}{F} \qquad u = \frac{T_c - T_f}{T_f}\gamma \qquad H = \frac{(-\Delta H)c_{Af}\gamma}{\rho C_p T_f}$$

Hence the model becomes

$$\frac{dx_1}{dt} = -x_1 + \mathrm{Da}(1 - x_1)\exp\left(\frac{x_2}{1 + x_2/\gamma}\right) + \xi_1(t)$$

$$\frac{dx_2}{dt} = -x_2(1 + \beta) + H\,\mathrm{Da}(1 - x_1)\exp\left(\frac{x_2}{1 + x_2/\gamma}\right) + \beta u + \xi_2(t)$$

where $\xi_1(t)$, $\xi_2(t)$ represent random process disturbances with covariance matrix $\mathbf{R}^{-1}$.

The temperature measurement is given by

$$y(t) = x_2(t) + \eta(t) = \mathbf{C}\mathbf{x} + \eta(t)$$

where $\eta(t)$ represents zero mean random measurement error having covariance Q^{-1} and the output matrix is

$$\mathbf{C} = [0 \quad 1]$$

To begin, *observability conditions* can be checked by linearizing about the steady state of interest

$$\mathbf{f}_{\mathbf{x}_s} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{ss} = \begin{bmatrix} \dfrac{-1}{1 - x_{1s}} & \dfrac{x_{1s}}{(1 + x_{2s}/\gamma)^2} \\ \dfrac{-Hx_{1s}}{1 - x_{1s}} & -(1+\beta) + \dfrac{Hx_{1s}}{(1 + x_{2s}/\gamma)^2} \end{bmatrix}$$

and testing the observability matrix

$$L_0 = \left[\mathbf{C}^T \mid \mathbf{f}_{x_s}^T \mathbf{C}^T\right] = \begin{bmatrix} 0 & \dfrac{-Hx_{1s}}{1 - x_{1s}} \\ 1 & -(1+\beta) + \dfrac{Hx_{1s}}{(1 + x_{2s}/\gamma)^2} \end{bmatrix}$$

which has rank 2 provided $Hx_{1s} \neq 0$. Thus the nonlinear system is expected to be *observable*.

The *extended Kalman filter* [Eqs. (5.3.68) and (5.3.69) with $\mathbf{h}_{xx} = \mathbf{0}$] was tested on this system by simulating the reactor on an analog computer and carrying out state estimation and feedback control from the digital computer (see Fig. 5.14). The parameters used were

$$\gamma = 13.4 \qquad H = 2.5 \qquad \text{Da} = 1.0 \qquad \beta = 0.5 \qquad Q = 2500$$

$$\mathbf{R}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.0001 \end{bmatrix} \qquad \mathbf{P}(0) = \begin{bmatrix} 1 \times 10^{-6} & 0 \\ 0 & 1.6 \times 10^{-7} \end{bmatrix}$$

The matrix $\mathbf{f}_{\mathbf{x}}$ resulting from linearization around the current estimate is

$$\hat{\mathbf{f}}_{\mathbf{x}} = \begin{bmatrix} \dfrac{\partial \hat{f}_1}{\partial x_1} & \dfrac{\partial \hat{f}_1}{\partial x_2} \\ \dfrac{\partial \hat{f}_2}{\partial x_1} & \dfrac{\partial \hat{f}_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 - \text{Da}\exp\left[\dfrac{\hat{x}_2}{1 + \dfrac{\hat{x}_2}{\gamma}}\right] & \dfrac{\text{Da}(1 - \hat{x}_1)\exp(\hat{x}_2/(1 + \hat{x}_2/\gamma))}{[1 + \hat{x}_2/\gamma]^2} \\ -H\,\text{Da}\exp\left(\dfrac{\hat{x}_2}{1 + \hat{x}_2/\gamma}\right) & -(1+\beta) + \dfrac{H\,\text{Da}(1 - \hat{x}_1)\exp(\hat{x}_2/(1 + \hat{x}_2/\gamma))}{[1 + \hat{x}_2/\gamma]^2} \end{bmatrix}$$

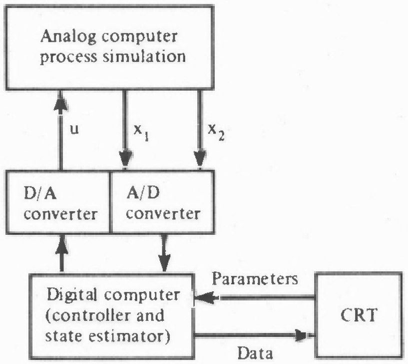


Figure 5.14 Simulation testing of the state estimation and control algorithm.

The filter equations then become

$$\dot{\hat{x}}_1 = -\hat{x}_1 + \mathrm{Da}(1 - \hat{x}_1)\exp\left(\frac{\hat{x}_2}{1 + \hat{x}_2/\gamma}\right) + P_{12}Q[y(t) - \hat{x}_2]$$

$$\dot{\hat{x}}_2 = -\hat{x}_2(1 + \beta) + H\,\mathrm{Da}(1 - \hat{x}_1)\exp\left(\frac{\hat{x}_2}{1 + \hat{x}_2/\gamma}\right) + \beta u + P_{22}Q[y(t) - \hat{x}_2]$$

while the covariance equations are

$$\dot{P}_{11} = 2\left(\frac{\partial f_1}{\partial x_1}P_{11} + \frac{\partial f_1}{\partial x_2}P_{12}\right) - Q(P_{12})^2$$

$$\dot{P}_{12} = \frac{\partial f_2}{\partial x_1}P_{11} + P_{12}\left(\frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) + P_{22}\frac{\partial f_1}{\partial x_2} - QP_{12}P_{22}$$

$$\dot{P}_{22} = 2\left(P_{12}\frac{\partial f_2}{\partial x_1} + P_{22}\frac{\partial f_2}{\partial x_2}\right) - Q(P_{22})^2 + R_{22}^{-1}$$

In this nonlinear problem both filter and covariance equations were solved on-line in real time.

The results when the state estimator is included in the feedback loop to provide estimates of conversion $\hat{x}_1$ are shown in Fig. 5.15. A single-loop PID controller

$$u = K_c\left(\epsilon + \frac{1}{\tau_I}\int \epsilon\, dt + \tau_d\frac{d\epsilon}{dt}\right)$$

was implemented. Here

$$\epsilon = x_{1d} - \hat{x}_1$$

is the deviation of the *estimated* conversion from the desired value. As can be seen in Fig. 5.15, the estimated and true values of $\hat{x}_1$ agree quite closely after a short transient, and the control loop, including the conversion

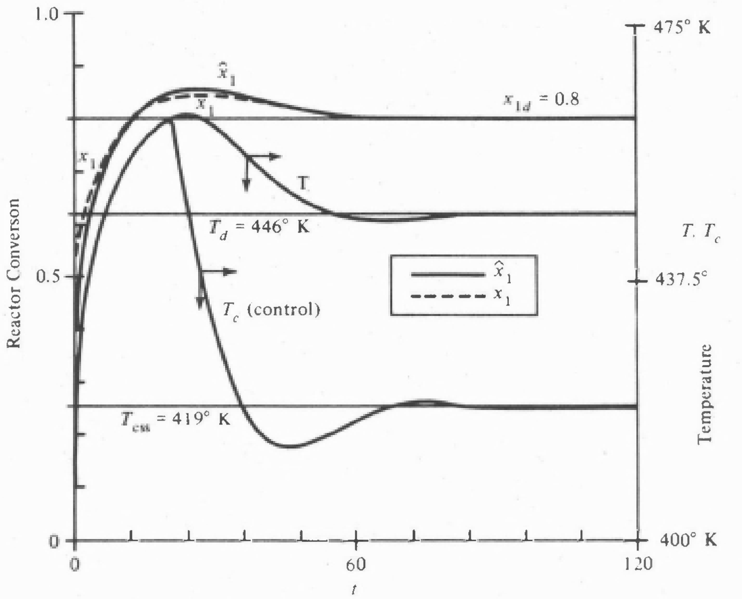


Figure 5.15 State estimator and feedback controller performance with temperature measurement only. $x_1(0)_{\text{actual}} = 0.5$; $\hat{x}_1(0) = 0.1$.

estimates, performed quite well. The variables T, T_c are plotted to show the reactor temperature and coolant temperature control action.

Nonlinear State Estimation with Discrete Time Data

For samples which are discrete in time the output device is

$$\mathbf{y}(t_k) = \mathbf{h}(\mathbf{x}(t_k), t_k) + \boldsymbol{\eta}_k \qquad k = 1, 2, \ldots \tag{5.3.71}$$

Although the estimation equation can be developed directly in this case (e.g., [4], p. 345), one may also use delta functions to obtain the equations from the continuous sampling case. Suppose in Eqs. (5.3.68) and (5.3.69) we let

$$\mathbf{Q}_k = \mathbf{Q}(t_k)\Delta t_k \tag{5.3.72}$$

and

$$\mathbf{Q}(t) = \sum_{k=1}^{M} \mathbf{Q}_k \delta(t - t_k) \tag{5.3.73}$$

where Δt_k is the sampling interval at time t_k and

$$t_f = \sum_{k=1}^{M} \Delta t_k \tag{5.3.74}$$

Then the discrete data nonlinear first-order extended Kalman filter is

$$\dot{\hat{\mathbf{x}}}(t|t_{k-1}) = \mathbf{f}(\hat{\mathbf{x}}, t) \tag{5.3.75}$$

$$\dot{\mathbf{P}}(t|t_{k-1}) = \hat{\mathbf{f}}_x\mathbf{P} + \mathbf{P}\mathbf{f}_x^T + \mathbf{R}^{-1}(t) \tag{5.3.76}$$

between observations $t_{k-1} < t < t_k$. At sampling points the updating expressions become [12]

$$\hat{\mathbf{x}}(t_k|t_k) = \hat{\mathbf{x}}(t_k|t_{k-1}) + \mathbf{K}(t_k)\left[\mathbf{y}(t_k) - \mathbf{h}(\hat{\mathbf{x}}(t_k|t_{k-1}), t_k)\right] \tag{5.3.77}$$

$$\mathbf{P}(t_k|t_k) = \mathbf{P}(t_k|t_{k-1}) - \mathbf{K}(t_k)\hat{\mathbf{h}}_x^T\mathbf{P}(t_k|t_{k-1}) \tag{5.3.78}$$

where

$$\mathbf{K}(t_k) = \mathbf{P}(t_k|t_{k-1})\hat{\mathbf{h}}_x\left[\hat{\mathbf{h}}_x\mathbf{P}(t_k|t_{k-1})\hat{\mathbf{h}}_x^T + \mathbf{Q}_k^{-1}\right]^{-1} \tag{5.3.79}$$

and $\hat{\mathbf{h}}_x$ is evaluated at $\hat{\mathbf{x}}(t_k|t_{k-1})$.

Nonlinear Observers

Just as for linear problems, observers may be constructed for nonlinear systems. However, because of the generality of the observation equations

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t)) \tag{5.3.80}$$

for nonlinear problems, the theory is not nearly so neat and compact as for linear problems. The reader should consult Refs. [21, 22] for some results in special cases.

5.4 STOCHASTIC FEEDBACK CONTROL FOR SYSTEMS DESCRIBED BY ORDINARY DIFFERENTIAL EQUATIONS

Stochastic control is concerned with the problem of controlling systems described by stochastic differential equations. For example, the system

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)\, dt + d\mathbf{v} \tag{5.4.1}$$

$$d\mathbf{w} = \mathbf{h}(\mathbf{x}, t)\, dt + d\mathbf{e} \tag{5.4.2}$$

is a set of nonlinear stochastic differential equations showing the response of the system state and output variables to random noise in the process $d\mathbf{v}$ and in the measuring device $d\mathbf{e}$. Here the variables $\mathbf{x}$, $\mathbf{w}$ are random variables and must be treated by special rules of stochastic calculus (e.g., the Ito or Stratonovich formulation). However, it is beyond the scope of this text to pursue these matters in depth, and thus we shall content ourselves with a formal treatment of the important results of interest. The interested reader should consult Refs. [4, 5, 7, 23–27] for more details of stochastic control.

The great bulk of useful results in stochastic control theory is for optimal stochastic feedback controllers [1–3, 5, 26, 27]. The basic design question is, "How should the stochastic controller differ from the deterministic controller in structure?" For linear systems, the answer is quite simple—there is usually very little difference. However, for nonlinear systems, the situation is a great deal more complex. To illustrate the linear case, let us treat the linear-quadratic problem in some detail.

For the case of additive white,* Gaussian process noise and measurement errors, and linear system and measurement equations, Eqs. (5.4.1) and (5.4.2) can be written

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu} + \boldsymbol{\xi}(t) \tag{5.4.3}$$

$$\mathbf{y}(t) = \mathbf{Cx} + \boldsymbol{\eta} \tag{5.4.4}$$

where formally the noise processes $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$ are $\boldsymbol{\eta}(t) \approx d\mathbf{e}/dt$, $\boldsymbol{\xi}(t) \approx d\mathbf{v}/dt$, and $\mathbf{y}(t) \approx d\mathbf{w}/dt$ with statistics given by Eq. (5.2.27). For linear-quadratic optimal control, the quadratic objective must be written in terms of expectations,

$$\min_{\mathbf{u}(t)} \left\{ I = \mathcal{E}\left[\tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}_f\mathbf{x}(t_f) + \tfrac{1}{2}\int_0^{t_f}(\mathbf{x}^T\mathbf{F}\mathbf{x} + \mathbf{u}^T\mathbf{E}\mathbf{u})\,dt \right] \right\} \tag{5.4.5}$$

Now if $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$ are independent, zero-mean, white-noise processes, then they contribute no bias into the control law, and one can show [1–3, 5, 23–27] that the *stochastic* feedback control law is the same as the *deterministic* feedback control law, i.e.,

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t) \tag{3.2.81}$$

where

$$\mathbf{K}(t) = \mathbf{E}^{-1}\mathbf{B}^T\mathbf{S}(t) \tag{3.2.82}$$

$$\frac{d\mathbf{S}}{dt} = -\mathbf{SA} - \mathbf{A}^T\mathbf{S} + \mathbf{SBE}^{-1}\mathbf{B}^T\mathbf{S} - \mathbf{F} \qquad \mathbf{S}(t_f) = \mathbf{S}_f \tag{3.2.80}$$

Thus if complete state measurements are available, $\mathbf{C} = \mathbf{I}$, the optimal stochastic controller has the same structure as the deterministic controller.

In the event that there is incomplete state measurement, then the optimal stochastic feedback controller must be coupled in some way to a state estimator. Thus, if the initial state $\mathbf{x}(0)$ is uncorrelated with $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$ and is Gaussian with mean and covariance given by

$$\mathcal{E}(\mathbf{x}_0) = \hat{\mathbf{x}}_0 \tag{5.4.6}$$

$$\mathcal{E}\left((\mathbf{x}(0) - \hat{\mathbf{x}}_0)(\mathbf{x}(0) - \hat{\mathbf{x}}_0)^T\right) = \mathbf{P}_0 \tag{5.4.7}$$

then the optimal stochastic feedback controller is

$$\mathbf{u} = -\mathbf{K}(t)\hat{\mathbf{x}}(t) \tag{5.4.8}$$

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{Bu} + \mathbf{PC}^T\mathbf{Q}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{5.4.9}$$

where $\mathbf{K}(t)$ is given by Eqs. (3.3.82) and (3.3.80) and

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{PA}^T + \mathbf{AP} - \mathbf{PC}^T\mathbf{QCP} + \mathbf{R}^{-1} \qquad \mathbf{P}(0) = \mathbf{P}_0 \tag{5.4.10}$$

Note that one simply uses the optimal estimates in the deterministic feedback control law. This *separation principle* or *certainty-equivalence principle* is a powerful result which means that the optimal controller and optimal estimator may be

* "White" noise contains all frequencies and is therefore totally uncorrelated in time.

completely separated in structure. Such separation principles apply to a rather broad class of linear systems [27], *but do not apply in general to nonlinear systems*. Thus the very neat theory for Gaussian, white-noise, linear systems is greatly complicated if:

1. The process is nonlinear.
2. The noise is colored (i.e., correlated in time) or non-Gaussian.

We shall not pursue these more complex cases here, but shall rely on certain approximations to allow suboptimal but acceptable controller designs for nonlinear stochastic processes. A practical discussion of these points may be found in [28].

Let us illustrate the results for linear systems with the following example problems.

Example 5.4.1 Let us consider the isothermal CSTR of Examples 3.2.5 and 5.2.4, in which the reaction $A \rightarrow B \rightarrow C$ is taking place. Let us assume that only x_2 (species B) is measured and that u_1 (the feed concentration of A) is the only control variable allowed. The modeling equations take the form

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 + u_1 + \xi_1(t) \qquad x_1(0) = x_{10} + \xi_{10} \tag{5.4.11}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + \xi_2(t) \qquad x_2(0) = x_{20} + \xi_{20} \tag{5.4.12}$$

$$y(t) = x_2(t) + \eta(t) \tag{5.4.13}$$

Now we know from Example 3.2.6 that the system is *completely controllable* with only $u_1(t)$ as a control. Similarly, in Example 5.2.2, we showed that the system is *completely observable* with only $x_2(t)$ as an output.

Now if we assume $\xi_1(t)$, $\xi_2(t)$, $\eta(t)$ are zero-mean, independent, Gaussian, white-noise processes and ξ_{10}, ξ_{20} are zero-mean, independent, Gaussian random variables, then the *separation theorem* applies and the optimal stochastic feedback controller can be broken into two parts: (1) a deterministic linear-quadratic feedback control law, and (2) an optimal least squares estimator for providing the state estimates. This structure may be seen in Fig. 5.16.

Therefore, if we wish to minimize the control objective

$$I_c = \mathcal{E}\left\{\frac{1}{2}\int_0^{t_f}\left[\mathbf{x}^T(t)\mathbf{F}\mathbf{x}(t) + Eu_1(t)^2\right] dt\right\} \tag{5.4.14}$$

then the optimal stochastic feedback control law is

$$u_1(t) = -\mathbf{K}(t)\hat{\mathbf{x}}(t) = -K_1(t)\hat{x}_1(t) - K_2(t)\hat{x}_2(t) \tag{5.4.15}$$

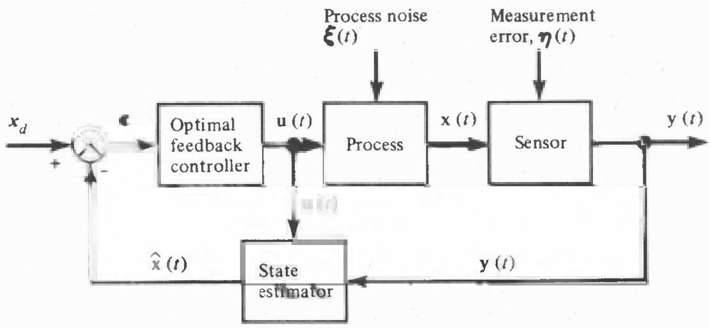


Figure 5.16 Structure of optimal stochastic controller.

where

$$\mathbf{K} = E^{-1}[S_{11}(t), S_{12}(t)] \tag{5.4.16}$$

or

$$K_1(t) = \frac{S_{11}(t)}{E} \qquad K_2(t) = \frac{S_{12}(t)}{E} \tag{5.4.17}$$

The parameters S_{ij} come from the solution to the Riccati equations

$$\dot{S}_{11} = 2S_{11}(1 + \mathrm{Da}_1) - 2S_{12}\,\mathrm{Da}_1 + \frac{1}{E}S_{11}^2 - F_{11} \qquad S_{11}(t_f) = 0 \tag{5.4.18}$$

$$\begin{aligned}\dot{S}_{12} &= S_{12}(2 + \mathrm{Da}_1 + \mathrm{Da}_3) - \mathrm{Da}_1 S_{22}\\ &\quad + \frac{1}{E}S_{11}S_{12} - F_{12} \qquad S_{12}(t_f) = 0\end{aligned} \tag{5.4.19}$$

$$\dot{S}_{22} = 2S_{22}(1 + \mathrm{Da}_3) + \frac{1}{E}S_{12}^2 - F_{22} \qquad S_{22}(t_f) = 0 \tag{5.4.20}$$

Thus one may precompute Eqs. (5.4.18) to (5.4.20) in order to obtain the optimal feedback controller gains.

The optimal estimates $\hat{x}_1(t)$, $\hat{x}_2(t)$ needed by the controller come from the filter equations, which are only slightly modified from Example 5.2.4, i.e.,

$$\dot{\hat{x}}_1 = -(1 + \mathrm{Da}_1)\hat{x}_1 + u_1(t) + P_{12}(t)Q(t)(y - \hat{x}_2) \tag{5.4.21}$$

$$\dot{\hat{x}}_2 = \mathrm{Da}_1\hat{x}_1 - (1 + \mathrm{Da}_3)\hat{x}_2 + P_{22}(t)Q(t)(y - \hat{x}_2) \tag{5.4.22}$$

where $u_1(t)$ is given by Eq. (5.4.15) and the estimate covariance may be precomputed from Eq. (5.2.80).

It is possible to combine state estimation and feedback control to produce proportional plus integral control. As an example [29], consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{v} \tag{5.4.23}$$

where v is a vector of "environmental disturbance" variables given by

$$\dot{\mathbf{v}} = \xi(t) \tag{5.4.24}$$

Here $\xi(t)$ represents some random disturbance. The system outputs are

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \eta \tag{5.4.25}$$

Now the steady-state linear-quadratic controller for the desired steady state $\mathbf{x}_d = \mathbf{0}$, $\mathbf{u}_d = \mathbf{0}$ becomes

$$\mathbf{u} = -\mathbf{K}_1\hat{\mathbf{x}} - \mathbf{K}_2\hat{\mathbf{v}} \tag{5.4.26}$$

and the state estimates of $\mathbf{x}$ and $\mathbf{v}$ are given by

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{D}\hat{\mathbf{v}} + \mathbf{K}_{e1}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{5.4.27}$$

$$\dot{\hat{\mathbf{v}}} = \mathbf{K}_{e2}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{5.4.28}$$

where $\mathbf{K}_{e1}$, $\mathbf{K}_{e2}$ become constants for long estimation time.

Now substitution of Eq. (5.4.28) into Eq. (5.4.26) yields the optimal, proportional plus integral control law

$$\mathbf{u} = -\mathbf{K}_1\hat{\mathbf{x}} + \mathbf{K}_2\mathbf{K}_{e2}\mathbf{C}\int_0^t \hat{\mathbf{x}}\,dt - \mathbf{K}_2\mathbf{K}_{e2}\int_0^t \mathbf{y}(t)\,dt \tag{5.4.29}$$

Let us illustrate this with an example.

Example 5.4.2 Let us consider the heating of a fluid in a stirred heating tank. The inlet temperature T_f is raised to T at the exit by heat transfer with a steam jacket at temperature T_w. A heat balance in the process yields

$$\rho C_p V \frac{dT}{dt'} = F\rho C_p(T_f - T) - hA(T - T_w) \tag{5.4.30}$$

Now suppose we let

$$x = T \qquad u = T_w \qquad v = T_f \qquad t = \frac{t'}{\theta} \qquad \alpha = \frac{hA}{F\rho C_p} \qquad \theta = \frac{V}{F}$$

$$\frac{dx}{dt} = -(1 + \alpha)x + v + \alpha u \tag{5.4.31}$$

where v, the inlet temperature, is assumed subject to disturbances given by

$$\frac{dv}{dt} = \xi(t) \tag{5.4.32}$$

where $\xi(t)$ represents some type of disturbance process. Let us assume that only $x(t)$ can be measured; thus

$$y = x + \eta \tag{5.4.33}$$

is the output device. Now the steady-state, linear-quadratic, optimal feedback control law is

$$u = u_d - K_1(\hat{x} - x_d) - K_2\hat{v} \tag{5.4.34}$$

where x_d, u_d represent the steady-state controller set point on x and u.

The state estimation equations are

$$\dot{\hat{x}} = -(1+\alpha)\hat{x} + \alpha u + \hat{v} + K_{e1}(y - \hat{x}) \tag{5.4.35}$$

$$\dot{\hat{v}} = K_{e2}(y - \hat{x}) \tag{5.4.36}$$

This leads immediately to the proportional plus integral controller

$$u(t) = u_d - K_1(\hat{x} - x_d) - K_2 K_{e2} \int_0^t (y - \hat{x})\, dt' \tag{5.4.37}$$

Note that the second integral term is designed to compensate for disturbances v which are estimated by the filter.

5.5 STATE ESTIMATION FOR SYSTEMS DESCRIBED BY FIRST-ORDER HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Let us consider the general class of systems described by

$$\frac{\partial \mathbf{x}(z,t)}{\partial t} = \mathbf{A}_1 \frac{\partial \mathbf{x}}{\partial z} + \mathbf{f}(\mathbf{x}(z,t), \mathbf{u}(z,t)) + \boldsymbol{\xi}(z,t) \tag{5.5.1}$$

$$\mathbf{x}(0,t) = \mathbf{B}_0 \mathbf{u}_0(t) \tag{5.5.2}$$

$$\mathbf{y}(t) = \mathbf{h}\big(\mathbf{x}(z_1^*, t), \mathbf{x}(z_2^*, t), \ldots, \quad \mathbf{x}(z_\gamma^*, t)\big) + \boldsymbol{\eta}(t) \tag{5.5.3}$$

where $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are additive, zero-mean noise processes. This is a very general class of equations, encompassing both linear and nonlinear systems and even systems with pure time delays. In what follows we shall consider conditions for *observability* and then discuss state estimation algorithms.

Observability

General conditions for *observability* for first-order hyperbolic systems have been discussed by Goodson and Klein [30], Yu and Seinfeld [11], and Thowsen and Perkins [31] and essentially require that every characteristic line intersect a measuring device and that a lumped parameter observability condition must be satisfied along these characteristic lines. To see this, consider the observation paths $w_i(t)$ in Fig. 5.17 beginning at $z = 1$ and moving across all characteristic lines in the domain R_α. Then we define outputs corresponding to the intersection of the characteristic lines and the observation paths:

$$\mathbf{y}_i(t) = \mathbf{C}_i \mathbf{x}(z,t)\big|_{z=w_i(t)} \qquad i = 1, 2, \ldots, \gamma$$

Finally one can apply a lumped parameter observability condition along the characteristics.

For illustration, we consider constant-coefficient linear systems having fixed sensors and only one set of characteristic lines. Equations (5.5.1) to (5.5.3) take

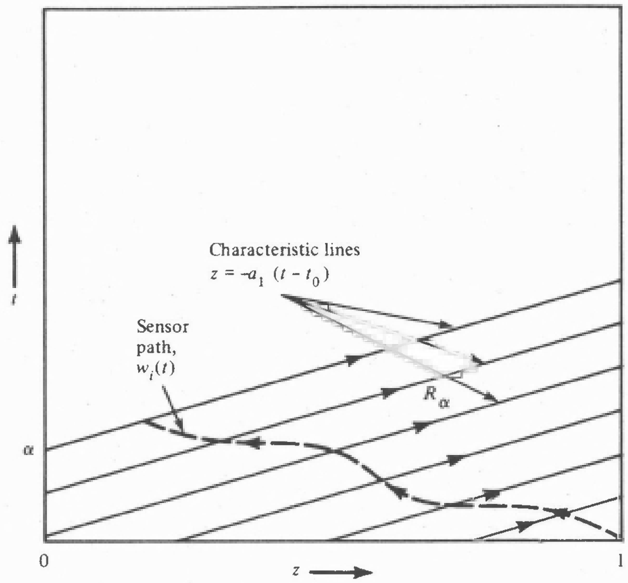


Figure 5.17 Characteristic lines and sensor path for first-order hyperbolic systems.

the form (where we suppress the controller for the moment)

$$\frac{\partial \mathbf{x}(z,t)}{\partial t} = a_1 \frac{\partial \mathbf{x}}{\partial z} + \mathbf{A}_0\mathbf{x} + \xi(z,t) \tag{5.5.4}$$

$$\mathbf{y}(t) = \sum_{i=1}^{\Upsilon} \mathbf{C}_i \mathbf{x}(z_i^*, t) + \eta(t) \tag{5.5.5}$$

where the solution takes the form

$$\left.\frac{d\mathbf{x}}{dt}\right|_0 = \mathbf{A}_0\mathbf{x} + \xi \tag{5.5.6}$$

along the characteristic lines

$$z = -a_1(t - t_0) \tag{5.5.7}$$

where $a_1 < 0$ in general for flow processes. Now for a *fixed* sensor to intersect each characteristic line, it must be located at $z = 1$ (see Fig. 5.17); otherwise the portion of the initial conditions between any sensor location z_i^* and $z = 1$ *are not observable.*

Now applying the lumped parameter observability conditions along the characteristics, one has the following conditions for observability.

In order for the system [Eqs. (5.5.4) and (5.5.5)] to be completely observable, it is necessary and sufficient that each characteristic line intersect a sensor and that the matrix

$$\mathbf{L}_{od} = \left[\mathbf{\Phi}(t_1, 0)^T\mathbf{C}_1^T \vdots \ldots \vdots \mathbf{\Phi}(t_\Upsilon, 0)^T\mathbf{C}_\Upsilon^T\right] \tag{5.5.8}$$

have rank n.

Here t_i is the time at which the particular characteristic line in question intersects the sensor at z_i^*. Thus

$$t_i = t_0 - z_i^*/a_1 \qquad i = 1, 2, \ldots, \gamma \tag{5.5.9}$$

The $n \times n$ matrix $\mathbf{\Phi}(t, 0)$ is the state transition matrix for the matrix $\mathbf{A}_0$.

Conditions for a wider class of first-order hyperbolic processes may be found in [11, 31].

Let us illustrate this result with an example.

Example 5.5.1 Let us consider the heat exchanger of Example 4.2.1 having fixed temperature sensors at

$$z_1^* = 0.25$$
$$z_2^* = 0.5$$
$$z_3^* = 0.75$$
$$z_4^* = 1.0$$

Recall that the modeling equations are

$$\frac{\partial x}{\partial t}(z, t) = -v\frac{\partial x}{\partial z} + a_0(x - u) \qquad x(0, t) = 0 \tag{4.2.25}$$

so that along the characteristic lines

$$t = t_0 + z/v \tag{5.5.10}$$

the solution is

$$\left.\frac{dx}{dt}\right|_0 = a_0 x - a_0 u \tag{5.5.11}$$

and

$$\Phi(t, 0) = e^{a_0 t} \tag{5.5.12}$$

with measuring device

$$\mathbf{y}(t) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x(0.25, t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x(0.5, t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x(0.75, t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x(1, t) \tag{5.5.13}$$

Thus the observability matrix becomes

$$\mathbf{L}_{od} = \left[\, e^{a_0 t_1} 000 \,\vdots\, 0 e^{a_0 t_2} 00 \,\vdots\, 00 e^{a_0 t_3} 0 \,\vdots\, 000 e^{a_0 t_4} \right] \tag{5.5.14}$$

which clearly has rank 1, and the system is observable.

Note that only z_4^* intersects all the characteristics, so the system is not *observable* without the outlet temperature sensor. If only z_2^* were used, for example, the shaded area in Fig. 5.18 bounded by $z = 1$, $t = 0$, and $t = (z - 0.5)/v$ would not be observable.

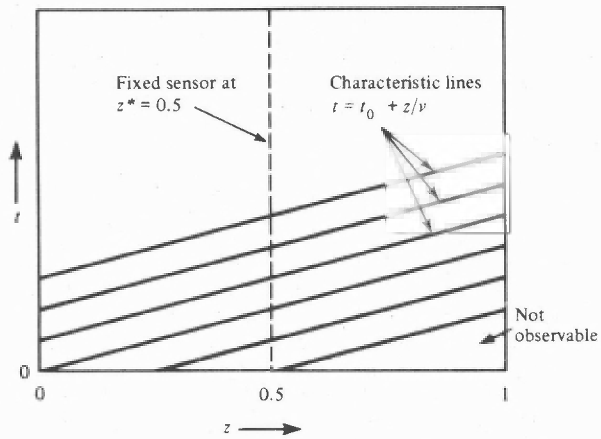


Figure 5.18 Characteristic lines and fixed temperature sensor at $z^* = 0.5$ for plug flow heat exchanger.

Note that from a practical feedback control point of view, the fixed sensor required for complete observability (i.e., at $z = 1$) is not very useful because one cannot take corrective control action. Thus state estimation of first-order hyperbolic systems is often directed at identifying inlet disturbances rather than at identifying the complete initial state. Therefore, partial observation of the process, concentrating on the section near the entrance (where identified disturbances can be compensated for by control action before the exit is reached), is often the practical goal. In this case sensors would be grouped toward the entrance, with perhaps only one sensor at $z = 1$ to allow a final check on the outlet condition (and satisfy the observability condition).

One should note that any time delay system can be represented by equations of the form of Eqs. (5.5.1) to (5.5.3). Explicit conditions for observability of time delay systems are discussed in [32].

State Estimation

Very general optimal state estimators for first-order partial differential equations have been developed and include a wide range of time delay and other hereditary systems as special cases [33]. In this section, however, we shall only present the results for systems of the form of Eqs. (5.5.1) to (5.5.3). The method of derivation follows closely the approach of Sec. 5.3 but requires considerably more algebra; therefore, we shall only present the results here and refer the reader to [33] for the derivation details.

The sequential state estimation algorithm which minimizes

$$I = \frac{1}{2}\int_0^{t_f}\left\{(\mathbf{y}-\mathbf{h})^T\mathbf{Q}(t)(\mathbf{y}-\mathbf{h})\,dt + \frac{1}{2}\int_0^{t}\int_0^{1}\int_0^{1}\left[\mathbf{x}_t(r,t) - \mathbf{A}_1\mathbf{x}_r(r,t) - \mathbf{f}\right]^T\right.$$

$$\left.\times \mathbf{R}(r,s,t)\left[\mathbf{x}_t(s,t) - \mathbf{A}_1\mathbf{x}_s(s,t) - \mathbf{f}\right]dt\,dr\,ds\right\} \tag{5.5.15}$$

is given by

$$\frac{\partial\hat{\mathbf{x}}(r,t)}{\partial t} = \mathbf{A}_1\frac{\partial\mathbf{x}}{\partial r}(r,t) + \hat{\mathbf{f}}(\hat{\mathbf{x}}(r,t),\mathbf{u}(r,t))$$

$$+\sum_{i=1}^{\gamma}\mathbf{P}(r,r_i^*,t)\frac{\partial\hat{\mathbf{h}}}{\partial\hat{\mathbf{x}}(r_i^*,t)}\mathbf{Q}(t)\left\{\mathbf{y} - \hat{\mathbf{h}}\left[\hat{\mathbf{x}}(r_1^*,t),\ldots,\hat{\mathbf{x}}(r_\gamma^*,t)\right]\right\} \tag{5.5.16}$$

with boundary condition

$$\hat{\mathbf{x}}(0,t) = \mathbf{B}_0\mathbf{u}_0(t) \tag{5.5.17}$$

and initial condition

$$\hat{\mathbf{x}}(r,0) = \hat{\mathbf{x}}_0(r)$$

The $n \times n$ differential sensitivity matrix $\mathbf{P}(r,s,t)$ is the solution of

$$\mathbf{P}_t(r,s,t) = \hat{\mathbf{f}}_x\mathbf{P} + \mathbf{P}\hat{\mathbf{f}}_x^T + \mathbf{A}_1\mathbf{P}_r + \mathbf{P}_s\mathbf{A}_1^T + \sum_{i=1}^{\gamma}\sum_{j=1}^{\gamma}\mathbf{P}(r,r_i^*,t)$$

$$\times\frac{\partial\left\{\left[\partial\hat{\mathbf{h}}/\partial\hat{\mathbf{x}}(r_i^*,t)\right]\mathbf{Q}(t)(\mathbf{y}-\hat{\mathbf{h}})\right\}}{\partial\hat{\mathbf{x}}(r_j^*,t)}\mathbf{P}(r_j^*,s,t) + \mathbf{R}_1^+(r,s,t) \tag{5.5.18}$$

$$\mathbf{P}(r,0,t) = \mathbf{P}(0,s,t) = \mathbf{0} \tag{5.5.19}$$

$$\mathbf{P}(r,s,0) = \mathbf{P}_0(r,s) \tag{5.5.20}$$

where for linear problems having white, Gaussian noise, $\mathbf{P}_0$ is the error covariance of the initial condition, $\mathbf{Q}^{-1}(t)$ is the covariance of the measurement error, and $\mathbf{R}_1^+(r,s,t)$ is the covariance of the process noise. For nonlinear problems, $\mathbf{P}_0$, $\mathbf{Q}^{-1}$, and $\mathbf{R}_1^+$ have no strict statistical significance and may be considered filter tuning parameters. Furthermore, $\mathbf{P}$ has no statistical significance and usually must be computed on-line in real time unless approximations are made. By contrast, for linear systems with white, Gaussian noise processes, the differential sensitivities $\mathbf{P}$ become true estimate covariances and may be precomputed off-line.

General results are also available for data taken at discrete time intervals; however, the reader should consult Ref. [34] for details. Also, work on observers for this class of systems has been reported [35].

Let us illustrate the application of this filtering result by considering an example problem.

Example 5.5.2 Let us consider the heat exchanger of the previous example where it is desired to estimate the temperature profile based on four fixed thermocouple sensors. Substituting Eqs. (4.2.25) and (5.5.13) into Eqs. (5.5.16) to (5.5.20), one obtains the estimation equations:

$$\frac{\partial \hat{x}(z,t)}{\partial t} = -v\frac{\partial \hat{x}(z,t)}{\partial z} + a_0(\hat{x} - u)$$
$$+ \sum_{i=1}^{4} P(r, r_i^*, t)\mathbf{C}_i^T\mathbf{Q}(t)\left[\mathbf{y} - \sum_{j=1}^{4} \mathbf{C}_j\mathbf{x}(r_j^*, t)\right] \tag{5.5.21}$$

$$\hat{x}(0,t) = 0 \qquad \hat{x}(r,0) = x_0(r) \tag{5.5.22}$$

$$P_t(r,s,t) = 2a_0P - v[P_r + P_s] + R_1^+(r,s,t)$$
$$- \sum_{i=1}^{4}\sum_{j=1}^{4} P(r, r_i^*, t)\mathbf{C}_i^T\mathbf{Q}(t)\mathbf{C}_jP(r_j^*, s, t) \tag{5.5.23}$$

$$P(r,0,t) = P(0,s,t) = 0 \qquad P(r,s,0) = P_0(r,s) \tag{5.5.24}$$

where $\mathbf{Q}^{-1}(t)$ is a 4×4 matrix of measurement error covariances, R_1^+ is a scalar process noise covariance, $P_0(r,s)$ is a covariance of the error in the initial state, and

$$\mathbf{C}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \mathbf{C}_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad \mathbf{C}_3 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \mathbf{C}_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{5.5.25}$$

This is a linear problem, and if we assume the measurement and process noise to be white and Gaussian, then $P(r,s,t)$ is the distributed covariance of the error estimates, defined by

$$P(r,s,t) = \mathcal{E}\{[\hat{x}(r,t) - x(r,t)][\hat{x}(s,t) - x(s,t)]\} \tag{5.5.26}$$

and can be precomputed off-line. Only the estimator equation, (5.5.21), must be solved on-line in real time.

5.6 STATE ESTIMATION FOR SYSTEMS DESCRIBED BY SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

By far the most common type of model arising from heat, mass, and momentum balances is the second-order partial differential equation. These equations occur in heat conduction, diffusion, and pressure-driven flow processes. For these reasons, state estimation applied to this class of problems has particular practical importance.

The class of systems we shall consider explicitly in this chapter take the form

$$\frac{\partial \mathbf{x}(z,t)}{\partial t} = \mathbf{f}[\mathbf{x}_{zz}, \mathbf{x}_z, \mathbf{x}(z,t), \mathbf{u}(z,t)] + \boldsymbol{\xi}(z,t) \tag{5.6.1}$$

with boundary conditions

$$\mathbf{b}_0[\mathbf{x}_z, \mathbf{x}, \mathbf{u}_0(t)] + \boldsymbol{\xi}_0(t) = \mathbf{0} \qquad z = 0 \tag{5.6.2}$$

$$\mathbf{b}_1[\mathbf{x}_z, \mathbf{x}, \mathbf{u}_1(t)] + \boldsymbol{\xi}_1(t) = \mathbf{0} \qquad z = 1 \tag{5.6.3}$$

These equations are capable of representing most one-dimensional* PDE systems of process control interest. The measurements can be considered either continuous in time,

$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(z_1^*, t), \mathbf{x}(z_2^*, t), \ldots, \mathbf{x}(z_\gamma^*, t)] + \boldsymbol{\eta}(t) \tag{5.6.4}$$

or discrete in time,

$$\mathbf{y}(t_k) = \mathbf{h}[\mathbf{x}(z_1^*, t_k), \mathbf{x}(z_2^*, t_k), \ldots, \mathbf{x}(z_\gamma^*, t_k)] + \boldsymbol{\eta}(t_k) \qquad k = 1, 2, \ldots \tag{5.6.5}$$

Equations (5.6.1) to (5.6.5) are defined for $t \geq 0$ and $0 < z < 1$, $\mathbf{x}(z, t)$ is an n vector, $\mathbf{y}$ is an l vector, while $\mathbf{b}_0$ and $\mathbf{b}_1$ are l_0 and l_1 vector functions, respectively. The functions $\boldsymbol{\xi}(z, t)$, $\boldsymbol{\xi}_0(t)$, $\boldsymbol{\xi}_1(t)$, $\boldsymbol{\eta}(t)$, $\boldsymbol{\eta}(t_k)$ represent zero-mean random processes with arbitrary statistical properties. The points $0 \leq z_1^* \leq z_2^* \ldots \leq z_\gamma^* \leq 1$ represent the sensor locations. The initial condition, $\mathbf{x}(z, 0)$ is generally unknown.

Before proceeding to state estimation algorithms, we shall consider the question of observability and the choice of sensor locations.

Observability and the Choice of Sensor Location

Just as in lumped parameter systems, conditions for observability are only well developed for linear systems and are usually "approximate observability" conditions which allow observation of the state to within an arbitrarily small error. In practice, this means that one uses modal analysis to analyze the system and observability conditions are stated in terms of the system eigenfunctions [30, 36–41]. In addition, just as the *controllability conditions* discussed in Chap. 4 were dependent on actuator position, the observability conditions depend strongly on the choice of sensor location.

To illustrate these points, let us consider the problem of the long, thin rod heated along its axis and having modeling equations

$$\frac{\partial x}{\partial t}(z, t) = \frac{\partial^2 x(z, t)}{\partial z^2} + u(z, t) \tag{5.6.6}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = 0 \tag{5.6.7}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{5.6.8}$$

* Problems involving more space dimensions require straightforward extensions of these estimation equations. An illustrative example is given in Chap. 6.

Further, let us assume a measuring device of the form

$$y_i(t) = x(z_i^*, t) \qquad i = 1, 2, \ldots, l \tag{5.6.9}$$

i.e., there are thermocouple measurements at a finite number of discrete points. Now the question is, "*How many measurements are needed and at what location are measurements required to ensure observability?*" Let us begin with a single measurement and test for observability.

From Sec. 4.2 we recall that the solution to Eqs. (5.6.6) to (5.6.8) takes the form

$$x(z, t) = a_0(t) + \sqrt{2} \sum_{n=1}^{N} a_n(t) \cos n\pi z \tag{5.6.10}$$

$$u(z, t) = b_0(t) + \sqrt{2} \sum_{n=1}^{N} b_n(t) \cos n\pi z \tag{5.6.11}$$

where

$$\frac{da_n(t)}{dt} = -n^2\pi^2 a_n + b_n(t) \qquad n = 0, 1, 2, \ldots, N \tag{5.6.12}$$

and

$$y_i(t) = a_0(t) + \sqrt{2} \sum_{n=1}^{N} a_n(t) \cos n\pi z_i^* \qquad i = 1, 2, \ldots, l \tag{5.6.13}$$

If we define the $N + 1$ vector $\mathbf{w}(t)$, $(N + 1) \times (N + 1)$ matrix $\mathbf{A}$, and $N + 1$ vector $\mathbf{v}$ as

$$\mathbf{w} = \begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & -\pi^2 & \cdots & \cdots & 0 \\ \cdots & \cdots & -4\pi^2 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & -N^2\pi^2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_0(t) \\ b_1(t) \\ \vdots \\ b_N(t) \end{bmatrix} \tag{5.6.14}$$

the system takes the form

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{v} \tag{5.6.15}$$

with output given by

$$\mathbf{y} = \mathbf{C}\mathbf{w} \tag{5.6.16}$$

where the $l \times (N + 1)$ matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{bmatrix} 1 & \sqrt{2}\cos \pi z_1^* & \sqrt{2}\cos 2\pi z_1^* & \cdots & \sqrt{2}\cos N\pi z_1^* \\ 1 & \sqrt{2}\cos \pi z_2^* & \sqrt{2}\cos 2\pi z_2^* & \cdots & \sqrt{2}\cos N\pi z_2^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \sqrt{2}\cos \pi z_l^* & \sqrt{2}\cos 2\pi z_l^* & \cdots & \sqrt{2}\cos N\pi z_l^* \end{bmatrix} \tag{5.6.17}$$

Now application of the approximate N mode observability conditions yields the condition that the $(N+1) \times (N+1)$ observability matrix $\mathbf{L}_0$ should have rank $N+1$, where

$$\mathbf{L}_0 = \left[\mathbf{C}^T \,\vdots\, \mathbf{A}^T\mathbf{C}^T \,\vdots\, \ldots \,\vdots\, (\mathbf{A}^T)^N \mathbf{C}^T\right] \tag{5.6.18}$$

For the case of only a *single sensor* z_1^*, the observability condition takes the form

$$\mathbf{L}_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \sqrt{2}\cos \pi z_1^* & -\pi^2\sqrt{2}\cos \pi z_1^* & \cdots & (-1)^N \pi^{2N}\sqrt{2}\cos \pi z_1^* \\ \sqrt{2}\cos 2\pi z_1^* & -4\pi^2\sqrt{2}\cos 2\pi z_1^* & \cdots & (-1)^N (4\pi^2)^N \sqrt{2}\cos 2\pi z_1^* \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{2}\cos N\pi z_1^* & -N^2\pi^2\sqrt{2}\cos N\pi z_1^* & \cdots & (-1)^N (N^2\pi^2)^N \sqrt{2}\cos N\pi z_1^* \end{bmatrix} \tag{5.6.19}$$

Thus $\mathbf{L}_0$ will have rank $N+1$ if and only if z_1^* is selected so that none of the eigenfunctions vanish, i.e.,

$$\cos n\pi z_1^* \neq 0 \qquad \text{for all } n = 1, 2, \ldots, N \tag{5.6.20}$$

This is a fairly severe restriction; for example, in the event that

$$z_1^* = \begin{cases} \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots, & \dfrac{1}{2N} \\ \frac{3}{4}, \frac{3}{8}, \frac{3}{10}, \ldots, & \dfrac{3}{2N} \\ \frac{5}{6}, \frac{5}{8}, \frac{5}{12}, \ldots, & \dfrac{5}{2N} \\ \vdots & \end{cases}$$

the system is *not observable*. In fact, Goodson and Klein [30] show that for increasing N the zeros of the eigenfunction move closer and closer to the external boundaries, as shown in Fig. 5.19. Thus one must place measurements to avoid these zeros. As N increases, the zeros within the envelope become more and more dense. There is, however, one significant advantage to this structure: *the surface measurement locations, which are the easiest to realize practically, always guarantee observability*. This is a consequence of the cosine eigenfunctions which arise because of the form of the boundary conditions, Eqs. (5.6.7) and (5.6.8). If instead the surface temperatures had been specified, the system would be unobservable at the external surface; however, a boundary heat flux measurement in this case would again make the system observable on the boundary.

In summary, we have the remarkable result that the heated rod is observable even with only a single temperature measurement if we avoid placing this sensor at the zeros of the eigenfunctions.

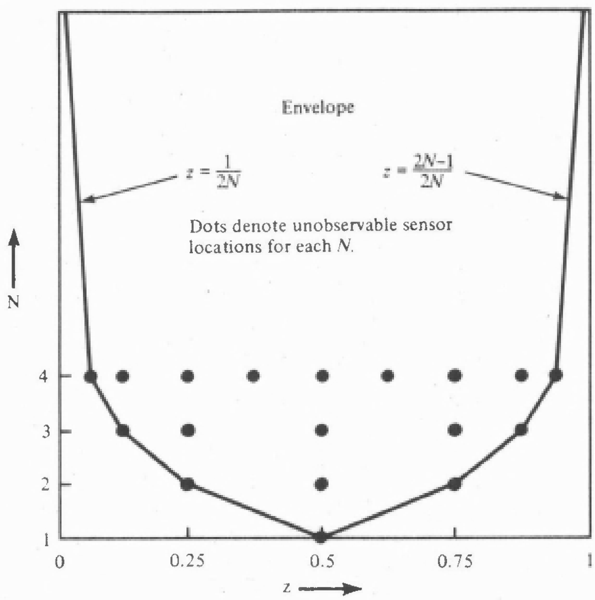


Figure 5.19 Influence of sensor location on observability of one-dimensional heated rod.

The choice of *optimal* sensor locations requires carrying the analysis a bit further, for there is an infinite variety of sensor locations satisfying the observability conditions. A number of computational algorithms have been proposed for determining the optimal sensor locations, and the reader is urged to consult the literature for a discussion of these [42–47].

For *nonlinear* problems, very few observability results exist, and one must be content with either linearizing the system and testing local observability or relaxing the nonlinearity and examining the observability of the related linear problem.

State Estimation

State estimation algorithms have been developed for a wide range of problems and by a wide variety of derivation methods. For linear problems having white, Gaussian noise, very complete rigorous results are available; however, for nonlinear problems or for problems with more complex noise processes, there are fewer rigorous results, but many general formal results. The reader is directed to a recent review [36] for a fuller discussion of these points and for a survey of practical applications.

In this section we shall present a very general nonlinear state estimator for the system of Eqs. (5.6.1) to (5.6.5). To begin, let us require that the state estimates minimize the error functional

$$
\begin{aligned}
I = &\int_0^{t_f} (\mathbf{y} - \mathbf{h})^T \mathbf{Q}(\mathbf{y} - \mathbf{h})\, dt \\
&+ \int_0^{t_f}\int_0^1\int_0^1 \left[\mathbf{x}_t(r, t) - \mathbf{f}(\mathbf{x}_{rr}, \mathbf{x}_r, \mathbf{x}, \mathbf{u})\right]^T \\
&\times \mathbf{R}(r, s, t)\left[\mathbf{x}_t(s, t) - \mathbf{f}(\mathbf{x}_{ss}, \mathbf{x}_s, \mathbf{x}, \mathbf{u})\right] dr\, ds\, dt \\
&+ \int_0^{t_f} \left[(\mathbf{b}_0^T \mathbf{R}_0 \mathbf{b}_0) + (\mathbf{b}_1^T \mathbf{R}_1 \mathbf{b}_1)\right] dt
\end{aligned}
\tag{5.6.21}
$$

where the matrices $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{Q}(t)$ are positive definite and symmetric. In the case of discrete time data, $\mathbf{Q}(t)$ is replaced by $\mathbf{Q}_d(t)$, where

$$
\mathbf{Q}_d(t) = \sum_{k=1}^{M} \mathbf{Q}(t_k)\delta(t - t_k)\Delta t_k \tag{5.6.22}
$$

and Δt_k is the sampling interval at time k. The quantity $\mathbf{R}(r, s, t)$ is defined by

$$
\int_0^1 \mathbf{R}^+(r, \rho, t)\mathbf{R}(\rho, s, t)\, d\rho = \mathbf{I}\delta(r - s) \tag{5.6.23}
$$

where $\mathbf{R}^+(r, s, t)$ is a positive definite, symmetric matrix function. There will be two cases to consider:

1. *Continuous time data.* For continuous time data, the state estimation equations take the form [34]

$$
\frac{\partial \hat{\mathbf{x}}}{\partial t}(z, t) = \hat{\mathbf{f}}(\hat{\mathbf{x}}_{zz}, \hat{\mathbf{x}}_z, \hat{\mathbf{x}}, \mathbf{u}) + \sum_{i=1}^{\Upsilon} \mathbf{P}(z, z_i^*, t) \frac{\partial \hat{\mathbf{h}}}{\partial \hat{\mathbf{x}}(z_i^*, t)} \mathbf{Q}(t)\left[\mathbf{y}(t) - \hat{\mathbf{h}}\right] \tag{5.6.24}
$$

with boundary conditions

$$
\hat{\mathbf{b}}_0(\hat{\mathbf{x}}, \hat{\mathbf{x}}_z, \mathbf{u}_0) = \mathbf{0} \qquad z = 0 \tag{5.6.25}
$$

$$
\hat{\mathbf{b}}_1(\hat{\mathbf{x}}, \hat{\mathbf{x}}_z, \mathbf{u}_1) = \mathbf{0} \qquad z = 1 \tag{5.6.26}
$$

where the differential sensitivities $\mathbf{P}(r, s, t)$ must satisfy

$$
\begin{aligned}
\mathbf{P}_t(r, s, t) = &\frac{\partial \hat{\mathbf{f}}}{\partial \hat{\mathbf{x}}}(\hat{\mathbf{x}}_{rr}, \hat{\mathbf{x}}_r, \hat{\mathbf{x}}, \mathbf{u})\mathbf{P}(r, s, t) + \frac{\partial \hat{\mathbf{f}}}{\partial \hat{\mathbf{x}}_r}\mathbf{P}_r(r, s, t) \\
&+ \frac{\partial \hat{\mathbf{f}}}{\partial \hat{\mathbf{x}}_{rr}}\mathbf{P}_{rr}(r, s, t) + \mathbf{P}(r, s, t)\frac{\partial \hat{\mathbf{f}}}{\partial \hat{\mathbf{x}}}(\hat{\mathbf{x}}_{ss}, \hat{\mathbf{x}}_s, \mathbf{x}, \mathbf{u})^T \\
&+ \mathbf{P}_s(r, s, t)\frac{\partial \hat{\mathbf{f}}^T}{\partial \hat{\mathbf{x}}_s} + \mathbf{P}_{ss}(r, s, t)\frac{\partial \hat{\mathbf{f}}^T}{\partial \hat{\mathbf{x}}_{ss}} \\
&+ \sum_{i=1}^{\Upsilon}\sum_{j=1}^{\Upsilon} \mathbf{P}(r, r_i^*, t)\frac{\partial\left\{\left[\partial \hat{\mathbf{h}}/\partial \hat{\mathbf{x}}(r_i^*, t)\right]\mathbf{Q}(t)(\mathbf{y} - \hat{\mathbf{h}})\right\}}{\partial \hat{\mathbf{x}}(r_j^*, t)}\mathbf{P}(r_j^*, s, t) \\
&+ \mathbf{R}^+(r, s, t)
\end{aligned}
\tag{5.6.27}
$$

as well as the boundary conditions at

$r = 0$

$$\frac{\partial \hat{\mathbf{b}}_0}{\partial \hat{\mathbf{x}}}\mathbf{P}(r, s, t) + \frac{\partial \hat{\mathbf{b}}_0}{\partial \hat{\mathbf{x}}_r}\mathbf{P}_r(r, s, t) + \mathbf{R}_0^{-1}(t)\left(\frac{\partial \hat{\mathbf{b}}_0}{\partial \hat{\mathbf{x}}_r}\right)^{-1}\hat{\mathbf{f}}^T_{\mathbf{x}_{rr}}\delta(s) = \mathbf{0} \tag{5.6.28}$$

$r = 1$

$$\frac{\partial \hat{\mathbf{b}}_1}{\partial \hat{\mathbf{x}}}\mathbf{P}(r, s, t) + \frac{\partial \hat{\mathbf{b}}_1}{\partial \hat{\mathbf{x}}_r}\mathbf{P}_r(r, s, t) - \mathbf{R}_1^{-1}(t)\left(\frac{\partial \hat{\mathbf{b}}_1}{\partial \hat{\mathbf{x}}_r}\right)^{-1}\hat{\mathbf{f}}^T_{\mathbf{x}_{rr}}\delta(s - 1) = \mathbf{0} \tag{5.6.29}$$

$s = 0$

$$\mathbf{P}(r, s, t)\frac{\partial \hat{\mathbf{b}}_0^T}{\partial \hat{\mathbf{x}}} + \mathbf{P}_s(r, s, t)\frac{\partial \hat{\mathbf{b}}_0^T}{\partial \hat{\mathbf{x}}_s} + \left[\mathbf{R}_0^{-1}(t)\left(\frac{\partial \hat{\mathbf{b}}_0}{\partial \hat{\mathbf{x}}_s}\right)^{-1}\hat{\mathbf{f}}^T_{\mathbf{x}_{ss}}\right]^T\delta(r) = \mathbf{0} \tag{5.6.30}$$

$s = 1$

$$\mathbf{P}(r, s, t)\frac{\partial \hat{\mathbf{b}}_1^T}{\partial \hat{\mathbf{x}}} + \mathbf{P}_s(r, s, t)\frac{\partial \hat{\mathbf{b}}_1^T}{\partial \hat{\mathbf{x}}_s} - \left[\mathbf{R}_1^{-1}(t)\left(\frac{\partial \hat{\mathbf{b}}_1}{\partial \hat{\mathbf{x}}_s}\right)^{-1}\hat{\mathbf{f}}^T_{x_{ss}}\right]^T\delta(r - 1) = \mathbf{0} \tag{5.6.31}$$

Note that the differential sensitivities are symmetric [i.e., $\mathbf{P}(r, s, t) = \mathbf{P}(s, r, t)^T$] and reduce to estimate covariances when the system becomes linear with Gaussian, white-noise processes.

2. *Discrete time data.* The state estimation equations [34] for discrete time data take the following form between samples:

$$\frac{\partial \hat{\mathbf{x}}(z, t)}{\partial t} = \mathbf{f}(\hat{\mathbf{x}}_{zz}, \hat{\mathbf{x}}_z, \hat{\mathbf{x}}, \mathbf{u}) \qquad \begin{array}{l} t_{k-1} < t < t_k \\ k = 1, 2, \ldots \end{array} \tag{5.6.32}$$

with boundary conditions given by Eqs. (5.6.25) and (5.6.26). At sampling points t_k the estimates are updated by

$$\begin{aligned}\hat{\mathbf{x}}(z, t_k^+) = \hat{\mathbf{x}}(z, t_k^-) + \sum_{i=1}^{\Upsilon} \Big\{ & \mathbf{P}(z, z_i^*, t_k^-)\hat{\mathbf{V}}_i(t_k^-) \\ & + \tfrac{1}{2}\mathbf{P}(z, z_i^*, t_k^-)\sum_{j=1}^{\Upsilon}\hat{\mathbf{V}}_{ij}(t_k^-)\delta\hat{\mathbf{x}}(z_j^*, t_k) \\ & + \tfrac{1}{2}\big[\delta\mathbf{P}(z, z_i^*, t_k)\hat{\mathbf{V}}_i(t_k^-)\Big\}\end{aligned} \tag{5.6.33}$$

The differential sensitivities $\mathbf{P}(r, s, t)$ are given by the following equation between samples:

$$\mathbf{P}_t(r, s, t) = \hat{\mathbf{f}}_{\hat{\mathbf{x}}}\mathbf{P} + \hat{\mathbf{f}}_{\hat{\mathbf{x}}_r}\mathbf{P}_r + \hat{\mathbf{f}}_{\hat{\mathbf{x}}_{rr}}\mathbf{P}_{rr} + \mathbf{P}\hat{\mathbf{f}}^T_{\hat{\mathbf{x}}} + \mathbf{P}_s\hat{\mathbf{f}}^T_{\hat{\mathbf{x}}_s} + \mathbf{P}_{ss}\hat{\mathbf{f}}^T_{\hat{\mathbf{x}}_{ss}} + \mathbf{R}^+ \qquad t_{k-1} < t < t_k \tag{5.6.34}$$

with boundary conditions determined by Eqs. (5.6.28) to (5.6.31). The updating of the differential sensitivities at the sampling points is given by

$$\mathbf{P}(r, s, t_k^+) = \mathbf{P}(r, s, t_k^-) + \sum_{i=1}^{\gamma} \sum_{j=1}^{\gamma} \left\{ \mathbf{P}(r, r_i^*, t_k^-)\hat{\mathbf{V}}_{ij}(t_k^-)\mathbf{P}(r_j^*, s, t_k^-) \right.$$

$$+ \tfrac{1}{2}\delta\mathbf{P}(r, r_i^*, t_k)\hat{\mathbf{V}}_{ij}(t_k^-)\mathbf{P}(r_j^*, s, t_k^-)$$

$$+ \tfrac{1}{2}\mathbf{P}(r, r_i^*, t_k^-)\hat{\mathbf{V}}_{ij}(t_k^-)\delta\mathbf{P}(r_j^*, s, t_k)$$

$$\left. + \tfrac{1}{2}\mathbf{P}(r, r_i^*, t_k^-)\left[\sum_{m=1}^{\gamma} \mathbf{V}_{ijm}(t_k^-)\delta\hat{\mathbf{x}}(r_m^*, t_k) \right]\mathbf{P}(r_j^*, s, t_k^-) \right\} \tag{5.6.35}$$

In these equations,

$$\begin{aligned}
\delta\hat{\mathbf{x}}(z, t_k) &= \hat{\mathbf{x}}(z, t_k^+) - \hat{\mathbf{x}}(z, t_k^-) \\
\delta\mathbf{P}(r, s, t_k) &= \mathbf{P}(r, s, t_k^+) - \mathbf{P}(r, s, t_k^-) \\
\hat{V}(t_k^-) &= -(\mathbf{y} - \hat{\mathbf{h}})^T \mathbf{Q}_k (\mathbf{y} - \hat{\mathbf{h}}) \\
\hat{\mathbf{V}}_i(t_k^-) &= \frac{\partial \hat{V}(t_k^-)}{\partial \hat{\mathbf{x}}(z_i^*, t_k^-)} \\
\hat{\mathbf{V}}_{ij}(t_k^-) &= \frac{\partial^2 \hat{V}(t_k^-)}{\partial \hat{\mathbf{x}}(z_i^*, t_k^-)\partial \hat{\mathbf{x}}(z_j^*, t_k^-)} \\
\hat{\mathbf{V}}_{ijm}(t_k^-) &= \frac{\partial^3 \hat{V}(t_k^-)}{\partial \hat{\mathbf{x}}(z_i^*, t_k^-)\partial \hat{\mathbf{x}}(z_j^*, t_k^-)\partial \hat{\mathbf{x}}(z_m^*, t_k^-)}
\end{aligned} \tag{5.6.36}$$

Example 5.6.1 Let us illustrate these results by applying them to the problem of a thin flat plate having heating $u(z, t)$ applied along the length and water cooling passed through the center of the slab. The system, as constructed by Mäder [48], is shown in Fig. 5.20. The mathematical model of the heated slab takes the form

$$\frac{\partial x(z, t)}{\partial t} = \alpha \frac{\partial^2 x(z, t)}{\partial z^2} - \beta x(z, t) + \gamma u(z, t) + \xi(z, t) \qquad \begin{matrix} t > 0 \\ 0 \le z \le 1 \end{matrix} \tag{5.6.37}$$

$$z = 0, 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{5.6.38}$$

with temperature sensors of the form

$$y_i(t) = x(z_i^*, t) + \eta_i(t) \qquad i = 1, 2, \ldots, l \tag{5.6.39}$$

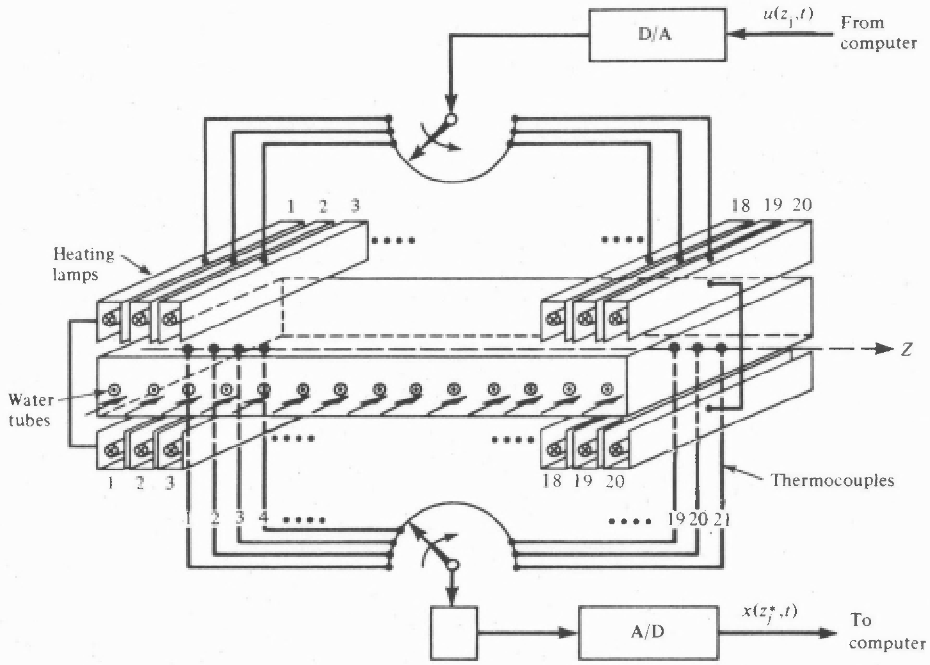


Figure 5.20 Experimental details of the heated slab [49]. *(Reproduced by permission of Pergamon Press Ltd.)*

Note that in Mäder's apparatus, there are 20 heating lamps which allow a good approximation to continuous heating $u(z, t)$. Also there are 21 thermocouples, so that the exact temperature profile may be known.*

First, it is easy to show that this system is observable if the thermocouples used by the state estimator do not all fall at the zeros of the system eigenfunctions, $\cos n\pi z$. In particular, if one chooses $z_1^* = 0$ as the single sensor to be used by the state estimator, then the system is observable and the state estimation equations take the form

$$\frac{\partial \hat{x}(z, t)}{\partial t} = \alpha \frac{\partial^2 \hat{x}(z, t)}{\partial z^2} - \beta \hat{x}(z, t) + \gamma u(z, t)$$
$$+ P(z, 0, t) Q(t) [y(t) - \hat{x}(0, t)] \tag{5.6.40}$$

with the boundary conditions of Eq. (5.6.38).

The problem is linear, and if one assumes $\xi(z, t)$, $\eta(t)$ to be Gaussian, white noise, then the differential sensitivities become estimate covariances of the form

$$P(z, z', t) = \mathcal{E}\{[x(z, t) - \hat{x}(z, t)][x(z', t) - \hat{x}(z', t)]\} \tag{5.6.41}$$

* This example is taken from [49] with permission of Pergamon Press Ltd.

The covariance equations arise directly from Eqs. (5.6.27) to (5.6.31) and take the form

$$P_t(z, z', t) = -2\beta P(z, z', t) + \alpha[P_{zz}(z, z', t) + P_{z'z'}(z, z', t)] - P(z, 0, t)QP(0, z', t) + R^+(z, z', t) \tag{5.6.42}$$

with boundary conditions

$$\begin{aligned}
z = 0: &\quad P_z(0, z', t) + \alpha R_0^{-1}\delta(z') = 0\\
z = 1: &\quad P_z(1, z', t) - \alpha R_1^{-1}\delta(z' - 1) = 0\\
z' = 0: &\quad P_{z'}(z, 0, t) + \alpha R_0^{-1}\delta(z) = 0\\
z' = 1: &\quad P_{z'}(z, 1, t) - \alpha R_1^{-1}\delta(z - 1) = 0
\end{aligned} \tag{5.6.43}$$

Now it is possible to use the eigenfunctions of the system

$$\begin{cases} \phi_0 = 1 \\ \phi_n = \sqrt{2}\cos n\pi z \qquad n = 1, 2, \ldots, N \end{cases} \tag{5.6.44}$$

to solve *both* the state estimation equations and the covariance equations. By making the substitutions

$$\hat{x}(z, t) = \sum_{n=0}^{N} a_n(t)\phi_n(z)$$

$$u(z, t) = \sum_{n=0}^{N} b_n(t)\phi_n(z) \tag{5.6.45}$$

$$P(z, z', t) = \sum_{n=0}^{N}\sum_{m=0}^{N} P_{nm}(t)\phi_n(z)\phi_m(z') \tag{5.6.46}$$

one can reduce the state estimation equations to the form

$$\frac{d\mathbf{w}}{dt} = \mathbf{A}\mathbf{w} + \mathbf{P}(t)\mathbf{C}Q(t)(y - \mathbf{C}\mathbf{w}) + \mathbf{v} \tag{5.6.47}$$

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{C}Q\mathbf{C}^T\mathbf{P} + \mathbf{D}(t) + \mathbf{D}_0(t) + \mathbf{D}_1(t) \tag{5.6.48}$$

where $\mathbf{w}$, $\mathbf{v}$, $\mathbf{C}$ are defined by Eqs. (5.6.14) and (5.6.17) and

$$\mathbf{D}_0 = \alpha^2\mathbf{\Phi}(0)\mathbf{R}_0^{-1}\mathbf{\Phi}^T(0) \tag{5.6.49}$$

$$\mathbf{D}_1 = \alpha^2\mathbf{\Phi}(1)\mathbf{R}_1^{-1}\mathbf{\Phi}(1)^T \tag{5.6.50}$$

Here $\mathbf{D}$ is the matrix of eigencoefficients d_{nm} of

$$R^+(z, z', t) = \sum_{n=0}^{N}\sum_{m=0}^{N} d_{nm}\phi_n(z)\phi_m(z') \tag{5.6.51}$$

and

$$\mathbf{\Phi}(z) = \begin{bmatrix} \phi_0(z) \\ \phi_1(z) \\ \vdots \\ \phi_N(z) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -\beta & 0 & \cdots & \cdots & 0 \\ 0 & -\beta - \pi^2 & \cdots & \cdots & 0 \\ 0 & \cdots & -\beta - 4\pi^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & -\beta - N^2\pi^2 \end{bmatrix} \tag{5.6.52}$$

Note that the $[(N + 1)^2 + (N + 1)]/2$ independent elements of the symmetric matrix $\mathbf{P}(t)$ may be precomputed off-line, while the $N + 1$ state estimation equations for $\mathbf{w}$ must be solved on-line.

These filter equations were implemented by Ajinkya et al. [49], and typical experimental results for the case of one thermocouple at $z = 0$ are shown in Fig. 5.21. Note that the filter converges to within 2°C of the exact profile in 1 min and to within 1°C in 3 min. More extensive experimental

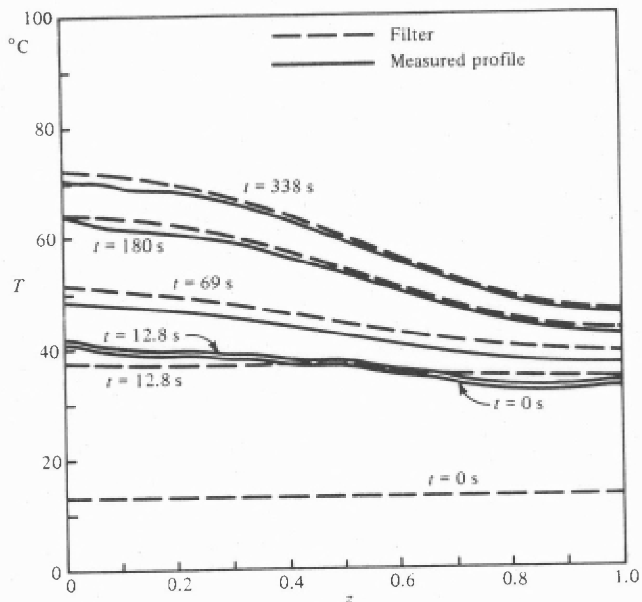


Figure 5.21 Open-loop filter performance with one measurement ($z = 0$). *(Reproduced by permission of Pergamon Press Ltd.)*

results may be seen in [49]. Only three eigenfunctions were used in this case ($N = 2$), so the covariance and state estimation equations together consist of nine ODEs. Even though the covariance equations were solved on-line in real time (for convenience), the total computations required less than $\frac{1}{2}$ percent of real time on the minicomputer.

Observers

State observers have been developed for distributed parameter systems (e.g., [50, 51]) and have the same basic properties as the lumped observers discussed in Sec. 5.2. As in the lumped parameter case, the purpose is to construct a deterministic estimator for unmeasured state variables. Let us illustrate the application of an observer to the example system described in Example 5.6.1.

Example 5.6.2 We shall neglect the noise in the equations for the heated slab so that the model becomes deterministic,

$$\frac{\partial x(z, t)}{\partial t} = \alpha \frac{\partial^2 x(z, t)}{\partial z^2} - \beta x(z, t) + \gamma u(z, t) \tag{5.6.53}$$

$$z = 0, 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{5.6.54}$$

$$y_i(t) = x(z_i^*, t) \qquad i = 1, 2, \ldots, l \tag{5.6.55}$$

One may show [50, 51] that the structure of the observer is the same as for the optimal state estimator, i.e.,

$$\hat{x}_t(z, t) = \alpha \hat{x}_{zz} - \beta \hat{x} + \gamma u(z, t) + \mathbf{g}^T(z)[\mathbf{y}(t) - \hat{\mathbf{y}}(t)] \tag{5.6.56}$$

where

$$\hat{\mathbf{y}}(t) = \begin{bmatrix} \hat{x}(z_1^*, t) \\ \vdots \\ \hat{x}(z_l^*, t) \end{bmatrix} = \int_0^1 \mathbf{C}(z)\hat{x}(z, t)\, dz \tag{5.6.57}$$

The boundary conditions are given by Eq. (5.6.54), and $\mathbf{g}(z)$ is an l vector of observer gains which must be chosen. The observer error may be defined as

$$e(z, t) = \hat{x}(z, t) - x(z, t) \tag{5.6.58}$$

and must satisfy

$$e_t(z, t) = \alpha e_{zz} - \beta e - \mathbf{g}^T(z) \int_0^1 \mathbf{C}(z') e(z', t)\, dz' \tag{5.6.59}$$

$$z = 0, 1 \qquad e_z = 0 \tag{5.6.60}$$

Note that Eq. (5.6.57) requires

$$\mathbf{C}(z) = \begin{bmatrix} \delta(z - z_1^*) \\ \delta(z - z_2^*) \\ \vdots \\ \delta(z - z_l^*) \end{bmatrix} \tag{5.6.61}$$

Now let us expand Eq. (5.6.59) in the system eigenfunctions

$$e(z, t) = \sum_{n=1}^{N} \epsilon_n(t)\phi_n(z) = \mathbf{\Phi}^T(z)\boldsymbol{\epsilon}(t) \tag{5.6.62}$$

where

$$\epsilon_n(t) = \int_0^1 \phi_n(z) e(z, t)\, dz \tag{5.6.63}$$

$$\phi_n(z) = \begin{cases} 1 & n = 0 \\ \sqrt{2} \cos n\pi z & n = 1, 2, \ldots, N \end{cases} \tag{5.6.64}$$

Then Eq. (5.6.59) reduces to

$$\dot{\boldsymbol{\epsilon}}(t) = (\alpha\mathbf{\Lambda} - \beta\mathbf{I} - \mathbf{GC})\boldsymbol{\epsilon}(t) \qquad \boldsymbol{\epsilon}(0) = \boldsymbol{\epsilon}_0 \tag{5.6.65}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & & & & \\ & -\pi^2 & & & \\ & & -4\pi^2 & & \mathbf{0} \\ \mathbf{0} & & & \ddots & \\ & & & & -N^2\pi^2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \phi_0(z_1^*) & \phi_1(z_1^*) & \cdots & \phi_N(z_1^*) \\ \phi_0(z_2^*) & \phi_1(z_2^*) & \cdots & \phi_N(z_2^*) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_0(z_l^*) & \cdots & \cdots & \phi_N(z_l^*) \end{bmatrix} \tag{5.6.66}$$

and the elements of $\mathbf{G}$, g_{nm}, are given by

$$g_{nm} = \int_0^1 \phi_n(z) g_m(z)\, dz \tag{5.6.67}$$

The observer will converge from some initial estimate error $\boldsymbol{\epsilon}_0$ to zero error if and only if all the eigenvalues of the matrix

$$\mathbf{E} = \alpha\mathbf{\Lambda} - \beta\mathbf{I} - \mathbf{GC} \tag{5.6.68}$$

have negative real parts. The speed of convergence is controlled by the magnitude of these eigenvalues. Thus observer design consists of choosing the elements g_{nm} such that the real parts of all the eigenvalues of $\mathbf{E}$ are large and negative. Köhne [50, 51] suggests several techniques for making this choice.

Köhne [51] has applied this observer experimentally with three sensors

$$z_1^* = 0 \qquad z_2^* = 0.5 \qquad z_3^* = 1.0$$

and with the observer weighting functions chosen as

$$g_1(z) = 1 - 2z$$

$$g_2(z) = \begin{cases} 2z & z \in [0, 0.5] \\ 2(1 - z) & z \in [0.5, 1] \end{cases}$$

$$g_3(z) = 2z - 1$$

and reported excellent results. Typical results from an initial guess some 45°C in error are shown in Fig. 5.22. Note that the observer converges to within 1°C of the true profile in about 80 s.

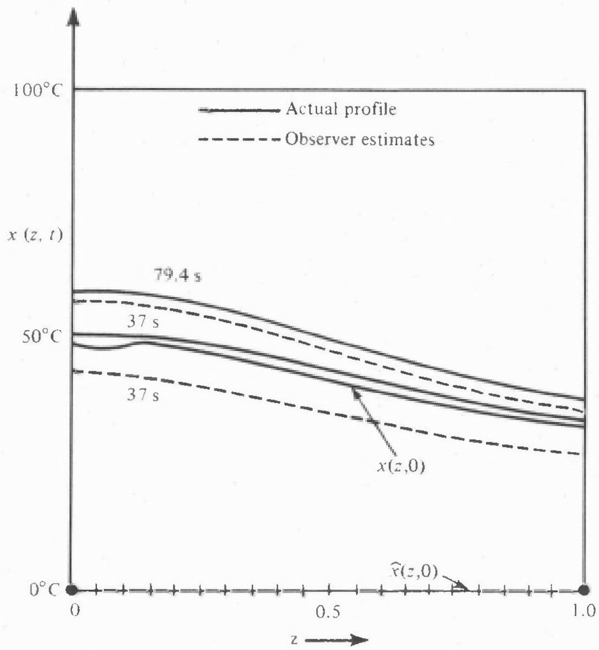


Figure 5.22 Observer performance with a heated slab plate [51]. *(Reproduced with permission of Springer-Verlag).*

5.7 STOCHASTIC FEEDBACK CONTROL FOR DISTRIBUTED PARAMETER SYSTEMS

Just as in the case of lumped parameter systems, distributed parameter stochastic feedback control is concerned with the control of systems having random disturbances in the process and in the measurements. The controller design involves choosing appropriate estimates of the states to be used in the control loop and to determine the best control structure. This is a relatively new and unexplored field for distributed parameter systems, but important results are known for linear systems. Good summaries of recent results may be found in Refs. [36, 52].

We shall illustrate the application of stochastic feedback control through an example problem.*

Example 5.7.1 Let us consider the heated slab shown in Fig. 5.20 and discussed in Example 5.6.1. The modeling equations are given by Eqs. (5.6.37) to (5.6.39), and it is desired to design a feedback control system for this stochastic process. One may show [36, 52] that for the white, Gaussian noise process $\xi(z, t)$, $\eta_i(t)$, $\xi_0(t)$, $\xi_1(t)$, the stochastic feedback controller which minimizes the expected quadratic objective

$$I = \mathcal{E}\left\{\frac{1}{2}\int_0^1\int_0^1 x(r, t_f)S_f(r, s)x(s, t_f)\, dr\, ds \right.$$
$$\left. +\frac{1}{2}\int_0^{t_f}\int_0^1\int_0^1 \left[x(r, t)F(r, s, t)x(s, t) + u(r, t)E(r, s, t)u(s, t)\right] dt\, dr\, ds\right\} \tag{5.7.1}$$

is the deterministic feedback control law

$$u(z, t) = -\gamma\int_0^1\int_0^1 E^*(z, s, t)S(s, \rho, t)\hat{x}(\rho, t)\, ds\, d\rho \tag{5.7.2}$$

where $\hat{x}(\rho, t)$ is the optimal least squares estimate found in the last section. The quantity E^* is a weighting factor defined by

$$\int_0^1 E^*(z, s, t)E(s, \rho, t)\, ds = \delta(z - \rho) \tag{5.7.3}$$

and the Riccati variables $S(s, \rho, t)$ are found just as in the deterministic case (see Sec. 4.3):

$$\begin{aligned} S_t(r, s, t) = &-\alpha^2\left[S_{ss} + S_{rr}\right] + 2\beta S \\ &+\gamma^2\int_0^1\int_0^1 S(r, z, t)E^*(z, \rho, t)S(\rho, s, t)\, ds\, d\rho \\ &-F(r, s, t) \end{aligned} \tag{5.7.4}$$

*This example is taken from Ref. [49] with the permission of Pergamon Press Ltd.

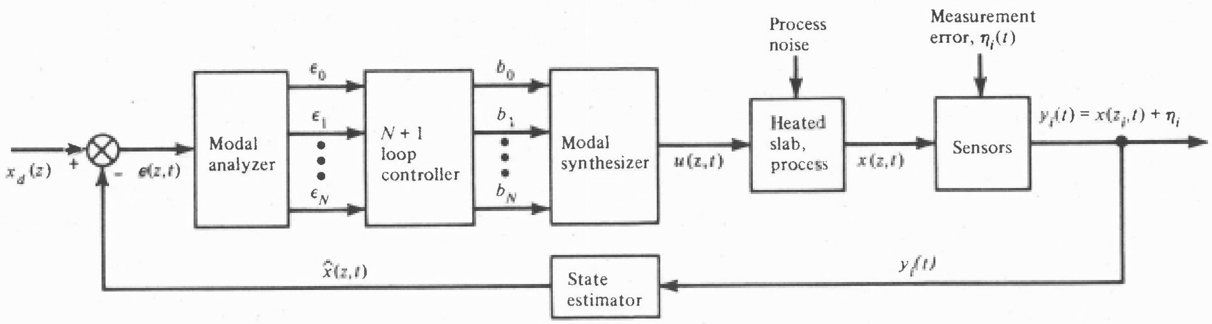


Figure 5.23 Structure of a stochastic modal feedback controller.

with boundary conditions

$$r = 0, 1 \qquad S_r = 0$$

$$s = 0, 1 \qquad S_s = 0 \tag{5.7.5}$$

and terminal condition

$$S(r, s, t_f) = S_f(r, s) \tag{5.7.6}$$

Thus just as in the lumped parameter case discussed in Sec. 5.4, the optimal linear-quadratic stochastic controller for distributed parameter systems is the *deterministic* feedback controller operating on the minimal least squares state estimates $\hat{x}(z, t)$. Hence, a *separation theorem* exists for this class of problems [52].

As an illustration of how a *suboptimal* stochastic feedback controller performs for this process, a simple modal stochastic feedback controller coupled to the optimal state estimator was tested experimentally [49]. The controller structure is shown in Fig. 5.23. The modal feedback controller consists of $N + 1$ single-loop controllers

$$\bar{b}_n(s) = g_n(s)\bar{\epsilon}_n(s) \qquad n = 0, 1, \ldots, N \tag{5.7.7}$$

In the present experimental example, the controllers were chosen to be proportional plus integral controllers, so that

$$\bar{b}_n(s) = K\left(1 + \frac{1}{\tau_I s}\right)\bar{\epsilon}_n(s) \tag{5.7.8}$$

The stochastic feedback controller performance can be seen in Fig. 5.24

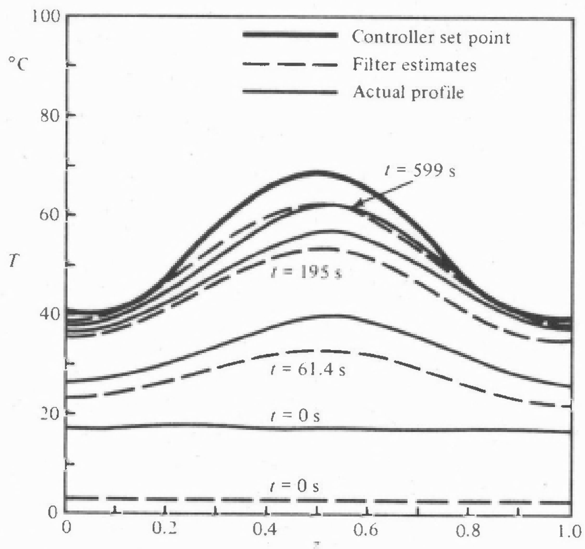


Figure 5.24 Closed-loop filter performance with one measurement ($z = 0$) and 8°C standard deviation added measurement errors [49]. *(Reproduced by permission of Pergamon Press Ltd.)*

when only one thermocouple at $z = 0$ is used and random errors of 8°C standard deviation are added to the thermocouple signal. Note that the filter converges rapidly and the stochastic feedback controller drives the temperature profile to the set point.

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PROBLEMS

5.1 The following kinetic scheme is being carried out in an isothermal batch chemical reactor. The scheme is characteristic of partial oxidation reactions found in industry. If we wish to estimate all the concentrations, we must determine the minimal number of species to be measured and which ones should be chosen for measurement. Assume that the initial concentrations are *unknown*. Thus this is simply a question of *observability*.

$$\begin{array}{l} A \xrightarrow{k_1} B \xrightarrow{k_2} C \\ \downarrow k_3 \\ C \end{array}$$

The mathematical model is

$$\frac{dx_1}{dt} = -(k_1 + k_3)x_1$$

$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2$$

$$\frac{dx_3}{dt} = k_3 x_1 + k_2 x_2$$

where

$$x_1 = [A] \qquad x_2 = [B] \qquad x_3 = [C]$$

(*a*) If continuous concentration measurements are available, determine the minimal set of measurements sufficient for observability.

(*b*) For discrete measurements at time t_k, $k = 1, 2, \ldots, N$, determine the minimal number of sample points N sufficient for observability. [Hint: Here $\mathbf{\Phi}(t, 0) = e^{\mathbf{A}t} = \mathbf{M}e^{\mathbf{\Lambda}t}\mathbf{M}^{-1}$, where $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues of $\mathbf{A}$ and $\mathbf{M}$ is the matrix of corresponding eigenvectors.]

5.2 For the batch kinetic scheme shown in Prob. 5.1, a least squares state estimator is to be developed.

(*a*) Write out the filtering and covariance equations assuming that both x_2 and x_3 are measured continuously. Which equations must be solved online and which may be solved off-line?

(*b*) Repeat part (*a*) when only discrete time measurements of x_2 and x_3 are possible.
(*c*) Carry out estimation simulation for part (*a*) when

$$k_1 = 1 \qquad k_2 = 2 \qquad k_3 = 0.5 \qquad x_1(0) = 1.0 \qquad x_2(0) = x_3(0) = 0$$

$$\hat{x}_1(0) = 0.8 \qquad \hat{x}_2(0) = 0.1 \qquad \hat{x}_3(0) = 0.1 \qquad \mathbf{Q}^{-1} = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix}$$

$$\mathbf{R}^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{P}(0) = \begin{bmatrix} 0.005 & 0 & 0 \\ 0 & 0.0001 & 0 \\ 0 & 0 & 0.0001 \end{bmatrix}$$

Use the model to generate data $y_1 = x_2 + \eta_1, y_2 = x_3 + \eta_2$ where the measurement error is chosen from a random number generator having zero mean and a Gaussian distribution with standard deviation $\sigma = 0.01$.

(*d*) Repeat the computations in part *c* for the discrete data filter.

5.3 Let us consider a kinetic scheme similar to that in Prob. 5.1 except that two of the reactions are now second order. The mathematical modeling equations may be written in terms of the species $x_1 = [A]$, $x_2 = [B]$, $x_3 = [C]$. As in the previous case, the initial compositions are unknown.

$$2A \underset{k_1}{\rightarrow} B \underset{k_2}{\rightarrow} C$$

$$k_3 \downarrow$$

$$C$$

$$\frac{dx_1}{dt} = -2(k_1 + k_3)x_1^2$$

$$\frac{dx_2}{dt} = k_1 x_1^2 - k_2 x_2$$

$$\frac{dx_3}{dt} = k_1 x_1^2 + k_2 x_2$$

(*a*) By linearization and using an observability test, determine observability when both *B* and *C* are measured continuously. Repeat for the case when only *C* is measured.

(*b*) Write out the extended Kalman filter equations for this problem as well as the equations for the differential sensitivities. Assume both x_2 and x_3 are measured continuously. Which equations must be solved on-line? Explain.

(*c*) Repeat part (*b*) when only discrete measurements of x_2 and x_3 are possible.

5.4 (*a*) For the heat exchanger problem discussed in Example 5.5.2, carry out the filter computations for the parameters.

$$v = 0.2 \qquad a_0 = -1.0 \qquad r_1^* = 0.25 \qquad r_2^* = 0.5 \qquad r_3^* = 0.75$$

$$r_4^* = 1.0 \qquad R_1^+ = 0 \qquad P_0 = 0.0025\delta(r - s) \qquad Q^{-1} = 0.0025$$

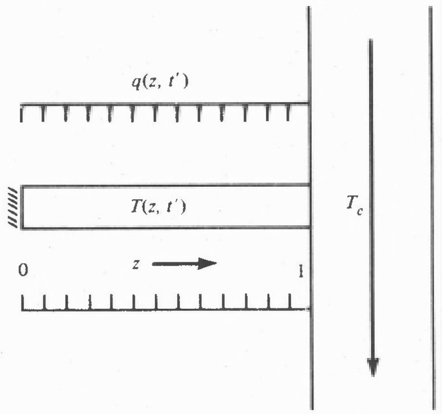
(*b*) Repeat the calculations with successively fewer sensors (i.e., three, two, and only one).

5.5 Consider the heating of a thin rod in a furnace with one end insulated and the other end ($z = 1$) inserted into a flowing liquid stream at constant temperature T_c. The modeling equations take the form

$$\rho C_p \frac{\partial T(z, t')}{\partial t'} = k \frac{\partial^2 T}{\partial z^2} + q(z, t') \qquad t' > 0, \quad 0 < z < 1$$

$$z = 0 \qquad \frac{\partial T}{\partial z} = 0$$

$$z = 1 \qquad -k \frac{\partial T}{\partial z} = h(T - T_c)$$



(*a*) Reduce the model to dimensionless form using the parameters

$$x = \frac{T - T_c}{T_c} \qquad u = \frac{q}{kT_c} \qquad t = \frac{t'k}{\rho C_p}$$

$$Bi = \frac{h}{k}$$

(*b*) Solve the modeling equations through an orthonormal expansion of the form

$$x(z, t) = \sum_{n=0}^{N} a_n(t)\phi_n(z)$$

$$u(z, t) = \sum_{n=0}^{N} b_n(t)\phi_n(z)$$

Determine $\phi_n(z)$ and the equations for $a_n(t)$.

(*c*) Determine the N-mode observability conditions for the case of only one sensor. For $N = 4$, determine which sensor locations must be avoided in order for one sensor to guarantee observability. Recommend a specific sensor location and justify your recommendation.

(*d*) Develop the state estimation equations for this problem given one sensor providing continuous temperature measurements. Which equations must be solved on-line and which ones may be solved off-line? Explain how you would use a modal decomposition to solve both the filter equations and the covariance equations.

(*e*) Carry out state estimation simulation with the following parameters: $Bi = 2.0$, u = constant $= 0.1$, $P_0 = 0.01\delta(r - s)$, $Q^{-1} = 0.0001$, $R^+ = 0$. Use one sensor location of your choice, and use modal decomposition to solve the filter and covariance equations. To obtain data, $x(r_1^*, t)$, use the model simulation and, if desired, add random measurement errors from a Gaussian random number generator having zero mean and $\sigma = 0.01$. Use $x(0) = 0$, $\hat{x}(0) = 0.1$ as actual and estimated initial conditions, respectively.