CHAPTER

FOUR

CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

4.1 INTRODUCTION

Distributed parameter systems are distinguished by the fact that the states, controls, and outputs may depend on spatial position. Thus the natural form of the system model is the partial differential equation, integral equation, or transcendental transfer function. One particularly important class of distributed parameter systems consists of those having pure time delays. There exists a wide range of industrially important distributed parameter control problems (see [1–4] for a selection); however, we shall choose two simple example problems to illustrate some of the fundamental concepts.

Example 4.1.1* Consider the problem of reheating a steel slab by thermal radiation (for rolling) in a batch furnace as sketched in Fig. 4.1. For proper rolling characteristics, it is necessary for the slab to have a specified temperature distribution $T_d(z)$. Thus our problem is to control the heat flux to the surface of the slab in such a way as to approach this desired temperature distribution in some optimal fashion.

* This example is taken from [5] and reprinted by permission of John Wiley and Sons, Inc.

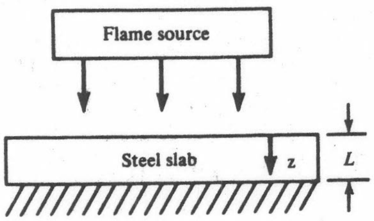

Figure 4.1 Radiant heating of a slab.

To be more precise, let us consider the modeling equations for the slab

$$\frac{\partial T(z,t)}{\partial t} = \frac{1}{\beta(T)} \frac{\partial[\alpha(T)\,\partial T(z,t)/\partial z]}{\partial z} \qquad \begin{array}{l} 0 \le z \le L \\ 0 \le t \le t_f \end{array} \tag{4.1.1}$$

$$\frac{\partial T(0,t)}{\partial z} = v(t) \tag{4.1.2}$$

$$\frac{\partial T(L,t)}{\partial z} = 0 \tag{4.1.3}$$

$$T(z,0) = T_0(z) \tag{4.1.4}$$

which reflect the fact that negligible heat is lost at the sides and bottom of the slab; by adjusting the flame, one can control the heat flux at the upper surface between bounds

$$v_* \le v(t) \le v^* \tag{4.1.5}$$

Note that the state variables $T(z, t)$ are spatially dependent, but that the control $v(t)$ is only time-dependent and is applied at the boundary. For distributed parameter systems one often must consider control variables appearing in the boundary conditions as well as those appearing in the differential equations.

The set points in such a control problem are also spatially dependent. For example, here one wishes to manipulate $v(t)$ so as to achieve a certain set point $T_d(z)$. If one wished to apply *optimal control* to this example problem, then the objective functional

$$I[v(t)] = \int_0^{t_f}\int_0^L [T(z,t) - T_d(z)]^2\,dt\,dz \tag{4.1.6}$$

if minimized, would cause $T(z, t)$ to be quickly driven toward the set point $T_d(z)$. This is an example of *open-loop control* applied at the boundary of a distributed process.

Example 4.1.2 Let us consider the problem of controlling the temperature in a stirred mixing tank such as that shown in Fig. 4.2. The temperature is regulated by a feedback controller which adjusts the fraction of hot stream $\lambda(t)$ which is fed to the tank. The tank level is controlled by an overflow weir, and the total inlet flow is kept constant by a flow regulator even when the ratio of hot- and cold-stream flow rates varies. Unfortunately, by poor

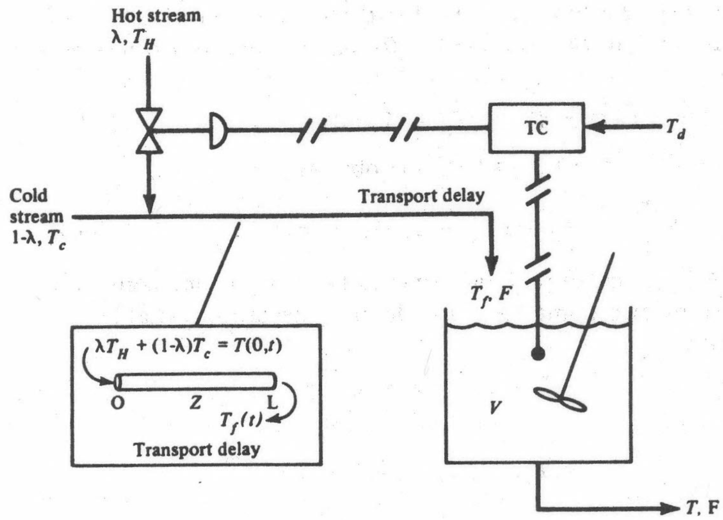

Figure 4.2 Control of the temperature in a stirred mixing tank with transport delays in the inlet piping.

design, the hot and cold streams are mixed some distance from the mixing tank, so that there is a transport delay in the inlet feed line.

An energy balance on the tank (assuming no heat losses to the environment) yields

$$\rho C_p V \frac{dT}{dt} = \rho C_p \left(F T_f(t) - FT \right) \tag{4.1.7}$$

where $T_f(t)$ represents the feed temperature at the mixing tank itself. It is necessary to calculate $T_f(t)$ in terms of the cold-stream temperature T_C, hot-stream temperature T_H, and fraction of hot-stream feed $\lambda(t)$. Also one must account for the fact that a transport delay occurs due to fluid flowing at flow rate F in a well-insulated pipe of length L and cross-sectional area a_c. An energy balance over this pipe [6] yields a model for the temperature profile in the pipe, $T_p(z, t)$,

$$\rho C_p \frac{\partial T_p(z, t)}{\partial t} + \rho C_p \frac{F}{a_c} \frac{\partial T_p(z, t)}{\partial z} = 0 \qquad 0 < z < L \tag{4.1.8}$$

The inlet to the pipe is

$$T_p(0, t) = \lambda(t) T_H + \left[1 - \lambda(t) \right] T_C \tag{4.1.9}$$

and the exit of the pipe is the inlet temperature to the mixing tank, given by

$$T_f(t) \equiv T_p(L, t) \tag{4.1.10}$$

Equation (4.1.8) is a first-order hyperbolic equation which models a pure transport delay. Fortunately it has a very simple solution,

$$T_p(L, t) = T(0, t - \alpha) \tag{4.1.11}$$

where $\alpha = La_c/F$ is the time required for the fluid to travel in the pipe. Thus making use of Eqs. (4.1.9) and (4.1.10), one obtains an expression for $T_f(t)$:

$$T_f(t) = T_C + (T_H - T_C)\lambda(t - \alpha) \qquad (4.1.12)$$

If this expression is used in Eq. (4.1.7), one obtains

$$\theta\frac{dT}{dt} = [T_C + (T_H - T_C)\lambda(t - \alpha) - T] \qquad (4.1.13)$$

where $\theta = V/F$ is the mean residence time in the tank. If one converts Eq. (4.1.13) to the transform domain by first defining deviation variables about some steady state

$$T = T_s \qquad \lambda = \lambda_s$$

that is,

$$y = T - T_s \qquad u = \lambda - \lambda_s$$

then

$$\frac{dy}{dt} = (T_H - T_C)u(t - \alpha) - y \qquad y(0) = 0 \qquad (4.1.14)$$

and transforming, one obtains

$$\bar{y}(s) = \frac{(T_H - T_C)e^{-\alpha s}}{\theta s + 1}\bar{u}(s) \qquad (4.1.15)$$

Note that except for the time delay, Eqs. (4.1.14) and (4.1.15) would be simple lumped parameter models. However, the presence of the time delay causes both theoretical and practical complications in control system design. Design procedures for this class of important problems shall be discussed in Sec. 4.5.

In the next section we shall introduce some basic concepts which are important in understanding the dynamics of distributed parameter systems. Then some simple controller design strategies for linear systems will be considered. Following this introductory material, the underlying theory and control system synthesis for the optimal control of distributed parameter systems will be presented. Finally, control system design procedures for nonlinear distributed parameter systems and for systems having pure time delays are discussed.

4.2 FEEDBACK CONTROL OF LINEAR DISTRIBUTED PARAMETER SYSTEMS

In carrying out the design of feedback controllers for distributed parameter systems, one can call on the optimal linear-quadratic controllers of the previous section as well as on a whole host of "suboptimal" design procedures similar to those discussed for lumped parameter systems in Chap. 3. There is a significant division of philosophy in the approaches taken to distributed parameter systems control. This is illustrated in Fig. 4.3. The easiest, most straightforward approach, termed *early lumping*, simply discretizes the distributed parameter system at the earliest opportunity into an approximate model consisting of a set of ordinary differential equations in time. Then the design methods of Chap. 3 are applied directly to accomplish controller design without recourse to distributed parameter systems theory at all. This approach has several disadvantages. First, conditions for controllability, stabilizability, etc., which should depend only on the placement of control actuators, can also depend on the method of lumping and the location of discretization points if early lumping is used. Second, one quickly loses the physical features of the problem through early lumping, and the ultimate controller design may be naïve and fail to take advantage of natural properties of the system.

The alternative approach, *late lumping*, takes full advantage of the available distributed parameter control theory and analyzes the full PDE model for controllability, stabilizability, best controller structure, etc. It is only at the last stages, after the controller design has been made, that the resulting process and control system equations are lumped for reasons of numerical integration in implementation. Late lumping allows the designer to take advantage of all the natural features of the problem and to understand the system structure much more completely. However, this approach requires a greater knowledge of distributed parameter systems control theory; hence this section shall be devoted to illustrating how late lumping may be applied to systems of engineering interest.

Because the properties of distributed parameter systems depend so strongly on the type of equations (parabolic, hyperbolic, elliptic, etc.), we shall discuss

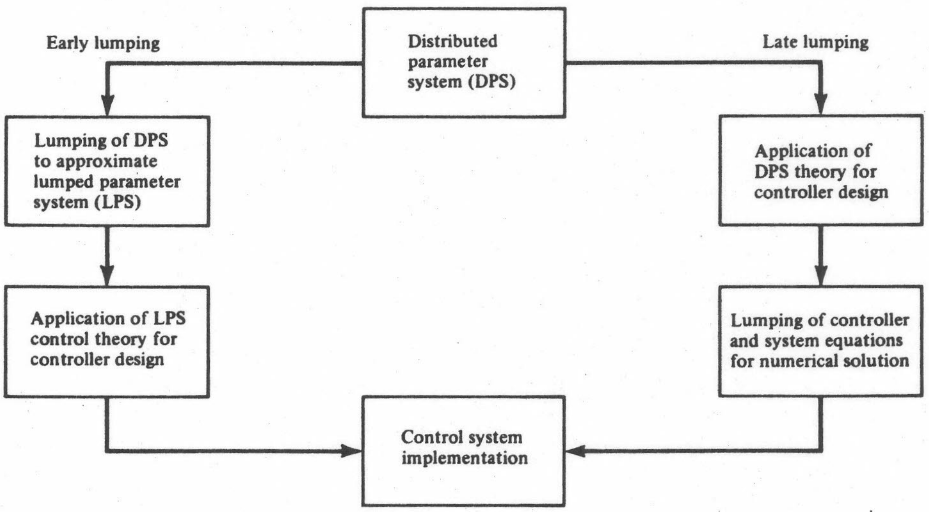

Figure 4.3 Design procedures via *early lumping* and *late lumping*.

those classes of equations that arise most often in process control and the applications of distributed parameter control theory to each.

First-Order Hyperbolic Systems

Let us consider the class of systems described by

$$\frac{\partial \mathbf{x}(z,t)}{\partial t} = \mathbf{A}_1 \frac{\partial \mathbf{x}}{\partial z} + \mathbf{A}_0 \mathbf{x}(z,t) + \mathbf{B}\mathbf{u}(z,t) \tag{4.2.1}$$

$$\mathbf{x}(0,t) = \mathbf{B}_0 \mathbf{u}_0(t) \tag{4.2.2}$$

$$\mathbf{y}(z,t) = \int_0^1 \mathbf{C}(z,r,t)\mathbf{x}(r,t)\,dr \tag{4.2.3}$$

Examples of this class of problems arise in the control of heat exchangers, chemical reactors, and other tubular processes [7]. Recall that Example 4.1.2 illustrates a very simple hyperbolic system.

There are several ways in which one can proceed in analyzing these processes. One approach is to use Laplace transforms in space,

$$\bar{\mathbf{x}}(p,t) = \int_0^\infty e^{-pz}\mathbf{x}(z,t)\,dz \tag{4.2.4}$$

so that for $\mathbf{A}_1$, $\mathbf{A}_0$, $\mathbf{B}$, and $\mathbf{B}_0$ constant, one obtains the *transform equations in space*

$$\frac{d\bar{\mathbf{x}}(p,t)}{dt} = p\mathbf{A}_1\bar{\mathbf{x}}(p,t) - \mathbf{A}_1\mathbf{B}_0\mathbf{u}_0(t) + \mathbf{A}_0\bar{\mathbf{x}}(p,t) + \mathbf{B}\bar{\mathbf{u}}(p,t) \tag{4.2.5}$$

or

$$\frac{d\bar{\mathbf{x}}(p,t)}{dt} = (p\mathbf{A}_1 + \mathbf{A}_0)\bar{\mathbf{x}}(p,t) - \mathbf{A}_1\mathbf{B}_0\mathbf{u}_0(t) + \mathbf{B}\bar{\mathbf{u}}(p,t) \tag{4.2.6}$$

In this manner the equations are reduced to ODEs in time. Now if one defines some set point $\mathbf{x}_d(z)$ and its transform

$$\bar{\mathbf{x}}_d(p) = \int_0^\infty e^{-pz}\mathbf{x}_d(z)\,dz \tag{4.2.7}$$

then one possible feedback control law would be

$$\bar{\mathbf{u}}(p,t) = -\mathbf{K}\bar{\mathbf{x}}(p,t) \tag{4.2.8}$$

$$\mathbf{u}_0(t) = -\mathbf{K}_0\bar{\mathbf{x}}(p,t) \tag{4.2.9}$$

so as to drive $\bar{\mathbf{x}}(p,t)$ to $\bar{\mathbf{x}}_d(p)$.

A second approach, which is probably easier, is to make use of the *Laplace transform in time*

$$\bar{\mathbf{x}}(z,s) = \int_0^\infty e^{-st}\mathbf{x}(z,t)\,dt \tag{4.2.10}$$

Then Eq. (4.2.1) becomes

$$\frac{d\bar{\mathbf{x}}}{dz} = \mathbf{A}_1^{-1}\{(s\mathbf{I} - \mathbf{A}_0)\bar{\mathbf{x}} - [\mathbf{x}_0(z) + \mathbf{B}\bar{\mathbf{u}}(z,s)]\} \tag{4.2.11}$$

$$\bar{\mathbf{x}}(0,s) = \mathbf{B}_0\bar{\mathbf{u}}_0(s) \tag{4.2.12}$$

This first-order ODE in the transformed variables then can be solved to yield

$$\bar{\mathbf{x}}(z, s) = \mathbf{\Phi}(z, s)\mathbf{B}_0\bar{\mathbf{u}}_0(s) - \mathbf{\Phi}(z, s)\int_0^z \mathbf{\Phi}^{-1}(r, s)\left[\mathbf{x}_0(r) + \mathbf{B}\bar{\mathbf{u}}(r, s)\right] dr \tag{4.2.13}$$

where $\mathbf{\Phi}(z, s)$ is the *fundamental matrix solution* found from the solution

$$\frac{d\mathbf{\Phi}(z, s)}{dz} = \mathbf{A}_1^{-1}(sI - \mathbf{A}_0)\mathbf{\Phi}(z, s) \tag{4.2.14}$$

$$\mathbf{\Phi}(0, s) = \mathbf{I} \tag{4.2.15}$$

Thus Eq. (4.2.13) is a linear input-output relation

$$\bar{\mathbf{x}}(z, s) = \mathbf{G}_0\bar{\mathbf{u}}_0(s) + \int_0^z \mathbf{G}(z, r, s)\bar{\mathbf{u}}(r, s)\, dr + \int_0^z \mathbf{G}_i(z, r, s)\mathbf{x}_0(r)\, dr \tag{4.2.16}$$

where

$$\begin{aligned} \mathbf{G}_0 &= \mathbf{\Phi}(z, s)\mathbf{B}_0 \\ \mathbf{G}(z, r, s) &= -\mathbf{\Phi}(z, s)\mathbf{\Phi}^{-1}(r, s)\mathbf{B} \\ \mathbf{G}_i(z, r, s) &= -\mathbf{\Phi}(z, s)\mathbf{\Phi}^{-1}(r, s) \end{aligned} \tag{4.2.17}$$

In the event that the control action is applied at a discrete number of points z_i or is independent of z, then Eq. (4.2.16) loses the integral signs and simple transfer functions arise.

A third approach to first-order hyperbolic PDE systems is through the *method of characteristics*. To illustrate this technique, let us assume that $\mathbf{A}_1$ in Eq. (4.2.1) has the simple form

$$\mathbf{A}_1 = -a\mathbf{I} \tag{4.2.18}$$

Thus Eq. (4.2.1) becomes

$$\frac{\partial \mathbf{x}(z, t)}{\partial t} + a\frac{\partial \mathbf{x}}{\partial z} = \mathbf{A}_0\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4.2.19}$$

$$\mathbf{x}(0, t) = \mathbf{B}_0\mathbf{u}_0(t) \tag{4.2.2}$$

Then by defining lines given by

$$\frac{dz}{dt} = a \tag{4.2.20}$$

or

$$t - \frac{1}{a}z = \text{const} \tag{4.2.21}$$

one obtains the solution of Eq. (4.2.19) as

$$\left.\frac{d\mathbf{x}}{dt}\right|_0 = \mathbf{A}_0\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4.2.22}$$

where the notation $|_0$ denotes the fact that the solution is taken along a characteristic line defined by Eq. (4.2.21). Therefore the repeated solution of

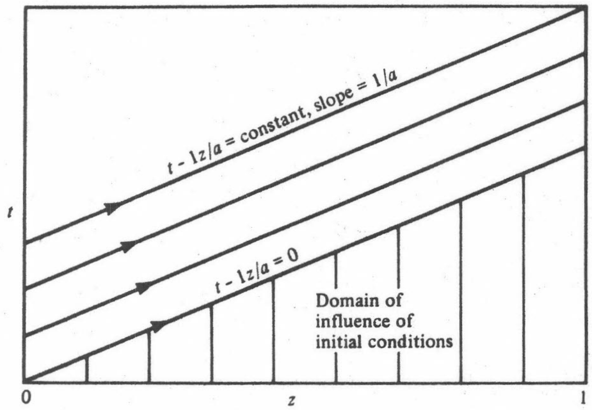

Figure 4.4 Characteristic lines of constant slope.

Eq.(4.2.22) at different values of t_0 will give the entire solution (see Fig. 4.4). If $\mathbf{A}_1$ is not of the form of Eq. (4.2.18), there will be characteristic lines for each element of $\mathbf{x}$, but the procedure can still be used.

The method of characteristics allows one to see several things very clearly. First, the initial conditions only have the domain of influence (shown in Fig. 4.4) below the line $t - z/a = 0$. For times greater than this, only the inlet conditions $\mathbf{x}(0, t)$ and the controls influence the solution. As a second point, note that solving the equations along a characteristic line corresponds to following the changes in an element of material moving from 0 to 1 with velocity a. Also note that discontinuities in the state variables arise in first-order hyperbolic systems when step changes are made at the boundary $z = 0$.

We shall now present an example problem to illustrate the methods discussed.

Example 4.2.1 Let us consider the feedback control of the steam-jacketed tubular heat exchanger shown in Fig. 4.5. Thermocouples measure the tube fluid temperature at four points, $T(0.25, t)$, $T(0.5, t)$, $T(0.75, t)$, and $T(1, t)$. These are used to determine the adjustment in the steam-jacket temperature $T_w(t)$ (through a steam inlet valve) in order to control the exchanger.

The mathematical model for the process takes the form

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z} = \frac{-hA}{\rho C_p}(T - T_w) \qquad T(0, t) = T_f \tag{4.2.23}$$

Now if we define the deviation variables

$$x = T - T_d(z) \qquad u = T_w - T_{wd} \qquad x_f = T_f - T_{f_d}$$

and parameters

$$a_0 = \frac{hA}{\rho C_p} \qquad a_1 = -v$$

where $T_d(z)$ is the desired temperature profile, T_{wd} is the steady-state steam-jacket temperature required to keep $T = T_d(z)$, and T_{f_d} is the nominal

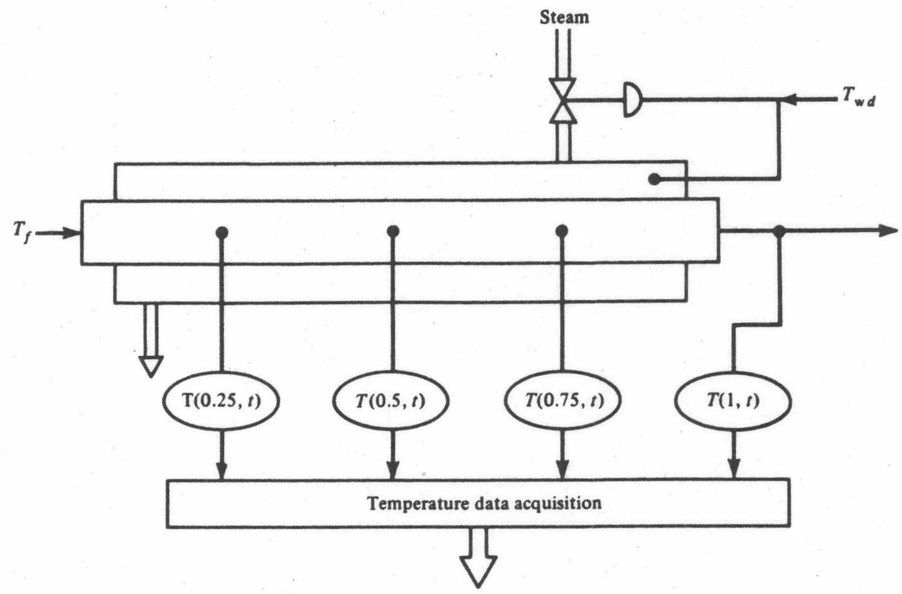

Figure 4.5 Control of a tubular heat exchanger with steam-jacket temperature control.

heat exchanger inlet temperature, then T_{wd}, T_d, and T_{f_d} satisfy

$$v\frac{\partial T_d}{\partial z} = \frac{-hA}{\rho C_p}(T_d - T_{wd}) \qquad T_d(0) = T_{f_d} \tag{4.2.24}$$

and the heat exchanger model is

$$\frac{\partial x}{\partial t} = a_1\frac{\partial x}{\partial z} - a_0 x + a_0 u \qquad x(0, t) = x_f(t) \tag{4.2.25}$$

For this system, we wish to design a feedback controller which measures the temperatures at four points, $T(0.25, t)$, $T(0.5, t)$, $T(0.75, t)$, and $T(1, t)$, and adjusts the steam-jacket temperature $T_w(t)$ to control the outlet temperature $T(1, t)$. The measured output variables are given by

$$\mathbf{y}(t) = \int_0^1 \mathbf{C}(r, t)x(r, t)\,dr \tag{4.2.26}$$

$$\mathbf{y} = \begin{bmatrix} x(0.25, t) \\ x(0.5, t) \\ x(0.75, t) \\ x(1, t) \end{bmatrix} \tag{4.2.27}$$

and

$$\mathbf{C}(r, t) = \begin{bmatrix} \delta(r - 0.25) \\ \delta(r - 0.5) \\ \delta(r - 0.75) \\ \delta(r - 1.0) \end{bmatrix} \tag{4.2.28}$$

Here $\delta(x)$ is the Dirac delta function. Let us now apply the *Laplace transform in time* to Eq. (4.2.25) to yield

$$s\bar{x}(z, s) - x_0(z) - a_1 \frac{d\bar{x}(z, s)}{dz} = -a_0\bar{x}(z, s) + a_0\bar{u}(s)$$

or, applying Eq. (4.2.16),

$$\bar{x}(z, s) = G(z, s)\bar{u}(s) + \int_0^z G_i(z, r, s)x_0(r)\,dr + G_0(z, s)\bar{x}_f(s) \tag{4.2.29}$$

where

$$G_0(z, s) = \exp\left[\frac{(s + a_0)z}{a_1}\right]$$

$$\begin{aligned} G(z, s) &= -\exp\left[\frac{(s + a_0)z}{a_1}\right]\left\{\int_0^z \exp\left[\frac{-(s + a_0)r}{a_1}\right] dr\right\}\frac{a_0}{a_1} \\ &= \frac{a_0}{s + a_0}\left[1 - \exp\left[\frac{(s + a_0)z}{a_1}\right]\right] \end{aligned} \tag{4.2.30}$$

$$G_i(z, r, s) = -\frac{1}{a_1}\exp\left[\left(\frac{s + a_0}{a_1}\right)(z - r)\right]$$

Thus one has a transfer function representation, and we may now use the design procedures of Chap. 3. Generally speaking, our control law should have the form

$$\bar{u}(s) = \mathbf{G}_c(s)\boldsymbol{\varepsilon}(s) \tag{4.2.31}$$

relating control action to our measured output variables. For a proportional controller this would take the form

$$\bar{u}(s) = \mathbf{K}(\mathbf{y} - \mathbf{y}_d) \tag{4.2.32}$$

where the designer must choose the individual components of $\mathbf{K}$. Such a controller structure is sketched in Fig. 4.6 for the case where $x_0(r) = 0$ and

$$\mathbf{K} = (K_{1/4}, K_{1/2}, K_{3/4}, K_1) \tag{4.2.33}$$

Let us note a few things about the transfer function for the heat exchanger. Assuming $x_f = 0$, $x_0(z) = 0$, let us look at the response of $x(1, t)$ to steam-jacket temperature

$$\bar{x}(1, s) = G(1, s)\bar{u}(s) \tag{4.2.34}$$

where the transfer function is

$$G(1, s) = \frac{a_0}{s + a_0}(1 - e^{a_0/a_1}e^{+s/a_1}) \tag{4.2.35}$$

Here there is a pure time delay* of magnitude $-1/a_1 = 1/v$ appearing in

* Recall that $a_1 = -v$.

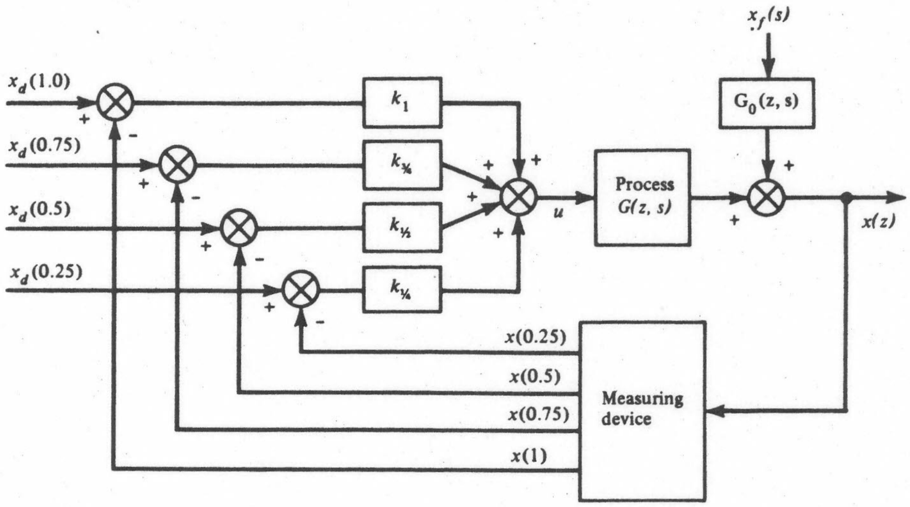

Figure 4.6 Proportional feedback control scheme for a tubular heat exchanger.

the transfer function. We shall discuss such delay problems further in Sec. 4.5.

Let us now consider the conditions for *controllability* and *stabilizability* for first-order hyperbolic partial differential equations. This question is much more complicated than for lumped parameter systems [8, 9], and complete results are not yet available. However, the basic requirement for controllability of first-order hyperbolic equations is that a control actuator intersect each characteristic line (see Fig. 4.4) and that a controllability condition along these characteristics be satisfied.

Second-Order Partial Differential Equations

Linear second-order PDEs can be classified according to values of the coefficients of the highest derivatives.* The linear scalar second-order PDE in variables t, z takes the form

$$a_{11}x_{tt} + 2a_{12}x_{tz} + a_{22}x_{zz} = F(x_t, x_z, x, t, z) \tag{4.2.36}$$

where the a_{ij} may be functions of t, z. One may form a *characteristic equation* by replacing all z derivatives on the left-hand side by $(-\lambda)$. Then one obtains

$$a_{11} - 2a_{12}\lambda + a_{22}\lambda^2 = 0 \tag{4.2.37}$$

which has roots

$$\lambda_1, \lambda_2 = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{22}} \tag{4.2.38}$$

* A particularly readable description of this may be found in [7, 10].

Table 4.1

Δ	Type	Characteristics
$\Delta < 0$	Elliptic	Complex
$\Delta = 0$	Parabolic	$\lambda_1 = \lambda_2$, real
$\Delta > 0$	Hyperbolic	$\lambda_1 \neq \lambda_2$, real

Now depending on the nature of the roots, the PDE will be *hyperbolic*, *parabolic*, or *elliptic*. If we recall that the nature of the roots is determined by the sign of the discriminant

$$\Delta = a_{12}^2 - a_{11}a_{22} \tag{4.2.39}$$

we are let to the classifications in Table 4.1.

These classifications also apply to first-order PDEs such as

$$a_1 x_t + a_2 x_z = F(\bar{x}, t, z) \tag{4.2.40}$$

which has the characteristic equation

$$a_1 - a_2\lambda = 0 \tag{4.2.41}$$

yielding a unique, real value for λ. Thus *all first-order equations are hyperbolic*. In addition, *all systems of first-order equations are hyperbolic*,* and *all second-order hyperbolic equations can be reduced to systems of first-order equations*. We have seen examples of first-order hyperbolic equations in the previous section.

Second-order hyperbolic equations arise in wave propagation problems. For example, the propagation of sound is modeled by

$$\frac{\partial^2 \zeta}{\partial t^2} = v_s^2 \frac{\partial^2 \zeta}{\partial z^2} \tag{4.2.42}$$

where v_s is the speed of sound and ζ is the sound amplitude. The characteristics of Eq. (4.2.42) are found from

$$v_s^2 \lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{v_s} \tag{4.2.43}$$

Thus there are two sets of characteristic lines in the t, z space for Eq. (4.2.42), one with slope $1/v_s$ and another with slope $-1/v_s$. These lines, shown in Fig. 4.7, represent the motion of sound waves being reflected from the boundaries. By making the substitution

$$x_1 = \frac{\partial \zeta}{\partial t}$$

$$x_2 = v_s \frac{\partial \zeta}{\partial z} \tag{4.2.44}$$

* The only exception is if a_2/a_1 is the same in each equation, in which case the roots of the characteristic equation (4.2.41) are identical, and the system becomes *parabolic*.

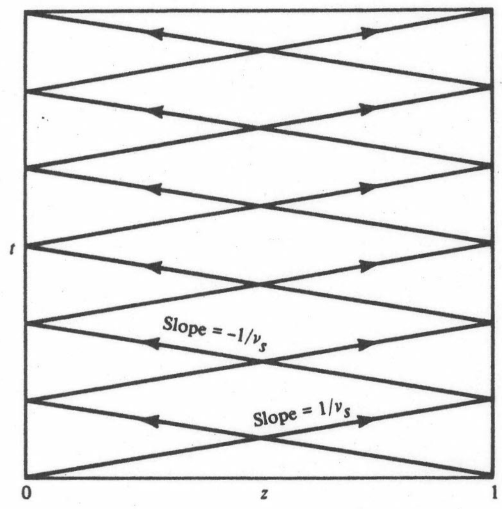

Figure 4.7 Characteristic lines for the sound propagation equation.

we can convert Eq. (4.2.42) to the first-order system

$$\frac{\partial x_1}{\partial t} + v_s \frac{\partial x_1}{\partial z} = 0$$

$$\frac{\partial x_2}{\partial t} - v_s \frac{\partial x_2}{\partial z} = 0 \qquad (4.2.45)$$

Thus any higher-order hyperbolic equations can be converted to a system of first-order equations and handled by the techniques of the last section.

Parabolic equations arise in processes with diffusion or heat conduction. For example, heat conduction in a one-dimensional solid is governed by the equation

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} \qquad (4.2.46)$$

This has the characteristic equation

$$\lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0 \qquad (4.2.47)$$

We shall discuss the treatment of parabolic equations in more detail in a later section.

Elliptic equations occur in multidimensional diffusion or heat transport problems such as steady-state conduction in a two-dimensional slab:

$$k\left(\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0 \qquad (4.2.48)$$

This problem has two space variables, so it does not fit our model, Eq. (4.2.36), exactly. However, the *characteristic equation* is

$$\lambda^2 + 1 = 0 \qquad (4.2.49)$$

and λ is definitely complex. In practice, elliptic equations involving *time* and a spatial variable rarely occur because physical systems seldom (if ever) have these modeling equations.

Classifications of second-order equations involving more than two independent variables are slightly more involved [10], but can be found by straightforward tests of the coefficients of the highest derivatives.

As in the case of first-order equations, the thrust of the analysis for second-order systems is to use an *exact* reduction of a distributed system to a lumped one and to take advantage of all the theory for lumped parameter systems. There are several means of doing this.

1. *The Laplace transform in time* can be used for second-order processes just as for first-order ones. These result in transfer functions involving spatial variables, often in infinite series form. In principle the lumped parameter design techniques can be applied to these transfer functions, although in practice there are difficulties. We shall discuss these points below.
2. *The method of characteristics* can be used with all hyperbolic equations and with some parabolic equations which have nonzero characteristics.* All these systems reduce to first-order equations and can be treated by the methods of the previous section.
3. *Modal analysis* is a very attractive method of treating PDEs which have a real, discrete spectrum of eigenvalues and which can be made self-adjoint. It is the natural reduction technique and works well with only a few modes if the eigenvalues are not bunched together.

Let us now discuss in more detail some of these exact lumping techniques.

Laplace Transform Methods

We have discussed the Laplace transform technique applied to first-order equations. Let us now show the form that this representation takes for parabolic equations. To illustrate this, let us consider the heat equation in a slab.

* Parabolic equations with nonzero characteristics are really degenerate cases of hyperbolic problems. For example, the parabolic system

$$\frac{\partial^2 x_1}{\partial t^2} + 2v\frac{\partial^2 x_1}{\partial t\,\partial z} = v^2\frac{\partial^2 x_1}{\partial z^2} = F \tag{4.2.50}$$

can be reduced to

$$\frac{\partial x_1}{\partial t} + v\frac{\partial x_1}{\partial z} = f_1$$

$$\frac{\partial x_2}{\partial t} + v\frac{\partial x_2}{\partial z} = f_2 \tag{4.2.51}$$

by defining x_2 appropriately. These equations could represent a tubular reactor in plug flow. For example, if $f_1 = x_2$, then the substitution

$$x_2 = \frac{\partial x_1}{\partial t} + v\frac{\partial x_1}{\partial z} \tag{4.2.52}$$

leads to Eq. (4.2.50).

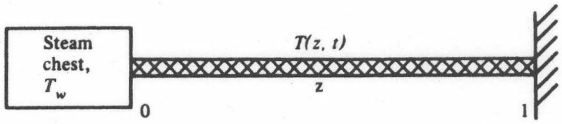

Figure 4.8 A one-dimensional heated rod.

Example 4.2.2 Let us consider the one-dimensional rod shown in Fig. 4.8. Heat is added from a steam chest at $z = 0$, and the $z = 1$ end is perfectly insulated. Let us define variables

$$x(z, t) = T - T_d$$
$$u(t) = T_w - T_{wd}$$

which represent deviations from the set-point values. In this case the model takes the form

$$\frac{\partial x(z, t)}{\partial t} = \frac{\partial^2 x(z, t)}{\partial z^2} \tag{4.2.53}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = \beta(x - u) \tag{4.2.54}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.2.55}$$

If one takes the Laplace transform with respect to time and assumes $x(z, 0) = 0$, then

$$s\bar{x}(z, s) = \frac{d^2\bar{x}}{dz^2} \tag{4.2.56}$$

$$z = 0 \qquad \frac{d\bar{x}}{dz} = \beta(\bar{x} - \bar{u}) \tag{4.2.57}$$

$$z = 1 \qquad \frac{d\bar{x}}{dz} = 0 \tag{4.2.58}$$

Equation (4.2.56) has the general solution

$$\bar{x}(z, s) = A \sinh\sqrt{s}\, z + B \cosh\sqrt{s}\, z \tag{4.2.59}$$

and the boundary condition of Eq. (4.2.58) yields

$$A + B \tanh\sqrt{s} = 0 \tag{4.2.60}$$

and Eq. (4.2.57) yields

$$\sqrt{s}\, A = \beta[B - \bar{u}(s)] \tag{4.2.61}$$

or

$$B = \frac{(\beta/\sqrt{s})\bar{u}(s)}{(\beta/\sqrt{s}) + \tanh\sqrt{s}} \tag{4.2.62}$$

Thus the solution is

$$\bar{x}(z, s) = \frac{\bar{u}(s)}{1 + (\sqrt{s}/\beta)\tanh\sqrt{s}}(-\tanh\sqrt{s}\,\sinh\sqrt{s}\,z + \cosh\sqrt{s}\,z) \tag{4.2.63}$$

Now as an example let us consider the control of the left-hand-side end temperature $x(0, t)$ by the steam-chest temperature $u(t)$. The system transfer function takes the form

$$\bar{x}(0, s) = \frac{\bar{u}(s)}{1 + (\sqrt{s}/\beta)\tanh\sqrt{s}} \tag{4.2.64}$$

and in principle, standard lumped parameter system controller design techniques may be used. However, there is no simple inversion of this complex transfer function. In fact, Eq. (4.2.64) can be expanded in an infinite series to yield

$$\bar{x}(z, s) = \bar{u}(s)\sum_{i=1}^{\infty}\frac{a_i}{s - \lambda_i} \tag{4.2.65}$$

where the λ_i are an infinite series of eigenvalues arising from the roots of the denominator of Eq. (4.2.64). The coefficients a_i are given by

$$a_i = \frac{2\beta\sqrt{\lambda_i}\cosh^2\sqrt{\lambda_i}}{\sqrt{\lambda_i} + \sinh\sqrt{\lambda_i}\cosh\sqrt{\lambda_i}} = \frac{2\beta\lambda_i}{\lambda_i - \beta(\beta + 1)} \tag{4.2.66}$$

Thus we see that for parabolic PDEs one obtains transcendental transfer functions which in general must be expanded into an infinite series of exponentials. However, if this series can be approximated by the first few terms, then normal lumped parameter transfer function design techniques may be directly applied.

Modal Analysis

A convenient and useful form of analysis of second-order equations is through modal decomposition. This form of analysis is possible when the second-order equation

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{Ax} + \mathbf{Bu} \tag{4.2.67}$$

has a spatial operator $\mathbf{A}$ which can be made self-adjoint and which has a real, discrete spectrum of eigenvalues. For example, in one dimension the operator

$$\mathbf{Ax}(z, t) = \mathbf{A}_2(z)\frac{\partial^2 \mathbf{x}}{\partial z^2} + \mathbf{A}_1(z)\frac{\partial \mathbf{x}}{\partial z} + \mathbf{A}_0(z)\mathbf{x} \tag{4.2.68}$$

would lead to a parabolic set of equations. It is also possible to extend these ideas to two space dimensions. For example,

$$\mathbf{A}\mathbf{x}(z, r, t) = \mathbf{A}_2(z)\frac{\partial^2 \mathbf{x}}{\partial z^2} + \mathbf{A}_1(z)\frac{\partial \mathbf{x}}{\partial z} + \mathbf{A}_0(z)\mathbf{x} + \mathbf{D}_2(r)\frac{\partial^2 \mathbf{x}}{\partial r^2} + \mathbf{D}_1(r)\frac{\partial \mathbf{x}}{\partial r} \tag{4.2.69}$$

would be one possible two-dimensional operator amenable to modal analysis. A wider discussion of these techniques may be found in Refs. [11–16].

Perhaps the best means of discussing the modal reduction of distributed systems is by considering a series of example problems. Let us begin by studying the control of the temperature distribution in a long, thin rod being heated in a multizone furnace, and shown in Fig. 4.9. The heating rate is defined as $q(z', t')$, and the modeling equation becomes

$$\rho C_p \frac{\partial T}{\partial t'} = k \frac{\partial^2 T}{\partial z'^2} + q(z', t') \qquad \begin{array}{l} t' > 0 \\ 0 < z' < l \end{array}$$

If one assumes negligible heat flux at the ends of the rod, the boundary conditions become

$$\frac{\partial T}{\partial z'} = 0 \qquad z' = 0, l$$

By putting the variables in dimensionless form,

$$t = \frac{t'k}{\rho C_p l^2} \qquad z = \frac{z'}{l} \qquad u(z, t) = \frac{q(z', t')l^2}{kT_0} \qquad x(z, t) = \frac{T(z, t)}{T_0}$$

where T_0 is some reference initial temperature, such as $T(0, 0)$, one obtains the equation

$$\frac{\partial x(z, t)}{\partial t} = \frac{\partial^2 x(z, t)}{\partial z^2} + u(z, t) \qquad \begin{array}{l} t > 0 \\ 0 < z < 1 \end{array} \tag{4.2.70}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.2.71}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.2.72}$$

$$x(z, 0) = x_0(z) \tag{4.2.73}$$

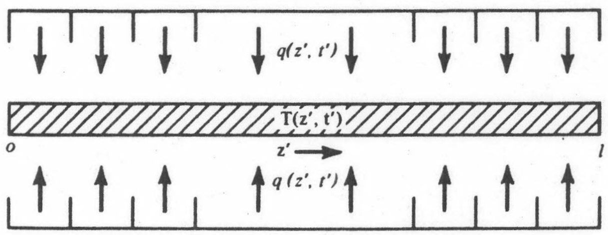

Figure 4.9 A long thin rod being heated in a multizone furnace.

Note that Eq. (4.2.70) is separable and can be treated by the technique of separation of variables. Thus we assume a solution of the form

$$x(z, t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(z) \tag{4.2.74}$$

where $a_n(t)$, $\phi_n(z)$ are a set of functions to be determined. We also assume that $u(z, t)$ can be represented in a separable fashion with the same functions $\phi_n(z)$.

$$u(z, t) = \sum_{n=0}^{\infty} b_n(t)\phi_n(z) \tag{4.2.75}$$

This will always be possible if $\phi_n(z)$, $n = 0, 1, \ldots,$ represent a complete set of basis functions.

Substituting Eqs. (4.2.74) and (4.2.75) into Eqs. (4.2.70) to (4.2.72) yields

$$\phi_n(z)\frac{da_n(t)}{dt} = a_n(t)\frac{d^2\phi_n(z)}{dz^2} + b_n(t)\phi_n(z) \qquad n = 0, 1, 2, \ldots \tag{4.2.76}$$

$$z = 0 \qquad \frac{d\phi_n}{dz} = 0 \tag{4.2.77}$$

$$z = 1 \qquad \frac{d\phi_n}{dz} = 0 \tag{4.2.78}$$

Dividing Eq. (4.2.76) by $a_n(t)\phi_n(z)$ produces

$$\frac{1}{a_n}\frac{da_n}{dt} = \frac{1}{\phi_n}\frac{d^2\phi_n}{dz^2} + \frac{b_n(t)}{a_n(t)}$$

which may be separated into only functions of t and only functions of z as follows:

$$\frac{1}{a_n}\frac{da_n}{dt} - \frac{b_n}{a_n} = -\lambda_n \tag{4.2.79}$$

$$\frac{1}{\phi_n}\frac{d^2\phi_n}{dz^2} = -\lambda_n \tag{4.2.80}$$

where λ_n is a constant. Let us now rewrite these as

$$\frac{da_n}{dt} + \lambda_n a_n = b_n(t) \qquad n = 0, 1, 2, \ldots \tag{4.2.81}$$

$$\frac{d^2\phi_n}{dz^2} + \lambda_n\phi_n = 0 \qquad n = 0, 1, 2, \ldots \tag{4.2.82}$$

Clearly, Eq. (4.2.81) allows us to calculate $a_n(t)$ if $b_n(t)$ and λ_n are known. Equation (4.2.82) is a *self-adjoint* differential equation. Let us now take a short excursion and discuss some basic concepts of differential equations.

The second-order differential operator defined over $0 < z < 1$

$$L(\cdot) = a_2(z)\frac{d^2(\cdot)}{dz^2} + a_1(z)\frac{d(\cdot)}{dz} + a_0(z)(\cdot) \tag{4.2.83}$$

has an adjoint operator $L^*(\cdot)$ defined [16, 17] so that for any two functions $y(z)$, $w(z)$, the relation

$$\int_0^1 y(z)L(w(z))\,dz = \int_0^1 w(z)L^*(y(z))\,dz \tag{4.2.84}$$

holds. For the differential operator, Eq. (4.2.83), the adjoint operator so defined is

$$L^*(\cdot) = \frac{d^2(a_2(z)(\cdot))}{dz^2} - \frac{d(a_1(z)(\cdot))}{dz} + a_0(z)(\cdot) \tag{4.2.85}$$

Now an operator which is identical to its adjoint is termed *self-adjoint*. Self-adjoint operators have some very nice properties, as we shall see shortly; thus, very often it is useful to put equations into self-adjoint form. For Eq. (4.2.83), this amounts to a change of variable [17]. If we define

$$r(z) = \exp\left[\int \frac{1}{a_2}(a_1 - \dot{a}_2)\,dz\right] \tag{4.2.86}$$

and change variables as

$$\hat{\Phi}_n = \Phi_n r(z) \tag{4.2.87}$$

then the operator $L\hat{\Phi}(z)$ in Eq. (4.2.83) will become self-adjoint.

Having established these concepts, let us now continue with the rod heat conduction problem. Recall that Eq. (4.2.82) is already self-adjoint and, together with the boundary condition, Eq. (4.2.77), yields the solution

$$\phi_n(z) = A_n \cos\sqrt{\lambda_n}\, z \tag{4.2.88}$$

Application of the boundary condition, Eq. (4.2.77), yields the condition

$$\sqrt{\lambda_n}\,\sin\sqrt{\lambda_n} = 0 \tag{4.2.89}$$

The only possible solutions to this are

$$\sqrt{\lambda_n} = n\pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

or

$$\lambda_n = n^2\pi^2 \qquad n = 0, 1, 2, \ldots \tag{4.2.90}$$

Here the λ_n are the *eigenvalues* of the system and the ϕ_n are the *eigenfunctions* or *modes* of the system. Because Eq. (4.2.82) is a homogeneous *self-adjoint* differential equation with homogeneous boundary conditions, the eigenfunctions are *orthogonal*. That is,

$$\int_0^1 \phi_n(z)\phi_m(z)\,dz = 0 \qquad \text{for } n \neq m \tag{4.2.91}$$

It is useful to choose the constant A_n in Eq. (4.2.88) to make the eigenfunctions *orthonormal*, i.e.,

$$\int_0^1 \phi_n(z)^2\,dz = 1 \tag{4.2.92}$$

To do this we simply substitute Eq. (4.2.88) into Eq. (4.2.91) to obtain

$$A_n^2 = \left[\int_0^1 \left(\cos\sqrt{\lambda_n}\, z\right)^2 dz\right]^{-1}$$

or

$$A_n = \begin{cases} 1 & n = 0 \\ \sqrt{2} & n = 1, 2, \ldots \end{cases} \tag{4.2.93}$$

Now due to the orthogonality of the eigenfunctions, one can immediately write down the following relationships:

$$\int_0^1 \phi_m(z)x(z, t)\, dz = \sum_{n=0}^{\infty} a_n(t)\int_0^1 \phi_n(z)\phi_m(z)\, dz$$
$$= a_m(t) \tag{4.2.94}$$

Thus, given any temperature distribution $x(z, t)$, it is possible to immediately determine the eigencoefficient $a_n(t)$. In particular, we can immediately represent the initial conditions $x_0(z)$ in the form of Eq. (4.2.74) by determining the coefficients

$$a_n(0) = \int_0^1 \phi_n(z)x_0(z)\, dz \tag{4.2.95}$$

By similar equations the coefficients $b_n(t)$ for the series representation of $u(z, t)$ in Eq. (4.2.75) are given by

$$b_n(t) = \int_0^1 \phi_n(z)u(z, t)\, dz \tag{4.2.96}$$

Thus the temperature distribution $x(z, t)$ resulting from some heat flux distribution $u(x, t)$ is given by the expression

$$x(z, t) = a_0(t) + \sqrt{2}\sum_{n=1}^{N} a_n(t)\cos n\pi z$$

Here $a_n(t)$ is determined from the solution of

$$\frac{da_n}{dt} = -n^2\pi^2 a_n + b_n(t)$$

where $a_n(0)$ is given by Eq. (4.2.95) and $b_n(t)$ by Eq. (4.2.96). The quantity N is the actual number of terms in the eigenfunction expansion necessary to provide a good approximation to the exact solution.

It is possible to use this modal representation in several ways.

1. *Simulation* The modal representation is a very efficient means of simulating the process when there is time-varying control action. Because the eigenvalues $\lambda_n = n^2\pi^2$ increase rapidly with increasing n, only a few eigenfunctions N are required for representing the the system behavior.

   For simulation, only N ordinary differential equations of the form of Eq. (4.2.81) must be solved sequentially (not simultaneously) for $a_n(t)$. In practice, $N = 2$ or 3 is often found to suffice, so that there is very little computational effort involved in simulation. As an example of the form of this solution, in the case where the heating rate $u(z, t)$ is constant in time (but possibly spatially varying), the problem in Eqs. (4.2.70) to (4.2.73) may be solved analytically to yield

$$a_n(t) = e^{-n^2\pi^2 t} a_n(0) + \frac{b_n}{n^2\pi^2}(1 - e^{-n^2\pi^2 t}) \qquad n = 0, 1, \ldots N$$

and

$$x(z, t) = a_0(0) + \sqrt{2} \sum_{n=1}^{N} \left[e^{-n^2\pi^2 t} a_n(0) + \frac{b_n}{n^2\pi^2}(1 - e^{-n^2\pi^2 t}) \right] \cos n\pi z$$

2. *Control* A second valuable use of modal decomposition is in the design of control structures. These applications have been discussed by Gilles [11], Gould [12], Wang [13], and Ajinkya et al. [15]. Let us consider the controller structure in Fig. 4.10, where we assume state variable outputs.

   The control $u(z, t)$ is applied to the plant, yielding the state $x(z, t)$. The actual state $x(z, t)$ and the desired state $x_d(z, t)$ are compared and the error fed to a *modal analyzer* consisting of Eq. (4.2.94). The resulting coefficients of the error signal

$$\varepsilon_n = a_{nd} - a_n \qquad n = 0, 1, \ldots, N$$

   are fed to an $N + 1$ lumped parameter variable feedback control scheme. The outputs of this are the controller coefficients $b_n(t)$. These are then fed to a modal synthesizer consisting of Eq. (4.2.75). This produces the control signal $u(z, t)$ which is fed to the plant. Notice that the multivariable controller for this linear problem consists of $N + 1$ single-loop controllers. This is because *there are no interactions* in the modal formulation for linear problems, i.e., the coefficient b_n only influences coefficient a_n.

In principle, this control scheme requires that the complete state $x(z, t)$ must be available as an output. In practice, of course, this is impossible. However, one can provide this information in several ways.

1. One can measure $x(z_i, t)$, $i = 1, 2, \ldots l$, at a large number of points and simply smooth these data to get $x(z, t)$.
2. One can measure $x(z_i, t)$ at only a few points (possibly only one) and use a

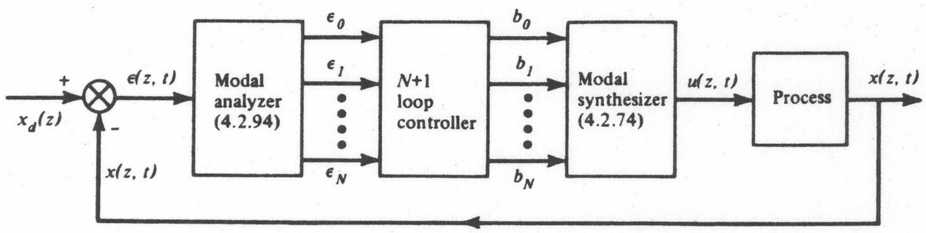

Figure 4.10 A distributed parameter modal feedback controller for distributed control $u(z, t)$.

state estimator to provide estimates of $x(z, t)$ (see Chap. 5 for a discussion of state estimation).

3. One can use a technique suggested by Gould [12] and replace Eq. (4.2.94) in the modal analyzer by the scheme described below.

First, measure $x(z_i, t)$ at $N + 1$ spatial positions; then

$$x(z_i, t) = \sum_{n=0}^{N} a_n(t)\phi_n(z_i) \qquad i = 1, 2, \ldots, N + 1 \tag{4.2.97}$$

If we define

$$\mathbf{x} = \begin{bmatrix} x(z_1, t) \\ x(z_2, t) \\ \vdots \\ x(z_{N+1}, t) \end{bmatrix} \qquad \mathbf{\Phi} = \begin{bmatrix} \phi_0(z_1) & \phi_0(z_2) & \cdots & \phi_0(z_{N+1}) \\ \phi_1(z_1) & \phi_1(z_2) & \cdots & \phi_1(z_{N+1}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N(z_1) & \cdots & \cdots & \phi_N(z_{N+1}) \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}$$

then Eq. (4.2.97) becomes

$$\mathbf{x} = \mathbf{\Phi a} \tag{4.2.98}$$

and if the sampling locations are well chosen, $\mathbf{\Phi}$ will not be singular and

$$\mathbf{a} = \mathbf{\Phi}^{-1}\mathbf{x} \tag{4.2.99}$$

This relation may be used in place of Eq. (4.2.94) in the modal analyzer. If measurements at more than $N + 1$ spatial positions are available, then one could use a least squares fit for $\mathbf{a}$ using

$$\mathbf{a} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{x}$$

in place of Eq. (4.2.99).

Schemes 1 and 2 have been tested experimentally and found to work well, but no experimental testing of scheme 3 seems to have been performed.

Example 4.2.3 Let us illustrate the application of modal feedback control by applying a proportional plus integral modal controller to the rod-heating problem. This means that in Eq. (4.2.81) the control law

$$b_n(t) = K_n\left(\varepsilon_n + \frac{1}{\tau_{I_n}}\int \varepsilon_n \, dt\right) \qquad n = 0, 1, 2, \ldots, N$$

would be applied to the Fourier coefficients, where

$$\varepsilon_n = a_{nd} - a_n \qquad n = 0, 1, 2, \ldots, N$$

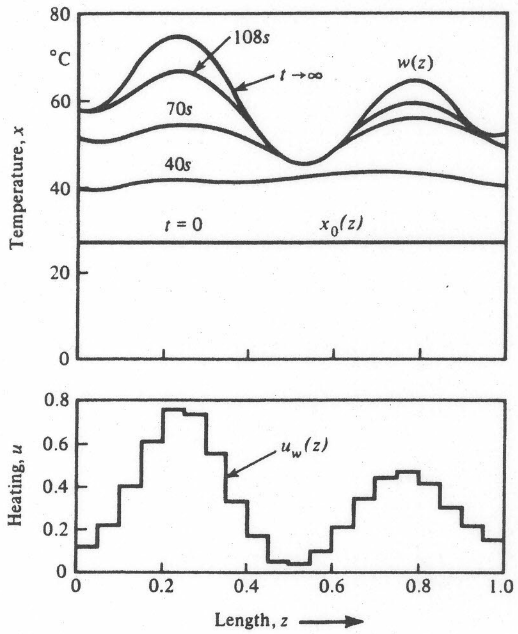

Figure 4.11 Experimental results of modal feedback PI control of a heated billet in a multizone furnace [18]. (*Reproduced by permission of Oldenbourg Verlag, G.M.B.H.*)

Mäder [18] has applied this control law experimentally to such a problem using the controller structure shown in Fig. 4.10. Some of his experimental results are shown in Fig. 4.11, where the modal PI controller takes the metal temperature from $x_0(z)$ at $t = 0$ to the desired profile $w(z)$ very quickly. He found that six eigenfunctions $\phi_n(z)$ were sufficient to provide a good representation of both the state and control variables.

Sometimes the control appearing in the differential equation will depend only on time, i.e., $u(t)$. In this case, the modal feedback control scheme shown in Fig. 4.10 must be modified to the form shown in Fig. 4.12, where a single controller of the form

$$u(t) = g(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N) \tag{4.2.100}$$

must be determined. There are many different ways of determining this control law; however, one simple way is to drive the instantaneous integral squared deviation

$$\varepsilon(t) = \int_0^1 [x_d(z) - x(z, t)]^2 \, dz \tag{4.2.101}$$

toward zero. Expanding Eq. (4.2.101) in the modal expansion, one obtains

$$\varepsilon(t) = \int_0^1 \left(\sum_{n=0}^{N} \varepsilon_n(t)\phi_n(z) \right)^2 dz \tag{4.2.102}$$

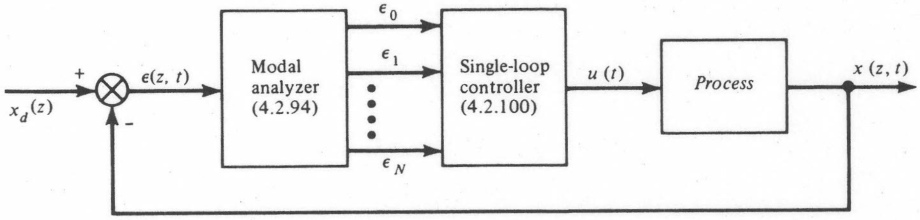

Figure 4.12 A distributed parameter modal feedback controller for time dependent control $u(t)$.

or by orthogonality,

$$\varepsilon(t) = \sum_{n=0}^{N} \varepsilon_n^2(t) \tag{4.2.103}$$

Hence a possible design would be a PI controller of the form

$$u(t) = K\left[\varepsilon(t) + \frac{1}{\tau_I}\int \varepsilon(t)\, dt\right] \tag{4.2.104}$$

Thus making $\varepsilon(t)$ as small as possible will cause $x(z, t)$ to approach $x_d(z)$ with minimal error in the least squares sense.

Let us now consider another example problem in which the differential operator is *non-self-adjoint* and the control is applied at the boundary. We shall discuss the control of a cylindrical ingot being heated in a furnace as shown in Fig. 4.13. We shall assume the top and bottom of the ingot receive negligible heating and the heating rate at the surface is uniform and may be controlled as a function of time. In addition, axial variations in temperature are neglected, and the problem is described by the one-dimensional, cylindrical heat equation

$$\rho C_p \frac{\partial T}{\partial t'} = \frac{k}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial T}{\partial r'}\right) \qquad \begin{matrix} 0 \le r' \le R \\ t' > 0 \end{matrix} \tag{4.2.105}$$

with boundary conditions

$r' = 0$:
$$\frac{\partial T}{\partial r'} = 0 \tag{4.2.106}$$

$r' = R$:
$$k\frac{\partial T}{\partial r'} = q(t') \tag{4.2.107}$$

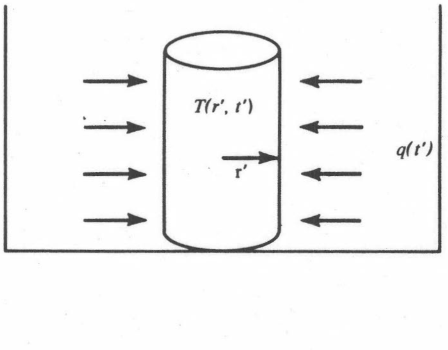

Figure 4.13 A cylindrical ingot being heated in a furnace.

The control problem is to adjust $q(t')$ so that the ingot achieves the desired temperature distribution $T(r', t')$. Note, however, two complications: (1) the operator in r is *non-self-adjoint*, and (2) the control is applied at the boundary.

First let us discuss how to treat *non-self-adjoint* systems. In general, one may invoke a change of variable, such as in Eq. (4.2.86). However, for equations of the form

$$L(\cdot) = \frac{1}{\rho(z)} \frac{d}{dz}\left[p(z)\frac{d(\cdot)}{dz}\right] + q(z)(\cdot) \qquad 0 \le z \le 1 \tag{4.2.108}$$

such as arise in cylindrical and spherical diffusion and heat conduction problems, it is possible to use Sturm-Liouville theory [16, 17]. This theory allows one to state that the system

$$Lx = \lambda x \tag{4.2.109}$$

[where L is defined in Eq. (4.2.108)] coupled with homogeneous boundary conditions, will have a discrete spectrum of eigenvalues λ_n and a corresponding set of eigenfunctions $\phi_n(z)$ which are orthogonal with respect to $\rho(z)$, i.e.,

$$\int_0^1 \rho(z)\phi_n(z)\phi_m(z)\,dz = 0 \qquad n \neq m \tag{4.2.110}$$

Now let us discuss the situation when the control is applied at the boundary, as in Eq. (4.2.107). This causes the boundary conditions to become nonhomogeneous and could lead to great theoretical complications. However, it is possible to make some transformations to eliminate this problem. Generally speaking, one may introduce the boundary control into the differential equation through the use of a Dirac delta function. For a general discussion of this, see [19–21]. To illustrate this approach, let us now proceed with our cylindrical ingot heating problem. Let us put the problem in more convenient form by defining

$$x = \frac{T}{T_0} \qquad t = \frac{t'k}{\rho C_p R^2} \qquad r = \frac{r'}{R} \qquad u(t) = \frac{q(t')R}{T_0 k} \tag{4.2.111}$$

where T_0 is some reference initial temperature [for example, $T_0 = T(R, 0)$, the initial surface temperature]. With these dimensionless variables the problem takes the form

$$\frac{\partial x(r,t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x(r,t)}{\partial r}\right) \qquad \begin{matrix} 0 \le r \le 1 \\ t > 0 \end{matrix} \tag{4.2.112}$$

with boundary conditions

$$r = 0 \qquad \frac{\partial x}{\partial r} = 0 \tag{4.2.113}$$

$$r = 1 \qquad \frac{\partial x}{\partial r} = u(t) \tag{4.2.114}$$

$$t = 0 \qquad x = x_0(r) \tag{4.2.115}$$

Now it is possible to construct the solution to this problem in terms of the *Green's function* [10, 16, 17] and to show that there is an equivalent problem to

Eqs. (4.2.112) to (4.2.115) with $u(t)$ appearing on the right-hand side of the differential equation (see Olivei [21] for an example). Let us now show a less rigorous shortcut to that equivalent problem. Let us add $u(t)$ to Eq. (4.2.112) with a Dirac delta function $\delta(r-1)$ so that

$$\frac{\partial x}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x(r,t)}{\partial r}\right) + \delta(r-1)u(t) \tag{4.2.116}$$

with boundary conditions

$$r = 0 \qquad \frac{\partial x}{\partial r} = 0 \tag{4.2.113}$$

$$r = 1^+ \qquad \frac{\partial x}{\partial r} = 0 \tag{4.2.117}$$

$$t = 0 \qquad x = x_0(r) \tag{4.2.115}$$

Now we can prove that this change is rigorous by integrating Eq. (4.2.116) across the infinitesimal interval $1^- < r < 1^+$:

$$\int_{1^-}^{1^+} r\cancelto{0}{\frac{\partial x}{\partial t}}\, dr = \int_{1^-}^{1^+} \frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right) dr + \int_{1^-}^{1^+} r\,\delta(r-1)u(t)\, dr \tag{4.2.118}$$

$$0 = \frac{\partial x}{\partial r}\bigg]_{1^-}^{1^+} + u(t) \tag{4.2.119}$$

but invoking Eq. (4.2.117), we see that Eq. (4.2.119) yields

$$\frac{\partial x}{\partial r} = u(t)$$

at $r = 1^-$, so the *formulations are equivalent*.

Let us now proceed to use separation of variables to solve Eqs. (4.2.113) and (4.2.115) to (4.2.117). If one assumes a solution of the form

$$x(r,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(r) \tag{4.2.120}$$

$$\delta(r-1)u(t) = \sum_{n=0}^{\infty} b_n(t)\phi_n(r) \tag{4.2.121}$$

and substitutes into Eqs. (4.2.113), and (4.2.115) to (4.2.117), the equations become

$$\phi_n(r)\frac{da_n}{dt} = \frac{a_n(t)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_n}{\partial r}\right) + b_n(t)\phi_n(r) \qquad n = 0, 1, 2, \ldots \tag{4.2.122}$$

with boundary conditions

$$r=0 \qquad \frac{d\phi_n}{dr} = 0 \tag{4.2.123}$$

$$r=1 \qquad \frac{d\phi_n}{dr} = 0 \tag{4.2.124}$$

By separation of variables we are led to the eigenvalue problem (where we choose the separation constant $-\lambda_n$ for convenience)

$$\frac{da_n}{dt} + \lambda_n a_n = b_n \qquad n = 0, 1, \ldots \tag{4.2.125}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_n}{dr}\right) + \lambda_n \phi_n = 0 \qquad n = 0, 1, \ldots \tag{4.2.126}$$

Now Eq. (4.2.126) is *non-self-adjoint*; however, it is in the Sturm-Liouville form [Eq. (4.2.108)], and Sturm-Liouville theory tells us that

$$\int_0^1 r\phi_n(r)\phi_m(r)\, dr = 0 \qquad n \neq m \tag{4.2.127}$$

so that $\phi_n^* = r\phi_n(r)$ are the eigenfunctions of the adjoint equation to Eq. (4.2.126), that is, of

$$\frac{d^2\phi_n^*}{dr^2} - \frac{1}{r}\frac{d\phi_n^*}{dr} + \left(\frac{1}{r^2} + \lambda_n\right)\phi_n^* = 0 \tag{4.2.128}$$

because in general the eigenfunctions of the operator and its adjoint operator are orthogonal,

$$\int_0^1 \phi_n^*(r)\phi_m(r)\, dr = 0 \qquad n \neq m \tag{4.2.129}$$

With this as background, let us proceed to solve Eq. (4.2.126) with the boundary conditions of Eqs. (4.2.123) and (4.2.124). Equation (4.2.126) can be put in the form of Bessel's equation [22],

$$r^2\frac{d^2\phi_n}{dr^2} + r\frac{d\phi_n}{dr} + r^2\lambda_n\phi_n = 0 \tag{4.2.130}$$

which has the general solution

$$\phi_n = A_n J_0\left(\sqrt{\lambda_n}\, r\right) + B_n Y_0\left(\sqrt{\lambda_n}\, r\right) \tag{4.2.131}$$

where $J_n(y)$ is a Bessel function of the first kind and of nth order and $Y_n(y)$ is a Bessel function of the second kind and nth order. If we apply the boundary condition of Eq. (4.2.123), we obtain

$$r = 0 \qquad \frac{d\phi_n}{dr} = -A_n\sqrt{\lambda_n}\, \cancel{J_1(0)}^{\,0} - B_n\sqrt{\lambda_n}\, Y_1(0) = 0 \tag{4.2.132}$$

which requires $B_n = 0$ because $Y_1(0) \neq 0$. Applying the boundary condition of Eq. (4.2.124) yields

$$r = 1 \qquad \frac{d\phi_n}{dr} = -A_n\sqrt{\lambda_n}\, J_1\left(\sqrt{\lambda_n}\right) = 0 \tag{4.2.133}$$

which leads to a definition of the constant λ_n.

$$J_1\left(\sqrt{\lambda_n}\right) = 0 \tag{4.2.134}$$

This has roots $\lambda_0 = 0$, $\sqrt{\lambda_1} = 3.83$, $\sqrt{\lambda_2} = 7.01$, $\sqrt{\lambda_3} = 10.17$, $\sqrt{\lambda_4} = 13.33$, etc. Thus one can calculate λ_n for n as large as desired from Eq. (4.2.134). We can now choose A_n so that the eigenfunctions are *orthonormal*, i.e.,

$$A_n^2 = \left[\int_0^1 rJ_0\left(\sqrt{\lambda_n}\, r\right)^2 dr\right]^{-1} \qquad n = 0, 1, 2, \ldots \tag{4.2.135}$$

or using Bessel function identities (as in [22]),

$$A_n = \left[\frac{J_0^2\left(\sqrt{\lambda_n}\right)}{2}\right]^{-1/2} = \frac{\sqrt{2}}{J_0\left(\sqrt{\lambda_n}\right)} \qquad n = 0, 1, \ldots \tag{4.2.136}$$

Thus

$$\phi_n(r) = \frac{\sqrt{2}\, J_0\left(\sqrt{\lambda_n}\, r\right)}{J_0\left(\sqrt{\lambda_n}\right)} \qquad n = 0, 1, 2, \ldots \tag{4.2.137}$$

Hence the solution to Eq. (4.2.112) takes the form of Eq. (4.2.120), where $\phi_n(r)$ is given by Eq. (4.2.137) and Eq. (4.2.125) must be solved for the $a_n(t)$.

Because of the orthogonality relationships, Eq. (4.2.127), one obtains inversion relations

$$a_n(t) = \int_0^1 \phi_n^*(r) x(r, t)\, dr = \int_0^1 r\phi_n(r) x(r, t)\, dr \tag{4.2.138}$$

and in particular, the initial conditions are

$$a_n(0) = \int_0^1 r\phi_n(r) x_0(r)\, dr \tag{4.2.139}$$

Also, the coefficients $b_n(t)$ can be found from

$$b_n(t) = \int_0^1 r\, \delta(r-1)\phi_n(r) u(t)\, dr = \phi_n(1) u(t) \tag{4.2.140}$$

$$b_n(t) = \sqrt{2}\, u(t) \qquad n = 0, 1, 2, \ldots \tag{4.2.141}$$

Thus the boundary control $u(t)$ affects all the modes in the same way, and Eq. (4.2.125) becomes

$$\frac{da_n}{dt} + \lambda_n a_n = \sqrt{2}\, u(t) \tag{4.2.142}$$

The modal control scheme for this problem will have the same form as Fig. 4.12, and the feedback control law is

$$u(t) = g(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N(t)) \tag{4.2.143}$$

This means that some weighting of the ε_i is necessary in the feedback control law, and some of the methods of Chap. 3 would be useful in that regard.

More detailed applications of these modal approaches are presented in Chap. 6.

Controllability and Stabilizability

As in the case of first-order hyperbolic PDE systems, controllability and stabilizability results are complex and depend strongly on the exact definition of what is meant by *controllability* or *stabilizability* [2, p. 138; 9; 23; 24]. For example, *exact* controllability requires the *exact* achievement of some final distributed state $x_d(z)$ from any initial distributed state $\mathbf{x}_0(z)$, and the requirements for exact controllability are quite stringent [9]. By contrast, *approximate controllability* only requires that the null initial state $\mathbf{x}_0(z) = \mathbf{0}$ be taken to within an arbitrarily small neighborhood of the final desired state $\mathbf{x}_d(z)$. For essentially all practical process control problems of interest, *approximate controllability* is sufficient for the adequate design of a controller. Thus we shall be concerned here with approximate controllability conditions.

Another difference between distributed parameter systems and lumped parameter systems is in controllability conditions when the control is applied at the boundary of a distributed parameter system. However, detailed consideration of the various controllability conditions is beyond the purpose of this book, and we shall simply illustrate the fundamental concepts through several examples. The approach we shall use is to develop *approximate controllability* and *approximate stabilizability* results by lumping the system through N-eigenfunction decomposition and then applying lumped parameter *controllability* and *stabilizability* theorems to the N ODEs in the eigencoefficients. This *N-mode controllability* can usually be extended to approximate controllability by letting $N \to \infty$. This approach is best discussed in terms of specific example systems. Thus let us consider the control of the axial temperature distribution in the long, thin rod with modeling equations given by Eqs. (4.2.70) to (4.2.72).

$$\frac{\partial x(z,t)}{\partial t} = \frac{\partial^2 x(z,t)}{\partial z^2} + u(z,t) \tag{4.2.70}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.2.71}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.2.72}$$

The modal decomposition using N eigenfunctions produces the solution

$$x(z,t) = \sum_{n=0}^{N} a_n(t)\phi_n(z) \tag{4.2.74}$$

where

$$\phi_n(z) = \begin{cases} 1 & n = 0 \\ \sqrt{2} \quad \cos n\pi z & n = 1, 2, \ldots, N \end{cases} \tag{4.2.144}$$

and

$$\dot{a}_n(t) = -n^2\pi^2 a_n + b_n(t) \qquad n = 0, 1, 2, \ldots, N \tag{4.2.145}$$

Now define the state variables

$$\mathbf{w} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & & & & \mathbf{0} \\ & -\pi^2 & & & \\ & & -4\pi^2 & & \\ & & & \ddots & \\ & \mathbf{0} & & & -N^2\pi^2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} \tag{4.2.146}$$

$$\mathbf{B} = \mathbf{I}$$

Then we can put Eq. (4.2.145) in the form

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{v}$$

and applying the lumped parameter controllability criterion, we get the controllability matrix

$$\mathbf{L}_c = [\mathbf{I} \mid \mathbf{A} \mid \ldots \mid \mathbf{A}^N] \tag{4.2.147}$$

which must have rank $N + 1$ for *approximate controllability*. However, the $(N + 1) \times (N + 1)$ identity matrix $\mathbf{I}$ has rank $N + 1$, so this system is *approximately controllable* for any number of eigenfunctions.

Now let us consider the cylindrical ingot heating problem with control on the boundary considered earlier. The modeling equations are

$$\frac{\partial x(r, t)}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right) \qquad \begin{matrix} 0 \le r \le 1 \\ t > 0 \end{matrix} \tag{4.2.112}$$

with boundary conditions

$$r = 0 \qquad \frac{\partial x}{\partial r} = 0 \tag{4.2.113}$$

$$r = 1 \qquad \frac{\partial x}{\partial r} = u(t) \tag{4.2.114}$$

The solution is

$$x(r, t) = \sum_{n=0}^{N} a_n(t)\phi_n(r) \tag{4.2.74}$$

where

$$\phi_n(r) = \frac{\sqrt{2}\, J_0\left(\sqrt{\lambda_n}\, r\right)}{J_0\left(\sqrt{\lambda_n}\right)} \tag{4.2.137}$$

and $a_n(t)$ comes from the solution to

$$\dot{a}_n = -\lambda_n a_n + \sqrt{2}\, u(t) \qquad n = 0, 1, 2, \ldots \tag{4.2.148}$$

If as before we define

$$\mathbf{w} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -\lambda_0 & & & & \mathbf{0} \\ & -\lambda_1 & & & \\ & & -\lambda_2 & & \\ & & & \ddots & \\ & \mathbf{0} & & & -\lambda_N \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ \vdots \\ \sqrt{2} \end{bmatrix}$$

then

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{b}u \tag{4.2.149}$$

and the $(N + 1) \times (N + 1)$ controllability matrix is

$$\mathbf{L}_c = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & -\sqrt{2}\,\lambda_1 & \sqrt{2}\,\lambda_1^2 & \ldots & \sqrt{2}\,(-\lambda_1)^N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sqrt{2} & -\sqrt{2}\,\lambda_N & \sqrt{2}\,\lambda_N^2 & \ldots & \sqrt{2}\,(-\lambda_N)^N \end{bmatrix} \tag{4.2.150}$$

and the system is *approximately controllable* so long as the eigenvalues are simple. Simple eigenvalues are a consequence of the Sturm-Liouville character of this example problem. This result is somewhat surprising when one thinks about achieving any given temperature profile with only surface heat flux control. However, having simple eigenvalues means that each mode is excited at a different rate by $u(t)$, and by suitable adjustment of $u(t)$, each $a_n(t)$ can be taken to the set-point value.

Feedback Control with Discrete Control Actuators

In all the discussions so far we have considered controls which acted either at boundaries or continuous in space. However, in a number of practical problems, control actuators can only be placed at a finite number of discrete points or zones along the length of the distributed system. As examples, consider the problem of heating a rod in a furnace with a small number of local heaters or the control of a packed-bed chemical reactor through interstage cooling. In such problems, the performance of the control system is strongly influenced by the location of these control actuators. In fact, it is possible to choose locations for which the system is *uncontrollable*, or, alternatively, locations which are *optimal* in some sense [25–29]. Let us illustrate these points through an example problem.

Example 4.2.4 Let us consider the rod heating problem modeled by Eq. (4.2.70) and shown in Fig. 4.9. Here let us assume that the heating control takes the form

$$u(z,t) = \sum_{k=1}^{M} g_k(z)u_k(t) \tag{4.2.151}$$

where the choice

$$g_k(z) = \delta(z - z_k^*) \tag{4.2.152}$$

corresponds to pointwise control at positions $z_1^*, z_2^*, \ldots, z_M^*$. Note that other functional forms $g_k(z)$ will lead to other forms of local control, for example,

$$g_k(z) = H(z - z_k^*) - H(z - z_{k+1}^*) \tag{4.2.153}$$

[where $H(z)$ is the Heaviside step function] would produce M zones of piecewise uniform heating in the interval $z_k^* < z < z_{k+1}^*$. These cases are illustrated in Fig. 4.14.

If we apply the controller [Eq. (4.2.151)], then the model equations become

$$\frac{\partial x(z,t)}{\partial t} = \frac{\partial^2 x(z,t)}{\partial z^2} + \sum_{k=1}^{M} g_k(z)u_k(t) \tag{4.2.154}$$

$$\frac{\partial x}{\partial z} = 0 \qquad z = 0, 1 \tag{4.2.155}$$

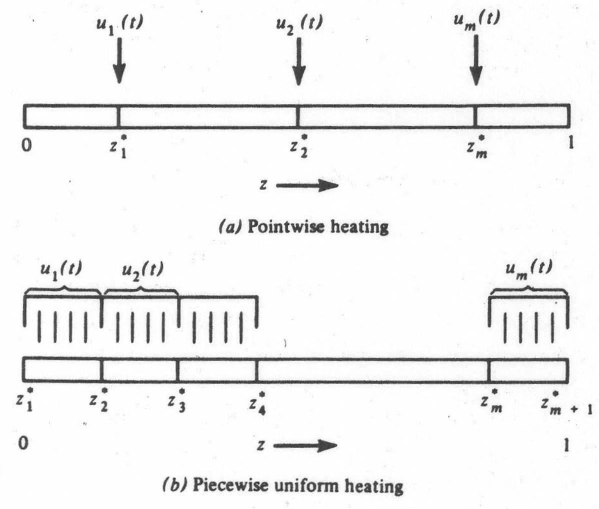

Figure 4.14 Examples of discrete control actuators.

Now let us illustrate how the selection of the location of the actuators influences the *controllability* of the system. Equations (4.2.154) and (4.2.155) may be reduced to the set of eigencoefficient equations

$$\dot{a}_n(t) = -n^2\pi^2 a_n + b_n(t) \qquad n = 0, 1, \ldots, N \tag{4.2.145}$$

where $b_n(t)$ is the eigencoefficient of the control

$$b_n(t) = \sum_{k=1}^{M} u_k(t) \int_0^1 \phi_n(z) g_k(z)\, dz \tag{4.2.156}$$

For the case where discrete pointwise actuators, Eq. (4.2.152), are used, then

$$b_n(t) = \sum_{k=1}^{M} \phi_n(z_k^*) u_k(t) \tag{4.2.157}$$

By defining $\mathbf{w}$, $\mathbf{A}$ from Eq. (4.2.146) and

$$\mathbf{B} = \begin{bmatrix} \phi_0(z_1^*) & \phi_0(z_2^*) & \cdots & \phi_0(z_M^*) \\ \phi_1(z_1^*) & \phi_1(z_2^*) & \cdots & \phi_1(z_M^*) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_N(z_1^*) & \cdots & \cdots & \phi_N(z_M^*) \end{bmatrix} \tag{4.2.158}$$

one obtains

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{B}\mathbf{u} \tag{4.2.159}$$

as an N-eigenfunction representation of the system. The $(N + 1)M \times (N + 1)$ controllability matrix for this system is

$$\mathbf{L}_c = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots \\ \phi_1(z_1^*) & \cdots & \phi_1(z_M^*) & -\pi^2\phi_1(z_1^*) & \cdots & -\pi^2\phi_1(z_M^*) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_N(z_1^*) & \cdots & \phi_N(z_M^*) & -N^2\pi^2\phi_N(z_1^*) & \cdots & -N^2\pi^2\phi_N(z_M^*) & \cdots \end{bmatrix} \tag{4.2.160}$$

and must have the rank $N + 1$ for controllability. Now if the z_k^* are chosen badly, one of the rows of $\mathbf{L}_c$ might vanish identically and the system would be uncontrollable. For example, if $N = 2$, $M = 2$, then

$$\mathbf{L}_c = \left[\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ \phi_1(z_1^*) & \phi_1(z_2^*) & -\pi^2\phi_1(z_1^*) & -\pi^2\phi_1(z_2^*) & \pi^4\phi_1(z_1^*) & \pi^4\phi_1(z_2^*) \\ \phi_2(z_1^*) & \phi_2(z_2^*) & -4\pi^2\phi_2(z_1^*) & -4\pi^2\phi_2(z_2^*) & 16\pi^4\phi_2(z_1^*) & 16\pi^4\phi_2(z_2^*) \end{array}\right] \tag{4.2.161}$$

must have rank 3. Here ϕ_1 and ϕ_2 are given by

$$\begin{aligned} \phi_1 &= \sqrt{2} \cos \pi z \\ \phi_2 &= \sqrt{2} \cos 2\pi z \end{aligned} \tag{4.2.162}$$

Now if one chooses $z_1^* = \frac{1}{4}$, $z_2^* = \frac{3}{4}$, then

$$\phi_2(z_1^*) = \phi_2(z_2^*) = 0 \qquad (4.2.163)$$

and $\mathbf{L}_c$ has only rank 2. Thus this choice of heater positions causes the system, Eqs. (4.2.154) and (4.2.155), to be *uncontrollable*. This example illustrates the rough rule of thumb that *for controllability one should avoid placing the control actuators at the zeros of the system eigenfunctions*. More will be said about optimal actuator placement in the next section.

4.3 OPTIMAL CONTROL THEORY AND PRACTICE*

One particularly important class of distributed parameter control system design procedures is *optimal control*. As in the case of lumped parameter systems, we shall begin our discussion of the optimal control of distributed parameter systems with the consideration of *open-loop* optimal control strategies. A very general class of such problems can be modeled by the partial differential equations

$$\mathbf{A}\frac{\partial \mathbf{x}}{\partial t} = \mathbf{f}\left(\mathbf{x}, \frac{\partial \mathbf{x}}{\partial z}, \frac{\partial^2 \mathbf{x}}{\partial z^2}, \mathbf{u}\right) \qquad 0 \le t \le t_f \qquad 0 \le z \le L \qquad (4.3.1)$$

where $\mathbf{x}(t, z)$ is an n vector of state variables, $\mathbf{u}(t, z)$ is a m vector of control variables, and $\mathbf{A}$ is an $n \times n$ matrix. To prevent matters from becoming too complex, we shall restrict ourselves to two independent variables, $0 \le t \le t_f$ and $0 \le z \le L$, although the analysis could be extended to more independent variables in a straightforward way [30, 31].

Equation 4.3.1 is the general representation of a very large number of practical problems. The drying of porous materials, the behavior of chemical reactor systems, and heat transfer problems like those described in the last section are only a few examples of problems having this form.

The boundary conditions associated with Eq. (4.3.1) depend on the particular problem being considered; however, normally there is an initial state

$$\mathbf{x}(z, 0) = \mathbf{w}(z) \qquad (4.3.2)$$

which may be available as a control variable. For example, the initial temperature distribution in the slab of the previous section might be subject to control by preheat. The system boundary conditions are usually split (for obvious physical reasons) and can take a variety of forms. We shall consider three separate cases of boundary conditions here.

Case 1 Some state variables x_s may have boundary conditions of the form

$$\frac{\partial x_s}{\partial z} = g_s(\mathbf{x}, \mathbf{v}(t)) \qquad \text{at } z = 0 \qquad (4.3.3a)$$

$$\frac{\partial x_s}{\partial z} = h_s(\mathbf{x}, \mathbf{y}(t)) \qquad \text{at } z = L \qquad (4.3.4a)$$

* Parts of this section are adapted from [5] with permission of John Wiley and Sons, Inc.

as, for example, when there is convective or radiant heat transfer at the surface.

Case 2 Other state variables x_r may have boundary conditions of the form

$$x_r(0, t) = \text{const} \tag{4.3.3b}$$

$$x_r(L, t) = \text{const} \tag{4.3.4b}$$

Case 3 Still others x_p may take the form

$$x_p(0, t) = v_p(t) \tag{4.3.3c}$$

$$x_p(L, t) = y_p(t) \tag{4.3.4c}$$

which allows the surface conditions to be controlled in an optimal fashion.

Here $\mathbf{v}(t)$ is a control operating at $z = 0$, and $\mathbf{y}(t)$ is a control operating at $z = L$. It can be seen that the boundary conditions given by Eqs. (4.1.2) and (4.1.3) are just special cases of this general form.

The optimal control problem for this system can be stated, in the most general way, as the desire to maximize the functional

$$I[\mathbf{u}(z, t), \mathbf{v}(t), \mathbf{y}(t), \mathbf{w}(z)] = \int_0^L G_1(\mathbf{x}(t_f, z), \mathbf{w}(z))\, dz + \int_0^{t_f} G_2(\mathbf{x}(L, t), \mathbf{x}(0, t), \mathbf{y}, \mathbf{v})\, dt + \int_0^L \int_0^{t_f} G\left(\mathbf{x}, \mathbf{u}, \frac{\partial \mathbf{x}}{\partial z}, \frac{\partial^2 \mathbf{x}}{\partial z^2}\right) dt\, dz \tag{4.3.5}$$

by choosing the controls $\mathbf{u}(z, t)$, $\mathbf{v}(t)$, $\mathbf{y}(t)$, $\mathbf{w}(z)$.

Necessary Conditions for Optimality

For the optimal control problem given by Eqs. (4.3.1) to (4.3.5), we shall now develop an informal derivation of the necessary conditions for optimality. As in the case of the maximum principle for ordinary differential equations, let us assume that we have a nominal set of optimal control trajectories $\bar{\mathbf{u}}(z, t)$, $\bar{\mathbf{v}}(t)$, $\bar{\mathbf{y}}(t)$, $\bar{\mathbf{w}}(z)$, and let us consider the effect of variations $\delta\mathbf{u}$, $\delta\mathbf{v}$, $\delta\mathbf{y}$, $\delta\mathbf{w}$ about these nominal trajectories. We shall begin by expanding Eq. (4.3.1) about the nominal trajectories to yield the perturbation equations

$$A_{ij}\frac{\partial(\delta x_j)}{\partial t} = \left(\frac{\partial f_i}{\partial x_j}\right)_. \delta x_j + \left(\frac{\partial f_i}{\partial u_k}\right)_. \delta u_k + \left(\frac{\partial f_i}{\partial(\dot{x}_j)}\right)_. \delta(\dot{x}_j) + \left(\frac{\partial f_i}{\partial(\ddot{x}_j)}\right)_. \delta(\ddot{x}_j) \tag{4.3.6}$$

where $(\ )_.$ signifies that the quantity is evaluated along the nominal trajectory. In addition,

$$\dot{x}_j \equiv \frac{\partial x_j}{\partial z} \qquad \ddot{x}_j \equiv \frac{\partial^2 x_j}{\partial z^2}$$

and we use the convention that a repeated subscript denotes a sum over that index; for example,

$$\frac{\partial f_i}{\partial x_k}\,\delta x_k \equiv \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}\,\delta x_k = \frac{\partial f_i}{\partial x_1}\,\delta x_1 + \frac{\partial f_i}{\partial x_2}\,\delta x_2 + \cdots + \frac{\partial f_i}{\partial x_n}\,\delta x_n$$

Equation (4.3.6) can be rewritten as

$$A_{ij}\frac{\partial(\delta x_j)}{\partial t} = \left(\frac{\partial f_i}{\partial x_k}\right)_. \delta x_k + \left(\frac{\partial f_i}{\partial u_k}\right)_. \delta u_k + \left[\frac{\partial f_i}{\partial(\dot{x}_j)}\right]_. \frac{\partial(\delta x_j)}{\partial z} + \left[\frac{\partial f_i}{\partial(\ddot{x}_j)}\right]_. \frac{\partial^2(\delta x_j)}{\partial z^2} \tag{4.3.7}$$

Expanding the objective [Eq. (4.3.5)] in the same way yields

$$\begin{aligned}
\delta I = {} & \int_0^L \left\{\left[\frac{\partial G_1}{\partial x_k(z, t_f)}\right]_. \delta x_k(z, t_f) + \left[\frac{\partial G_1}{\partial w_j(z)}\right]_. \delta w_j(z)\right\} dz \\
& + \int_0^{t_f}\left\{\left[\frac{\partial G_2}{\partial x_k(L, t)}\right]_. \delta x_k(L, t) + \left[\frac{\partial G_2}{\partial x_k(0, t)}\right]_. \delta x_k(0, t)\right. \\
& \left. + \left[\frac{\partial G_2}{\partial y_j(t)}\right]_. \delta y_j(t) + \left[\frac{\partial G_2}{\partial v_j(t)}\right]_. \delta v_j(t)\right\} dt \\
& + \int_0^{t_f}\int_0^L \left\{\left(\frac{\partial G}{\partial x_k}\right)_. \delta x_k + \left(\frac{\partial G}{\partial u_i}\right)_. \delta u_i\right. \\
& \left. + \left[\frac{\partial G}{\partial(\dot{x}_j)}\right]_. \frac{\partial(\delta x_j)}{\partial z} + \left[\frac{\partial G}{\partial(\ddot{x}_j)}\right]_. \frac{\partial^2(\delta x_i)}{\partial z^2}\right\} dz\, dt
\end{aligned} \tag{4.3.8}$$

Now let us use a distributed Lagrange multiplier (called an adjoint variable) $\lambda_k(z, t)$ to form the quantity

$$\begin{aligned}
\int_0^{t_f}\int_0^L \Bigg(\lambda_i(z, t)\Bigg\{ & A_{ij}\frac{\partial(\delta x_j)}{\partial t} - \left(\frac{\partial f_i}{\partial x_k}\right)_. \delta x_k - \left(\frac{\partial f_i}{\partial u_k}\right)_. \delta u_k \\
& - \left[\frac{\partial f_i}{\partial(\dot{x}_j)}\right]_. \frac{\partial(\delta x_j)}{\partial z} - \left[\frac{\partial f_i}{\partial(\ddot{x}_j)}\right]_. \frac{\partial^2(\delta x_j)}{\partial z^2}\Bigg\}\Bigg) dz\, dt = 0
\end{aligned} \tag{4.3.9}$$

which can be subtracted from Eq. (4.3.8) to yield

$$\delta I = \int_0^{t_f}\left\{\left[\frac{\partial G_2}{\partial x_k(L,t)}\right]_* \delta x_k(L,t) + \left[\frac{\partial G_2}{\partial x_k(0,t)}\right]_* \delta x_k(0,t) + \left[\frac{\partial G_2}{\partial y_j(t)}\right]_* \delta y_j(t) + \left[\frac{\partial G_2}{\partial v_j(t)}\right]_* \delta v_j(t)\right\} dt$$
$$+ \int_0^L \left\{\left[\frac{\partial G_1}{\partial x_k(z,t_f)}\right]_* \delta x_k(z,t_f) + \left[\frac{\partial G_1}{\partial w_j(z)}\right]_* \delta w_j(z)\right\} dz$$
$$+ \int_0^{t_f}\int_0^L \left\{\left(\frac{\partial H}{\partial x_k}\right)_* \delta x_k + \left(\frac{\partial H}{\partial u_i}\right)_* \delta u_i + \left[\frac{\partial H}{\partial(\dot{x}_j)}\right]_* \frac{\partial(\delta x_j)}{\partial z} + \left[\frac{\partial H}{\partial(\ddot{x}_j)}\right]_* \frac{\partial^2(\delta x_j)}{\partial z^2} - \lambda_i\left[A_{ij}\frac{\partial(\delta x_j)}{\partial t}\right]\right\} dz\, dt \tag{4.3.10}$$

where the quantity H (known as the Hamiltonian) is defined as

$$H = G + \lambda_i f_i \tag{4.3.11}$$

If we integrate the last three terms by parts so that

$$\int_0^{t_f}\int_0^L \left[\frac{\partial H}{\partial \dot{x}_j}\frac{\partial(\delta x_j)}{\partial z}\right] dz\, dt = \int_0^{t_f}\left\{\left[\frac{\partial H}{\partial \dot{x}_j}\delta x_j\right]_0^L - \int_0^L \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial \dot{x}_j}\right)\delta x_j\, dz\right\} dt \tag{4.3.12}$$

$$\int_0^{t_f}\int_0^L \left[\frac{\partial H}{\partial(\ddot{x}_j)}\frac{\partial^2(\delta x_j)}{\partial z^2}\right] dz\, dt = \int_0^{t_f}\left\{\left[\frac{\partial H}{\partial(\ddot{x}_j)}\frac{\partial(\delta x_j)}{\partial z}\right]_0^L - \left[\frac{\partial}{\partial z}\left(\frac{\partial H}{\partial \ddot{x}_j}\right)\delta x_j\right]_0^L + \int_0^L \frac{\partial^2(\partial H/\partial \ddot{x}_j)}{\partial z^2}\delta x_j\, dz\right\} dt \tag{4.3.13}$$

$$\int_0^{t_f}\int_0^L \lambda_i\left[A_{ij}\frac{\partial(\delta x_j)}{\partial t}\right] dt\, dz = \int_0^L \left\{\lambda_i A_{ij}\,\delta x_j\right]_0^{t_f} - \int_0^{t_f}\frac{\partial(\lambda_i A_{ij})}{\partial t}\delta x_j\, dt\right\} dz \tag{4.3.14}$$

then Eq. (4.3.10) becomes

$$\delta I = \int_0^{t_f}\int_0^L \Bigg\{\Bigg[\left(\frac{\partial H}{\partial x_k}\right)_. - \frac{\partial(\partial H/\partial \dot{x}_k)_.}{\partial z} + \frac{\partial^2(\partial H/\partial \ddot{x}_k)_.}{\partial z^2} + \frac{\partial(\lambda_i A_{ik})}{\partial t}\Bigg]\delta x_k$$

$$+ \left(\frac{\partial H}{\partial u_i}\right)_. \delta u_i\Bigg\}\, dz\, dt + \int_0^{t_f}\Bigg(\Bigg\{\left[\frac{\partial G_2}{\partial x_k(L,t)}\right]_. + \left(\frac{\partial H}{\partial \dot{x}_k}\right)_. - \frac{\partial(\partial H/\partial \ddot{x}_k)_.}{\partial z}\Bigg\}\delta x_k(L,t)$$

$$+ \left(\frac{\partial G_2}{\partial y_j}\right)_. \delta y_j(t) + \left[\frac{\partial G_2}{\partial v_j(t)}\right]_. \delta v_j(t) + \left[\frac{\partial H}{\partial(\ddot{x}_j)}\right]_. \left[\frac{\partial(\delta x_j)}{\partial z}\right]_0^L$$

$$+ \Bigg\{\left[\frac{\partial G_2}{\partial x_j(0,t)}\right]_. - \left(\frac{\partial H}{\partial \dot{x}_j}\right)_. + \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial \ddot{x}_j}\right)_.\Bigg\}\delta x_j(0,t)\Bigg)\, dt$$

$$+ \int_0^L \Bigg(\Bigg\{\left[\frac{\partial G_1}{\partial x_k(t_f, z)}\right]_. - \lambda_i A_{ik}\Bigg\}\delta x_k(z, t_f) + \Bigg\{\left[\frac{\partial G_1}{\partial w_k(z)}\right]_. + \lambda_i A_{ik}\Bigg\}\delta w_k(z)\Bigg)\, dz \tag{4.3.15}$$

To remove the explicit dependence of δI on $\delta x(z, t)$, let us define the adjoint variables $\lambda_i(z, t)$ by

$$\boxed{\frac{\partial(\lambda_i A_{ik})}{\partial t} = -\Bigg[\left(\frac{\partial H}{\partial x_k}\right)_. - \frac{\partial(\partial H/\partial \dot{x}_k)_.}{\partial z} + \frac{\partial^2(\partial H/\partial \ddot{x}_k)_.}{\partial z^2}\Bigg] \qquad k = 1, 2, \ldots, n} \tag{4.3.16}$$

which causes the first term in Eq. (4.3.15) to vanish.

Now let us consider the three separate cases that can arise from the boundary conditions [Eqs. (4.3.3) and (4.3.4)].

Case 1 For those state variables having boundary conditions Eqs. (4.3.3*a*) and (4.3.4*a*), the boundary condition variations become

$$\frac{\partial(\delta x_i(0,t))}{\partial z} = \Bigg\{\left[\frac{\partial g_i}{\partial x_j(0,t)}\right]_. \delta x_j(0,t) + \left[\frac{\partial g_i}{\partial v_j(t)}\right]_. \delta v_j(t)\Bigg\} \tag{4.3.17}$$

$$\frac{\partial(\delta x_i(L,t))}{\partial z} = \Bigg\{\left[\frac{\partial h_i}{\partial x_j(L,t)}\right]_. \delta x_j(L,t) + \left[\frac{\partial h_i}{\partial y_j(t)}\right]_. \delta y_j(t)\Bigg\} \tag{4.3.18}$$

Case 2 For those state variables with boundary conditions of the form of Eqs. (4.3.3*b*) and (4.3.4*b*), the variations

$$\left.\frac{\partial(\delta x_i)}{\partial z}\right]_0^L$$

are free and the variations

$$\delta x_i(0, t) \qquad \delta x_i(L, t)$$

vanish.

Case 3 For those state variables with boundary conditions Eqs. (4.3.3c) and (4.3.4c), the variations

$$\frac{\partial(\delta x_i)}{\partial z}\bigg]_0^L$$

are free and

$$\delta x_i(0, t) = \delta v_i(t), \qquad \delta x_i(L, t) = \delta y_i(t) \tag{4.3.19}$$

If we denote the state variables in Case 1 by index s, those in Case 2 by index r, and those in Case 3 by index p, we can rewrite Eq. (4.3.15) as

$$\begin{aligned}
\delta I = {} & \int_0^{t_f}\int_0^L \left(\frac{\partial H}{\partial u_i}\right)_{\cdot} \delta u_i \, dz \, dt + \int_0^L \left(\left\{ \left[\frac{\partial G_1}{\partial x_k(z, t_f)} \right]_{\cdot} - \lambda_i A_{ik} \right\} \delta x_k(z, t_f) \right. \\
& \left. + \left\{ \left[\frac{\partial G_1}{\partial w_k(z)} \right]_{\cdot} + \lambda_i A_{ik} \right\} \delta w_k(z) \right) dz + \int_0^{t_f} \left(\left(\frac{\partial H_2}{\partial v_s} \right)_{\cdot} \delta v_s(t) + \left(\frac{\partial H_3}{\partial y_s} \right)_{\cdot} \delta y_s \right. \\
& + \left\{ \left[\frac{\partial H_2}{\partial x_s(0, t)} \right]_{\cdot} - \left[\frac{\partial H(0, t)}{\partial \dot{x}_s} \right]_{\cdot} + \frac{\partial}{\partial z} \left(\frac{\partial H}{\partial \ddot{x}_s} \right)_{\cdot} \right\} \delta x_s(0, t) \\
& \left. + \left\{ \left[\frac{\partial H_3}{\partial x_s(L, t)} \right]_{\cdot} + \left[\frac{\partial H(L, t)}{\partial \dot{x}_s} \right]_{\cdot} - \frac{\partial}{\partial z} \left(\frac{\partial H}{\partial \ddot{x}_s} \right)_{\cdot} \right\} \delta x_s(L, t) \right) dt \\
& + \int_0^{t_f} \left(\left\{ \left[\frac{\partial G_2}{\partial x_r(0, t)} \right]_{\cdot} - \left(\frac{\partial H}{\partial \dot{x}_r} \right)_{\cdot} + \left(\frac{\partial H}{\partial \ddot{x}_r} \right)_{\cdot} \right\} \delta x_r(0, t) \right. \\
& \left. + \left\{ \left[\frac{\partial G_2}{\partial x_r(L, t)} \right]_{\cdot} + \left(\frac{\partial H}{\partial \dot{x}_r} \right)_{\cdot} - \frac{\partial(\partial H/\partial \ddot{x}_r)_{\cdot}}{\partial z} \right\} \delta x_r(L, t) + \left(\frac{\partial H}{\partial \ddot{x}_r} \right)_{\cdot} \frac{\partial(\delta x_r)}{\partial z} \bigg]_0^L \right) dt \\
& + \int_0^{t_f} \left\{ \left[\left(\frac{\partial G_2}{\partial v_p} \right)_{\cdot} - \left(\frac{\partial H}{\partial \dot{x}_p} \right)_{\cdot} + \frac{\partial}{\partial z} \left(\frac{\partial H}{\partial \ddot{x}_p} \right)_{\cdot} \right] \delta v_p(t) \right. \\
& \left. + \left[\left(\frac{\partial G_2}{\partial y_p} \right)_{\cdot} + \left(\frac{\partial H}{\partial \dot{x}_p} \right)_{\cdot} - \frac{\partial(\partial H/\partial \ddot{x}_p)_{\cdot}}{\partial z} \right] \delta y_p(t) + \left(\frac{\partial H}{\partial(\ddot{x}_p)} \right)_{\cdot} \frac{\partial(\delta x_p)}{\partial z} \bigg]_0^L \right\} dt
\end{aligned} \tag{4.3.20}$$

where we have defined additional Hamiltonians as

$$H_1 \equiv G_1 + \lambda_i A_{ik} w_k \tag{4.3.21}$$

$$H_2 \equiv G_2 - \frac{\partial H}{\partial \ddot{x}_i}(0, t) g_i \tag{4.3.22}$$

$$H_3 \equiv G_2 + \frac{\partial H(L, t)}{\partial \ddot{x}_i} h_i \tag{4.3.23}$$

Now to cause the coefficients of the arbitrary variations

$$\delta x_s(0, t) \qquad \delta x_s(L, t) \qquad \left.\frac{\partial(\delta x_r)}{\partial z}\right]_0^L \qquad \left.\frac{\partial(\delta x_p)}{\partial z}\right]_0^L$$

to vanish, we must specify the following boundary conditions on the adjoint variables.

For Case 1 boundary conditions:

$$\left\{\frac{\partial H_2}{\partial x_s(0, t)} - \frac{\partial H(0, t)}{\partial \dot{x}_s} + \frac{\partial}{\partial z}\left[\frac{\partial H(0, t)}{\partial \ddot{x}_s}\right]\right\} = 0 \tag{4.3.24}$$

$$\left\{\frac{\partial H_3}{\partial x_s(L, t)} + \frac{\partial H(L, t)}{\partial \dot{x}_s} - \frac{\partial}{\partial z}\left[\frac{\partial H(L, t)}{\partial \ddot{x}_s}\right]\right\} = 0 \tag{4.3.25}$$

For Cases 2 and 3 boundary conditions:

$$\left.\frac{\partial H}{\partial \ddot{x}_r}\right]_0^L = \left.\frac{\partial H}{\partial \ddot{x}_p}\right]_0^L = 0 \tag{4.3.26}$$

In addition, if the terminal state $\mathbf{x}(z, t_f)$ is completely unspecified, the terminal conditions on $\boldsymbol{\lambda}$ become

$$\lambda_i(z, t_f) A_{ik} = \frac{\partial G_1}{\partial x_k(z, t_f)} \qquad k = 1, 2, \ldots, n \tag{4.3.27}$$

It should be noted that if the partial differential equations are not second-order in some of the state variables $x_q(z, t)$, then $\partial H/\partial \ddot{x}_q \equiv 0$, and Case 2 or 3 boundary conditions are possible only at one side. If, for example, $x_q(L, t)$ was unspecified, then the coefficient of $\delta x_q(L, t)$ in Eq. (4.3.20) must vanish. The boundary condition on $\lambda_q(L, t)$ would then be

$$\frac{\partial G_2}{\partial x_q(L, t)} + \frac{\partial H}{\partial \dot{x}_q} = 0 \tag{4.3.28}$$

Thus these results apply to both first- and second-order partial differential equations. Applying these results reduces the variation in I to

$$
\begin{aligned}
\delta I = {} & \int_0^{t_f}\int_0^L \left(\frac{\partial H}{\partial u_i}\right)_{\cdot} \delta u_i \, dz \, dt + \int_0^{t_f}\left\{\left[\frac{\partial H_2}{\partial v_s(t)}\right]_{\cdot} \delta v_s(t) + \left[\frac{\partial H_3}{\partial y_s(t)}\right]_{\cdot} \delta y_s(t)\right\} dt \\
& + \int_0^{t_f}\left\{\left[\left(\frac{\partial G_2}{\partial v_p}\right)_{\cdot} - \left(\frac{\partial H}{\partial \dot{x}_p}\right)_{\cdot} + \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial \ddot{x}_p}\right)_{\cdot}\right] \delta v_p(t) \right. \\
& \left. + \left[\left(\frac{\partial G_2}{\partial y_p}\right)_{\cdot} + \left(\frac{\partial H}{\partial \dot{x}_p}\right)_{\cdot} - \frac{\partial(\partial H/\partial \ddot{x}_p)_{\cdot}}{\partial z}\right] \delta y_p(t)\right\} dt \\
& + \int_0^L \left\{\left[\frac{\partial H_1}{\partial w_i(z)}\right]_{\cdot} \delta w_i(z)\right\} dz
\end{aligned}
\tag{4.3.29}
$$

where the influence of the variations $\delta\mathbf{u}$, $\delta\mathbf{v}$, $\delta\mathbf{y}$, $\delta\mathbf{w}$ on the objective δI is now clear. Since the variations δu_i, δv_j, δy_k, δw_i are all arbitrary, a necessary condition for $\delta I \leq 0$ and the nominal policies $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{y}}$, $\bar{\mathbf{w}}$ to be optimal is that the coefficients of the variations vanish. Thus we can collect our results into the following weak maximum principle:

Theorem In order for the control trajectories $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{y}}$, and $\bar{\mathbf{w}}$ to be optimal for the problem defined by Eqs. (4.3.1) to (4.3.5) and subject to the upper- and lower-bound constraints

$$
\begin{aligned}
u_{i*} &\leq u_i \leq u_i^* \\
v_{j*} &\leq v_j \leq v_j^* \\
y_{k*} &\leq y_k \leq y_k^* \\
w_{l*} &\leq w_l \leq w_l^*
\end{aligned}
\tag{4.3.30}
$$

it is necessary that

$$
\left(\frac{\partial H}{\partial u_i}\right)_{\cdot} = 0 \tag{4.3.31}
$$

for $\bar{u}_i(z, t)$ unconstrained and H be a maximum when $\bar{u}_i(z, t)$ is constrained. If $\bar{u}_i$ is only a function of z, then

$$
\int_0^{t_f}\left(\frac{\partial H}{\partial u_i}\right)_{\cdot} dt = 0 \tag{4.3.32}
$$

must hold for unconstrained $\bar{u}_i(z)$ and $\int_0^{t_f} H\, dt$ must be maximized with respect to constrained $\bar{u}_i(z)$. Similarly, if $\bar{u}_i$ is only a function of t, then

$$
\int_0^{L}\left(\frac{\partial H}{\partial u_i}\right)_{\cdot} dz = 0 \tag{4.3.33}
$$

must hold for unconstrained $\bar{u}_i(t)$ and $\int_0^L H\, dz$ must be a maximum with respect to constrained $\bar{u}_i(t)$.

Furthermore, it is necessary that

$$\left(\frac{\partial H_1}{\partial w_l}\right) = 0 \tag{4.3.34}$$

$$\left(\frac{\partial H_2}{\partial v_s}\right) = 0 \tag{4.3.35}$$

$$\left\{\frac{\partial G_2}{\partial v_p} - \frac{\partial H(0, t)}{\partial \dot{x}_p} + \frac{\partial}{\partial z}\left[\frac{\partial H(0, t)}{\partial \ddot{x}_p}\right]\right\} = 0 \tag{4.3.36}$$

$$\left(\frac{\partial H_3}{\partial y_s}\right) = 0 \tag{4.3.37}$$

$$\left[\frac{\partial G_2}{\partial y_p} + \frac{\partial H(L, t)}{\partial \dot{x}_p} - \frac{\partial(\partial H(L, t)/\partial \ddot{x}_p)}{\partial z}\right] = 0 \tag{4.3.38}$$

must hold for unconstrained $w_l(z)$, $v_s(t)$, $v_p(t)$, $y_s(t)$, $y_p(t)$, respectively, and these quantities must be nonnegative at the upper bounds on the controls and nonpositive at the lower bounds. If any of the w_l, v_s, v_p, y_s, y_p are unconstrained constant parameters, then the necessary conditions become

$$\int_0^L \left(\frac{\partial H_1}{\partial w_l}\right) dz = 0 \tag{4.3.39}$$

$$\int_0^{t_f} \left(\frac{\partial H_2}{\partial v_s}\right) dt = 0 \tag{4.3.40}$$

$$\int_0^{t_f} \left\{\frac{\partial G_2}{\partial v_p} - \frac{\partial H(0, t)}{\partial \dot{x}_p} + \frac{\partial}{\partial z}\left[\frac{\partial H(0, t)}{\partial \ddot{x}_p}\right]\right\} dt = 0 \tag{4.3.41}$$

$$\int_0^{t_f} \left(\frac{\partial H_3}{\partial y_s}\right) dt = 0 \tag{4.3.42}$$

$$\int_0^{t_f} \left[\frac{\partial G_2}{\partial y_p} + \frac{\partial H(L, t)}{\partial \dot{x}_p} - \frac{\partial(\partial H(L, t)/\partial \ddot{x}_p)}{\partial z}\right] dt = 0 \tag{4.3.43}$$

The adjoint variables $\lambda_i(z, t)$ are defined by Eqs. (4.3.16) and (4.3.24) to (4.3.28), and H, H_1, H_2, H_3 by Eqs. (4.3.11) and (4.3.21) to (4.3.23).

We hope that the reader was not unduly intimidated by the apparent complexity of the theorem. The rather involved nature of these expressions is caused by the fact that we wish to present a fairly general statement of the necessary conditions for optimality for the system described by Eq. (4.3.1). The hope is that the reader can apply the results of the theorem directly to many real problems and will have to derive the necessary conditions only for very unusual problems not falling within this framework.

In order to illustrate the application of these general results to a particular problem, we shall produce the necessary conditions for optimality for the slab-heating problem discussed in Sec. 4.1.

Example 4.3.1 From the general formulation, produce the necessary conditions for optimality of the heat flux program $v(t)$ for the optimization problem described by Eqs. (4.1.1) to (4.1.6). Let us assume for the moment that the coefficients α and β are constant.

SOLUTION First we shall define the needed Hamiltonians:

$$H = [T - T_d(z)]^2 + \lambda(z, t)\frac{\alpha}{\beta}\frac{\partial^2 T}{\partial z^2}$$

$$H_2 = -\frac{\alpha}{\beta}\lambda(0, t)v(t)$$

$$H_3 = 0$$

Then the necessary condition [from Eq. (4.3.35)] for $v(t)$ to be optimal is that

$$v(t) = \begin{cases} v^* & \text{for } \dfrac{\alpha}{\beta}\lambda(0, t) > 0 \\ v_* \leq v \leq v^* & \text{for } \dfrac{\alpha}{\beta}\lambda(0, t) = 0 \\ v_* & \text{for } \dfrac{\alpha}{\beta}\lambda(0, t) < 0 \end{cases}$$

where the adjoint equation [from Eq. (4.3.16)] is

$$\frac{\partial \lambda(z, t)}{\partial t} = -\left[2(T - T_d) + \frac{\alpha}{\beta}\frac{\partial^2 \lambda(z, t)}{\partial z^2}\right]$$

Clearly the terminal state $\lambda(z, t_f)$ is unspecified and the boundary conditions are Case 1, so that the boundary conditions on λ [from Eqs. (4.3.24), (4.3.25), and (4.3.27)] become

$$\frac{\alpha}{\beta}\frac{\partial}{\partial z}[\lambda(0, t)] = 0$$

$$\frac{\alpha}{\beta}\frac{\partial}{\partial z}[\lambda(L, t)] = 0$$

$$\lambda(z, t_f) = 0$$

Thus we have specified the necessary conditions for $v(t)$ to be optimal by simply plugging into the general equations given in the theorem. We note the fact, which is of considerable practical interest, that the optimal heat flux must either correspond to the upper bound (the maximum allowable value) or be zero. The only exception to this stipulation is the case when $\lambda(0, t) = 0$. Thus we have learned the form of the optimal program without performing any calculations. One could readily test likely candidates for the optimal program $v(t)$ by solving the given adjoint partial differential equations and examining the behavior of $\lambda(0, t)$.

Some Computational Procedures

Just as Pontryagin's maximum principle formed the basis of computational approaches to the solution of lumped parameter optimal control problems in Chap. 3, the distributed maximum principle of this section forms the basis of a number of computational procedures for distributed parameter optimal control problems. The most commonly applied method is the *control vector iteration technique*. This procedure is very similar to the one described in Chap. 3 and makes use of the fact that if the initial estimates $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$, $\bar{\mathbf{y}}$, and $\bar{\mathbf{w}}$ are nonoptimal, then a gradient correction

$$\delta u_i(z, t) = \varepsilon_0\left(\frac{\partial H}{\partial u_i}\right). \tag{4.3.44}$$

$$\delta w_l(z) = \varepsilon_1\left(\frac{\partial H_1}{\partial w_l}\right). \tag{4.3.45}$$

$$\delta v_s(t) = \varepsilon_2\left(\frac{\partial H_2}{\partial v_s}\right). \tag{4.3.46}$$

$$\delta v_p(t) = \varepsilon_3\left[\frac{\partial G_2}{\partial v_p} - \frac{\partial H}{\partial \dot{x}_p} + \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial \ddot{x}_p}\right)\right]. \tag{4.3.47}$$

$$\delta y_s(t) = \varepsilon_4\left(\frac{\partial H_3}{\partial y_s}\right). \tag{4.3.48}$$

$$\delta y_p(t) = \varepsilon_5\left[\frac{\partial G_2}{\partial y_p} + \frac{\partial H}{\partial \dot{x}_p} - \frac{\partial(\partial H/\partial \ddot{x}_p)}{\partial z}\right]. \tag{4.3.49}$$

will show the greatest local improvement in δI for sufficiently small positive ε_0, ε_1, ε_2, ε_3, ε_4, ε_5. The detailed algorithm then is

1. Guess $u_i(z, t)$, $v_j(t)$, $y_k(t)$, $w_l(z)$, $0 \le t \le t_f$, $0 \le z \le L$.
2. Solve the state Eq. (4.3.1) together with the boundary conditions [Eqs. (4.3.2) to (4.3.4)]. Compute I from Eq. (4.3.5).
3. Solve the adjoint Eqs. (4.3.16) together with the boundary condition [Eqs. (4.3.24) to (4.3.28)].
4. Correct $u_i(z, t)$, $v_j(t)$, $y_k(t)$, $w_l(z)$ by Eqs. (4.3.44) to (4.3.49), where the ε_i are so chosen as to maximize I. A multivariable search may be used, or alternatively we may assume $\varepsilon_i = a_i\varepsilon_0$, $i = 1, 2, \ldots, 5$, and perform an initial scaling of the a_i followed by a single variable search on ε_0 at each iteration.
5. Return to step 2 and iterate.

Just as in the lumped parameter optimal control problems, these procedures progress very rapidly in the initial stages, but slow down considerably as the optimum is approached. Thus efforts are being made to extend second-order ascent procedures as well as conjugate gradient methods to these problems.

From a practical standpoint, computational difficulties would arise (caused largely by inadequate computer memory) if we were to tackle problems in several dimensions with a large number of control and state variables using this technique. We note, however, that it is quite feasible to carry out the optimal control of systems modeled by partial differential equations and having a number of state and control variables. Indeed, a host of such problems have been tackled by chemical and control engineers; some references will be made to such work in subsequent sections of this chapter.

For practical reasons we shall restrict ourselves, in the illustrative examples to be presented, to systems described by partial differential equations with relatively few state and control variables.

To demonstrate this control vector iteration procedure, we shall determine the optimal inlet temperature control for a train of packed bed reactors whose catalyst is subject to deactivation (see [32, 33] for the treatment of similar problems).

Example 4.3.2 Let us consider the problem of disposing of exhaust gases from a smelting or other ore-processing operation. One solution which has been employed to avoid the air pollution resulting from SO_2 and other noxious components in the stack gases is to oxidize the material (e.g., transform SO_2 to SO_3 for the production of sulfuric acid). Let us consider, furthermore, that this oxidation is to be carried out over some catalyst which is subject to deactivation with time. Because the reaction is exothermic and is assumed to be reversible, a number of adiabatic stages are employed with interstage cooling, as shown in Fig. 4.15. We assume that species A is the reactant and B is the oxidation product. Thus the reaction

$$A \underset{k_2}{\overset{k_1}{\rightleftarrows}} B$$

is to be carried out in the three adiabatic packed bed reactors sketched in

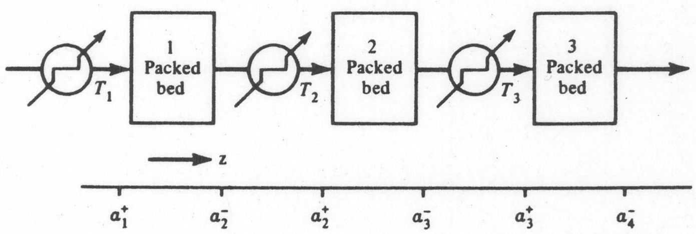

Figure 4.15 Optimization of the reactors used for SO_2 oxidation.

Fig. 4.15. The modeling equations are given as

$$u\frac{\partial C_B}{\partial z'}(z', t') = \psi(z', t')[k_1(T)(C_T - C_B) - k_2(T)C_B]$$

$$\text{(Mass balance on the product)} \qquad \begin{array}{l} 0 \le t' \le t_f \\ 0 \le z' \le L \end{array} \tag{4.3.50}$$

$$\rho C_p u \frac{\partial T(z', t')}{\partial z'} = (-\Delta H)\psi(z', t')[k_1(T)(C_T - C_B) - k_2(T)C_B]$$

$$\text{(Heat balance)} \qquad \begin{array}{l} 0 \le t' \le t_f \\ 0 \le z' \le L \end{array} \tag{4.3.51}$$

$$\begin{array}{ll} T(\alpha_1'^+, t') = T_1 & T(\alpha_2'^+, t') = T_2 \\ T(\alpha_3'^+, t') = T_3 & C_B(0, t') = C_{Bf} \end{array} \tag{4.3.52}$$

which represent steady-state material and heat balances in the reactor train. The quantities α_i' denote the points of separation between the beds, C_T is the total feed concentration, and C_{Bf} is the feed concentration of B. The total catalyst lifetime is t_f, and the total reactor length is L. The reaction rate constants are given by $k_i = A_{i0}e^{-E_i/RT}$, $i = 0, 1, 2$. The decline in catalyst activity $\psi(z', t')$ at each point in the bed can be described by

$$\frac{\partial \psi(z', t')}{\partial t'} = -k_0(T)\psi^2 \qquad \begin{array}{l} 0 \le t' \le t_f \\ 0 \le z' \le L \end{array} \tag{4.3.53}$$

where the initial activity is taken to be unity for fresh catalyst, that is,

$$\psi(z', 0) = 1.0 \tag{4.3.54}$$

Thus if the time scale for catalyst decay is much longer than the time scale for the dynamics of the reactor, then Eqs. (4.3.50) to (4.3.54) are the modeling equations for the system.

Let us suppose that we wish to control the interstage coolers (i.e., the inlet temperatures T_1, T_2, T_3) so as to maximize the conversion of A over the catalyst lifetime t_f. However, due to heat exchange constraints, it is assumed that the possible inlet temperatures are bounded by $T_* \le T_i \le T^*$. This is a practical optimal control problem because by raising the inlet temperature, we both increase the conversion of A from Eqs. (4.3.50) and (4.3.51) and hasten the deactivation of the catalyst through Eq. (4.3.53). Thus there is an optimal inlet temperature control strategy $T_1(t')$, $T_2(t')$, $T_3(t')$ which must be determined, and we do this by applying the control vector iteration technique to the problem.

Solution Let us first recognize that Eqs. (4.3.50) and (4.3.51) are not independent, but can be related by the transformation

$$T(z', t') = T_i + \left(\frac{-\Delta H}{\rho C_p}\right)[C_B(z, t) - C_B(\alpha_i', t)] \qquad i = 1, 2, 3 \tag{4.3.55}$$

because of the adiabatic operation.

Now we define the new variables

$$x_1(z, t) = \frac{C_B}{C_T} \qquad x_2(z, t) = \psi(z, t) \qquad u_1 = \frac{RT_1}{E_1} \qquad u_2 = \frac{RT_2}{E_1}$$

$$u_3 = \frac{RT_3}{E_1} \qquad p = \frac{E_1}{E_0} \qquad p_1 = \frac{E_2}{E_1} \tag{4.3.56}$$

$$\tau_k = \frac{RT}{E_1} \qquad \beta_i = \frac{A_{i0}LC_T}{u} \qquad i = 1, 2 \qquad \rho = A_0 t_f \qquad z = \frac{z'}{L} \qquad t = \frac{t'}{t_f}$$

$$\alpha_i = \frac{\alpha_i'}{L} \qquad x_{1f} = \frac{C_{Bf}}{C_T} \qquad J = \frac{(-\Delta H)RC_T}{\rho C_p E_1} \qquad u_{k*} = \frac{RT_*}{E_1} \qquad u_k^* = \frac{RT^*}{E_1}$$

so that the modeling equations become

$$0 = -\frac{\partial x_1(z, t)}{\partial z} + x_2(z, t)\left[\beta_1 e^{-1/\tau_k}(1 - x_1) - \beta_2 e^{-p_1/\tau_k} x_1\right] \qquad \begin{array}{l} 0 \le t \le 1 \\ 0 \le z \le 1 \end{array} \tag{4.3.57}$$

or

$$0 = -\frac{\partial x_1(z, t)}{\partial z} + \hat{f}(x_1, x_2, u_k)$$

which describes the reactor conversion. The catalyst activity can be determined from

$$\frac{\partial x_2(z, t)}{\partial t} = -\rho (x_2)^2 e^{-(p\tau_k)^{-1}} = \hat{g}(x_1, x_2, u_k) \qquad \begin{array}{l} 0 \le t \le 1 \\ 0 \le z \le 1 \end{array} \tag{4.3.58}$$

where

$$x_1(0, t) = x_{1f} \qquad x_2(z, 0) = 1.0 \tag{4.3.59}$$

and Eq. (4.3.55) becomes

$$\tau_k(z, t) = u_k(t) + J\left[x_1(z, t) - x_1(\alpha_k, t)\right] \qquad k = 1, 2, 3 \tag{4.3.60}$$

The objective functional, which is the cumulative conversion of A over a catalyst lifetime, now becomes

$$I = \int_0^1 x_1(1, t)\, dt \tag{4.3.61}$$

The Hamiltonians of interest, H and H_3, become

$$H = \lambda_1(z, t)\left[-\frac{\partial x_1(z, t)}{\partial z} + \hat{f}(x_1, x_2, u_k)\right] + \lambda_2(z, t)\hat{g}(x_1, x_2, u_k) \tag{4.3.62}$$

$$H_3 = x_1(1, t) \tag{4.3.63}$$

where the adjoint variables are given [see Eq. (4.3.16)] by

$$0 = -\left[\frac{\partial \lambda_1(z, t)}{\partial z} + \lambda_1 \frac{\partial \hat{f}}{\partial x_1} + \lambda_2 \frac{\partial \hat{g}}{\partial x_1}\right] \tag{4.3.64}$$

$$\frac{\partial \lambda_2(z, t)}{\partial t} = -\lambda_1 \frac{\partial \hat{f}}{\partial x_2} - \lambda_2 \frac{\partial \hat{g}}{\partial x_2} \tag{4.3.65}$$

with boundary conditions [see Eqs. (4.3.27) and (4.3.28)]

$$\lambda_1(1, t) = 1 \tag{4.3.66}$$

$$\lambda_2(z, 1) = 0 \tag{4.3.67}$$

The computational procedure then is as follows:

1. Guess $u_k(t)$, $0 \le t \le 1$, $k = 1, 2, 3$.
2. Solve the state Eqs. (4.3.57) and (4.3.58) forward in z, t using the method of characteristics (or finite differences); compute I.
3. Solve the adjoint Eqs. (4.3.64) and (4.3.65) backward in z, t.
4. Correct the controls $u_k(t)$ by

$$\underset{\text{new}}{u_k(t)} = \underset{\text{old}}{u_k(t)} + \varepsilon_0 \int_{\alpha_k}^{\alpha_{k+1}} \left(\frac{\partial H}{\partial u_k} \right) dz \tag{4.3.68}$$

   where $k = 1, 2, 3$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{2}{3}$, $\alpha_4 = 1$, and ε_0 is determined by a one-dimensional search.
5. Return to step 2 and iterate.

It is important to note that because there are three beds, control u_1 only applies over $0 \le z < \frac{1}{3}$, u_2 over $\frac{1}{3} \le z < \frac{2}{3}$, and u_3 over $\frac{2}{3} \le z < 1$. This explains the limits on the integral in Eq. (4.3.68). This computational algorithm was applied for the set of parameters $\beta_1 = 5.244 \times 10^5$, $\beta_2 = 2.28 \times 10^9$, $\rho = 1300$, $u_{k*} = 0.070$, $u_k^* = 0.080$, $p = 1.648$, $p_1 = 1.666$, $J = 0.005$, $x_{if} = 0$, and the result after five iterations is shown in Fig. 4.16. The

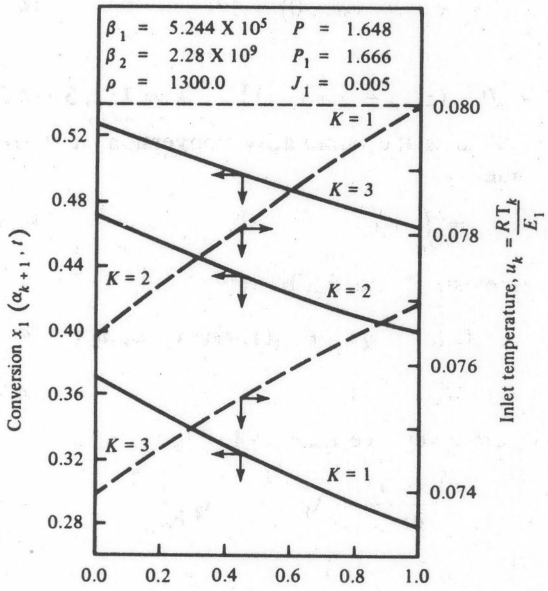

Figure 4.16 Optimal inlet temperature progression for Example 4.3.2, an oxidation reaction.

In principle a direct substitution approach similar to that discussed in Chap. 3 could be used for these problems. However, the uncoupling problem of explicitly representing the optimal control in terms of the state and adjoint variables can rarely be done in practice; thus direct-substitution methods usually cannot be applied.

There have been a large number of applications of these computational procedures to practical problems. The reader is directed to Refs. [1, 2, 36–40] for general surveys.

Optimal Feedback Control of Linear Distributed Parameter Systems—The Linear-Quadratic Problem

Just as the linear-quadratic problem led to an optimal feedback control law for lumped parameter systems in Chap. 3, there are similar results for distributed parameter systems. To illustrate this *general result* [2, 3], we shall explicitly consider an example of parabolic second-order partial differential equations.*

Let us consider the linear state equations

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{A}_2 \frac{\partial^2 \mathbf{x}}{\partial z^2} + \mathbf{A}_1 \frac{\partial \mathbf{x}}{\partial z} + \mathbf{A}_0 \mathbf{x} + \mathbf{B}\mathbf{u}(z, t) \tag{4.3.73}$$

with boundary conditions

$$\frac{\partial \mathbf{x}}{\partial z} + \mathbf{D}_0 \mathbf{x} = \mathbf{B}_0 \mathbf{u}_0(t) \qquad z = 0 \tag{4.3.74}$$

$$\frac{\partial \mathbf{x}}{\partial z} + \mathbf{D}_1 \mathbf{x} = \mathbf{B}_1 \mathbf{u}_1(t) \qquad z = 1 \tag{4.3.75}$$

Now the quadratic objective functional has the form

$$\begin{aligned} I = {} & \tfrac{1}{2}\int_0^1 \int_0^1 \mathbf{x}(r, t_f)^T \mathbf{S}_f(r, s)\mathbf{x}(s, t_f)\, dr\, ds \\ & + \tfrac{1}{2}\int_0^{t_f} \int_0^1 \int_0^1 \left[\mathbf{x}(r, t)^T \mathbf{F}(r, s, t)\mathbf{x}(s, t) + \mathbf{u}(r, t)^T \mathbf{E}(r, s, t)\mathbf{u}(s, t)\right] dr\, ds\, dt \\ & + \tfrac{1}{2}\int_0^{t_f} \left[\mathbf{u}_0^T(t)\mathbf{E}_0(t)\mathbf{u}_0(t) + \mathbf{u}_1^T(t)\mathbf{E}_1(t)\mathbf{u}_1(t)\right] dt \end{aligned} \tag{4.3.76}$$

Now let us apply the maximum principle of the previous section to this problem. Let

$$\begin{aligned} H(r, t) = {} & \tfrac{1}{2}\int_0^1 (\mathbf{x}^T \mathbf{F}\mathbf{x} + \mathbf{u}^T \mathbf{E}\mathbf{u})\, ds + \boldsymbol{\lambda}^T(r, t)[\mathbf{A}_2 \ddot{\mathbf{x}}(r, t) \\ & + \mathbf{A}_1 \dot{\mathbf{x}}(r, t) + \mathbf{A}_0 \mathbf{x}(r, t) + \mathbf{B}\mathbf{u}(r, t)] \end{aligned} \tag{4.3.77}$$

$$H_2(t) = \tfrac{1}{2}\mathbf{u}_0^T \mathbf{E}_0 \mathbf{u}_0 - \boldsymbol{\lambda}^T(0, t)\mathbf{A}_2[-\mathbf{D}_0 \mathbf{x}(0, t) + \mathbf{B}_0 \mathbf{u}_0(t)] \tag{4.3.78}$$

$$H_3(t) = \tfrac{1}{2}\mathbf{u}_1^T \mathbf{E}_1 \mathbf{u}_1 + \boldsymbol{\lambda}^T(1, t)\mathbf{A}_2[-\mathbf{D}_1 \mathbf{x}(1, t) + \mathbf{B}_1 \mathbf{u}_1(t)] \tag{4.3.79}$$

$$H_1(r) = \tfrac{1}{2}\int_0^1 \mathbf{x}(r, t_f)^T \mathbf{S}_f(r, s)\mathbf{x}(s, t_f)\, ds \tag{4.3.80}$$

* As we shall show, these results also apply to hyperbolic first-order PDE systems.

Then the necessary conditions for optimality become

$$\frac{\partial H(r, t)}{\partial \mathbf{u}(r, t)} = \int_0^1 \mathbf{E}\mathbf{u}(s, t)\, ds + \mathbf{B}^T\boldsymbol{\lambda}(r, t) = \mathbf{0} \tag{4.3.81}$$

$$\frac{\partial H_2}{\partial \mathbf{u}_0} = \mathbf{E}_0\mathbf{u}_0 - \mathbf{B}_0^T\mathbf{A}_2^T\boldsymbol{\lambda}(0, t) = \mathbf{0} \tag{4.3.82}$$

$$\frac{\partial H_3}{\partial \mathbf{u}_1} = \mathbf{E}_1\mathbf{u}_1 + \mathbf{B}_1^T\mathbf{A}_2^T\boldsymbol{\lambda}(1, t) = \mathbf{0} \tag{4.3.83}$$

If $\mathbf{u}$ depends only on time, then the necessary conditions for optimality become

$$\int_0^1 \frac{\partial H(r, t)}{\partial \mathbf{u}(t)}\, dr = \int_0^1 \left[\int_0^1 \mathbf{E}(r, s, t)\mathbf{u}(t)\, ds + \mathbf{B}^T\boldsymbol{\lambda}(r, t) \right] dr = \mathbf{0} \tag{4.3.84}$$

For $\mathbf{u}(r, t)$, inverting Eq. (4.3.81) leads to

$$\mathbf{u}(r, t) = -\int_0^1 \mathbf{E}^*(r, s, t)\mathbf{B}^T\boldsymbol{\lambda}(s, t)\, ds \tag{4.3.85}$$

where $\mathbf{E}^*(r, s, t)$ is the inverse of $\mathbf{E}(r, s, t)$, defined by

$$\int_0^1 \mathbf{E}^*(r, s, t)\mathbf{E}(s, \rho, t)\, ds = \delta(r - \rho)\mathbf{I} \tag{4.3.86}$$

For $\mathbf{u}(t)$, Eq. (4.3.84) leads to

$$\mathbf{u}(t) = -\left[\int_0^1 \int_0^1 \mathbf{E}(r, s, t)\, ds\, dr \right]^{-1} \int_0^1 \mathbf{B}^T\boldsymbol{\lambda}(r, t)\, dr \tag{4.3.87}$$

Similarly,

$$\mathbf{u}_0(t) = \mathbf{E}_0^{-1}\mathbf{B}_0^T\mathbf{A}_2^T\boldsymbol{\lambda}(0, t) \tag{4.3.88}$$

$$\mathbf{u}_1(t) = -\mathbf{E}_1^{-1}\mathbf{B}_1^T\mathbf{A}_2^T\boldsymbol{\lambda}(1, t) \tag{4.3.89}$$

Now the adjoint variable $\boldsymbol{\lambda}(r, t)$ is given by

$$\frac{d\boldsymbol{\lambda}(r, t)}{dt} = -\int_0^1 \mathbf{F}(r, s, t)\mathbf{x}(s, t)\, ds - \mathbf{A}_0^T\boldsymbol{\lambda} + \mathbf{A}_1^T\frac{\partial \boldsymbol{\lambda}}{\partial z} - \mathbf{A}_2^T\frac{\partial^2 \boldsymbol{\lambda}}{\partial z^2} \tag{4.3.90}$$

with boundary conditions

$$\boldsymbol{\lambda}(t_f, r) = \int_0^1 \mathbf{S}_f(r, s)\mathbf{x}(s, t_f)\, ds$$

$$\mathbf{D}_0^T\mathbf{A}_2^T\boldsymbol{\lambda}(0, t) - \mathbf{A}_1^T\boldsymbol{\lambda}(0, t) + \mathbf{A}_2^T\frac{\partial \boldsymbol{\lambda}(0, t)}{\partial z} = 0 \tag{4.3.91}$$

$$-\mathbf{D}_1^T\mathbf{A}_2^T\boldsymbol{\lambda}(1, t) + \mathbf{A}_1^T\boldsymbol{\lambda}(1, t) - \mathbf{A}_2^T\frac{\partial \boldsymbol{\lambda}(1, t)}{\partial z} = 0$$

Now if we *define* a variable $\mathbf{S}(r, s, t)$ by the Riccati transformation

$$\boldsymbol{\lambda}(r, t) = \int_0^1 \mathbf{S}(r, s, t)\mathbf{x}(s, t)\, ds \tag{4.3.92}$$

we can substitute Eq. (4.3.92) into Eq. (4.3.90) to yield

$$\int_0^1 [\dot{\mathbf{S}}\mathbf{x}(s,t) + \mathbf{S}\dot{\mathbf{x}}(s,t)]\,ds = \text{LHS} \tag{4.3.93}$$

$$\begin{aligned}\text{RHS} = \int_0^1 \{[-\mathbf{F}(r,s,t)\mathbf{x}(s,t) - \mathbf{A}_0^T\mathbf{S}\mathbf{x}(s,t)\\ + \mathbf{A}_1^T\mathbf{S}_r\mathbf{x}(s,t) - \mathbf{A}_2^T\mathbf{S}_{rr}\mathbf{x}(s,t)]\}\,ds\end{aligned} \tag{4.3.94}$$

$$\text{LHS} = \int_0^1 \left\{\dot{\mathbf{S}}\mathbf{x} + \mathbf{S}\left[\mathbf{A}_2\frac{\partial^2\mathbf{x}}{\partial s^2} + \mathbf{A}_1\frac{\partial \mathbf{x}}{\partial s} + \mathbf{A}_0\mathbf{x} + \mathbf{B}\mathbf{u}(s,t)\right]\right\} ds \tag{4.3.95}$$

Now

$$\begin{aligned}\int_0^1 \mathbf{S}(r,s,t)\mathbf{A}_2\frac{\partial x^2}{\partial s^2}\,ds &= \mathbf{S}(r,s,t)\mathbf{A}_2\frac{\partial \mathbf{x}}{\partial s}\Big]_0^1 - \int_0^1 \mathbf{S}_s\mathbf{A}_2\frac{\partial \mathbf{x}}{\partial s}\,ds\\ &= \mathbf{S}(r,1,t)\mathbf{A}_2[\mathbf{B}_1\mathbf{u}_1 - \mathbf{D}_1x(1,t)]\\ &\quad - \mathbf{S}(r,0,t)\mathbf{A}_2[\mathbf{B}_0\mathbf{u}_0\\ &\quad - \mathbf{D}_0\mathbf{x}(0,t)] - \mathbf{S}_sA_2\mathbf{x}]_0^1 + \int_0^1 \mathbf{S}_{ss}\mathbf{A}_2\mathbf{x}(s,t)\,ds\end{aligned} \tag{4.3.96}$$

Similarly

$$\int_0^1 \mathbf{S}\mathbf{A}_1\frac{\partial \mathbf{x}}{\partial s}\,ds = \mathbf{S}\mathbf{A}_1\mathbf{x}]_0^1 - \int_0^1 \mathbf{S}_s\mathbf{A}_1\mathbf{x}(s,t)\,ds \tag{4.3.97}$$

Thus

$$\begin{aligned}\text{LHS} = \int_0^1 [(\dot{\mathbf{S}} + \mathbf{S}_{ss}\mathbf{A}_2 - \mathbf{S}_s\mathbf{A}_1 + \mathbf{S}\mathbf{A}_0)\mathbf{x}(s,t) + \mathbf{S}\mathbf{B}\mathbf{u}(s,t)]\,ds\\ -[\mathbf{S}(r,1,t)\mathbf{A}_2\mathbf{D}_1 - \mathbf{S}(r,1,t)\mathbf{A}_1 + \mathbf{S}_s(r,1,t)\mathbf{A}_2]\mathbf{x}(1,t)\\ +[\mathbf{S}(r,0,t)\mathbf{A}_2\mathbf{D}_0 - \mathbf{S}(r,0,t)\mathbf{A}_1 + \mathbf{S}_s(r,0,t)\mathbf{A}_2]\mathbf{x}(0,t)\\ +\mathbf{S}(r,1,t)\mathbf{A}_2\mathbf{B}_1\mathbf{u}_1 - \mathbf{S}(r,0,t)\mathbf{A}_2\mathbf{B}_0\mathbf{u}_0\end{aligned} \tag{4.3.98}$$

Now

$$\begin{aligned}\int_0^1 \mathbf{S}\mathbf{B}\mathbf{u}(s,t)\,ds &= -\int_0^1\int_0^1 \mathbf{S}(r,s,t)\mathbf{B}\mathbf{E}^*(s,\rho,t)\mathbf{B}^T\boldsymbol{\lambda}(\rho,t)\,ds\,d\rho\\ &= -\int_0^1\int_0^1\int_0^1 \mathbf{S}(r,s,t)\mathbf{B}\mathbf{E}^*(s,\rho,t)\mathbf{B}^T\mathbf{S}(\rho,z,t)\mathbf{x}(z,t)\,dz\,ds\,d\rho\\ &= -\int_0^1\left[\int_0^1\int_0^1 \mathbf{S}(r,z,t)\mathbf{B}\mathbf{E}^*(z,\rho,t)\mathbf{B}^T\mathbf{S}(\rho,s,t)\,dz\,d\rho\right]\mathbf{x}(s,t)\,ds\end{aligned} \tag{4.3.99}$$

$$\mathbf{S}(r,1,t)\mathbf{A}_2\mathbf{B}_1\mathbf{u}_1 = -\int_0^1 \mathbf{S}(r,1,t)\mathbf{A}_2\mathbf{B}_1\mathbf{E}_1^{-1}\mathbf{B}_1^T\mathbf{A}_2^T\mathbf{S}(1,s,t)\mathbf{x}(s,t)\,ds \tag{4.3.100}$$

$$\mathbf{S}(r,0,t)\mathbf{A}_2\mathbf{B}_0\mathbf{u}_0 = \int_0^1 \mathbf{S}(r,0,t)\mathbf{A}_2\mathbf{B}_0\mathbf{E}_0^{-1}\mathbf{B}_0^T\mathbf{A}_2^T\mathbf{S}(0,s,t)\mathbf{x}(s,t)\,ds \tag{4.3.101}$$

Now combining RHS and LHS and collecting coefficients of $\mathbf{x}(s, t)$, one obtains the Riccati equation:

$$\begin{aligned}\mathbf{S}_t(r, s, t) = &-\mathbf{S}_{ss}\mathbf{A}_2 - \mathbf{A}_2^T\mathbf{S}_{rr} + \mathbf{S}_s\mathbf{A}_1 + \mathbf{A}_1^T\mathbf{S}_r \\ &-\mathbf{S}\mathbf{A}_0 - \mathbf{A}_0^T\mathbf{S} + \int_0^1\int_0^1 \mathbf{S}(r, z, t)\mathbf{B}\mathbf{E}^*(z, \rho, t)\mathbf{B}^T\mathbf{S}(\rho, s, t)\,dz\,d\rho \\ &+\mathbf{S}(r, 1, t)\mathbf{A}_2\mathbf{B}_1\mathbf{E}_1^{-1}\mathbf{B}_1^T\mathbf{A}_2^T\mathbf{S}(1, s, t) \\ &+\mathbf{S}(r, 0, t)\mathbf{A}_2\mathbf{B}_0\mathbf{E}_0^{-1}\mathbf{B}_0^T\mathbf{A}_2^T\mathbf{S}(0, s, t) - \mathbf{F}(r, s, t)\end{aligned} \tag{4.3.102}$$

With the coefficients of $\mathbf{x}(1, t)$, $\mathbf{x}(0, t)$ in Eq. (4.3.98) yielding the boundary conditions

$$\mathbf{S}_s(r, 1, t)\mathbf{A}_2 + \mathbf{S}(r, 1, t)(\mathbf{A}_2\mathbf{D}_1 - \mathbf{A}_1) = \mathbf{0} \tag{4.3.103}$$

$$\mathbf{S}_s(r, 0, t)\mathbf{A}_2 + \mathbf{S}(r, 0, t)(\mathbf{A}_2\mathbf{D}_0 - \mathbf{A}_1) = \mathbf{0} \tag{4.3.104}$$

and the terminal condition [see Eqs. (4.3.91) and (4.3.92)]

$$\mathbf{S}(r, s, t_f) = \mathbf{S}_f(r, s) \tag{4.3.105}$$

one can show that the symmetry conditions

$$\mathbf{S}(r, s, t) = \mathbf{S}(s, r, t)^T \tag{4.3.106}$$

must hold. Thus we now have the feedback control law for $\mathbf{u}(z, t)$ as

$$\mathbf{u}(z, t) = -\int_0^1\int_0^1 \mathbf{E}^*(z, s, t)\mathbf{B}^T\mathbf{S}(s, \rho, t)\mathbf{x}(\rho, t)\,ds\,d\rho \tag{4.3.107}$$

and for $\mathbf{u}(t)$ from

$$\mathbf{u}(t) = -\left[\int_0^1\int_0^1 \mathbf{E}(r, s, t)\,ds\,dr\right]^{-1}\int_0^1\int_0^1 \mathbf{B}^T\mathbf{S}(s, \rho, t)\mathbf{x}(\rho, t)\,ds\,d\rho \tag{4.3.108}$$

The boundary controls take the form

$$\mathbf{u}_0(t) = \mathbf{E}_0^{-1}\mathbf{B}_0^T\mathbf{A}_2^T\int_0^1 \mathbf{S}(0, s, t)\mathbf{x}(s, t)\,ds \tag{4.3.109}$$

$$\mathbf{u}_1(t) = -\mathbf{E}_1^{-1}\mathbf{B}_1^T\mathbf{A}_2^T\int_0^1 \mathbf{S}(1, s, t)\mathbf{x}(s, t)\,ds \tag{4.3.110}$$

where $\mathbf{S}(s, \rho, t)$ can be precomputed off-line from Eqs. (4.3.102) to (4.3.106).

Thus the linear quadratic problem leads to an optimal feedback control law even in the case of PDE systems.

Let us now illustrate the specific form of the optimal control law with more detailed examples.

Example 4.3.3 Let us consider the control of the temperature distribution in a long, thin rod being heated in a multizone furnace. This is similar to the example problem discussed in the last section. The modeling equations are

$$\rho C_p \frac{\partial T(z', t')}{\partial t'} = k\frac{\partial^2 T}{\partial z'^2} + q(z', t') \qquad T(z', 0) = T_0 \tag{4.3.111}$$

where $q(z', t')$ represents the heat flux from the different zones of the furnace and the boundary conditions

$$z' = 0 \qquad z' = l \qquad \frac{\partial T}{\partial z'} = 0$$

arise when we assume negligible heat loss at the ends of the rod. Defining dimensionless variables

$$t = \frac{t'k}{\rho C_p} \qquad z = \frac{z'}{l} \qquad u'(z, t) = \frac{q(z', t')l^2}{kT_0} \qquad x'(z, t) = \frac{T(z', t')}{T_0}$$

where T_0 is the initial uniform temperature in the rod, one obtains

$$\frac{\partial x'(z, t)}{\partial t} = \frac{\partial^2 x'(z, t)}{\partial z^2} + u'(z, t) \tag{4.3.112}$$

$$z = 0 \qquad \frac{\partial x'}{\partial z} = 0 \tag{4.3.113}$$

$$z = 1 \qquad \frac{\partial x'}{\partial z} = 0 \tag{4.3.114}$$

$$t = 0 \qquad x' = 1 \tag{4.3.115}$$

If we define the objective functional (to be minimized) as

$$I = \tfrac{1}{2}\int_0^1 \left[x'(z, t_f) - x'_d(z)\right]^2 S_f \, dz + \tfrac{1}{2}\int_0^{t_f}\int_0^1 \left\{F\left[x'(z, t) - x'_d(z)\right]^2 + E\left[u'(z, t) - u'_d(z)\right]^2\right\} dt \, dz \tag{4.3.116}$$

where $x'_d(z)$ is the desired final temperature profile and $u'_d(z)$ is the final steady-state control profile required to hold $x'_d(z)$, i.e.,

$$u'_d(z) \equiv -\frac{\partial^2 x'_d(z)}{\partial z^2} \tag{4.3.117}$$

and defining $x = x' - x'_d$, $u = u' - u'_d$, then Eqs. (4.3.112) to (4.3.116) become

$$\frac{\partial x(z, t)}{\partial t} = \frac{\partial^2 x(z, t)}{\partial z^2} + u(z, t) \tag{4.3.118}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.3.119}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.3.120}$$

$$x(z, 0) = 1 - x'_d(z) \tag{4.3.121}$$

$$I = \tfrac{1}{2} S_f \int_0^1 x^2(z, t_f) \, dz + \tfrac{1}{2}\int_0^{t_f}\int_0^1 \left[Fx^2(z, t) + Eu^2(z, t)\right] dt \, dz \tag{4.3.122}$$

This is the same form as the linear-quadratic problem posed above, where

$$S_f(r, s) = S_f\,\delta(r - s)$$

$$F(r, s, t) = F\,\delta(r - s)$$

$$E(r, s, t) = E\,\delta(r - s) \Rightarrow E^*(r, s, t) = E^{-1}\,\delta(r - s) \tag{4.3.123}$$

$$A_2 = 1 \qquad A_1 = 0 \qquad A_0 = 0 \qquad B = 1$$

$$D_0 = B_0 = D_1 = B_1 = 0$$

Thus the optimal feedback control law for this problem is

$$u(z, t) = -\int_0^1 E^{-1}S(z, \rho, t)x(\rho, t)\,d\rho \tag{4.3.124}$$

where $S(r, s, t)$ is given by

$$S_t(r, s, t) = -S_{ss} - S_{rr} + \int_0^1 S(r, \rho, t)E^{-1}S(\rho, s, t)\,d\rho - F\,\delta(r - s) \tag{4.3.125}$$

with boundary conditions

$$S_s(r, 1, t) = S_s(r, 0, t) = S_r(1, s, t) = S_r(0, s, t) = 0 \tag{4.3.126}$$

and

$$S(r, s, t_f) = S_f\,\delta(r - s) \tag{4.3.127}$$

Thus one can precompute $S(r, s, t)$ from Eqs. (4.3.125) to (4.3.127) and use it in the optimal feedback law, Eq. (4.3.124).

Example 4.3.4 Let us now consider the feedback control of the steam-jacketed tubular heat exchanger shown in Fig. 4.5 and discussed in Example 4.2.1. Thermocouples measure the tube fluid temperature at four points, $T(0.25, t)$, $T(0.5, t)$, $T(0.75, t)$, and $T(1, t)$, and adjust the steam-jacket temperature $T_w(t)$ (through a steam inlet valve) in order to control the exchanger.

Recall that the mathematical model for the process takes the form

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z} = \frac{hA}{\rho C_p}(T - T_w) \qquad T(0, t) = T_f \tag{4.3.128}$$

Now if we define the deviation variables

$$x = T - T_d(z) \qquad u = T_w - T_{wd} \qquad a_0 = \frac{hA}{\rho C_p}$$

where $T_d(z)$ is the desired temperature profile and T_{wd} is the steady-state steam jacket temperature required to keep $T = T_d(z)$; i.e., T_{wd} satisfies

$$v\frac{\partial T_d}{\partial z} = \frac{hA}{\rho C_p}(T_d - T_{wd}) \qquad T_d(0) = T_f \tag{4.3.129}$$

then the linear quadratic problem takes the form

$$\min_{u(t)}\left(I = \tfrac{1}{2}\int_0^1 x(z, t_f)^2 S_f \, dz + \tfrac{1}{2}\int_0^{t_f}\left\{\int_0^1 \left[x(z, t)^2 F\right] dz + Eu(t)^2\right\} dt\right) \tag{4.3.130}$$

where

$$\frac{\partial x}{\partial t} = -v\frac{\partial x}{\partial z} + a_0 x - a_0 u \qquad x(0, t) = 0 \tag{4.3.131}$$

Again, this is in the linear-quadratic form of Eqs. (4.3.73) to (4.3.76) where

$$\begin{aligned}
&S_f(r, s) = S_f\,\delta(r - s)\\
&F(r, s, t) = F\,\delta(r - s)\\
&E(r, s, t) = E \Rightarrow E^*(r, s, t) = E^{-1}\\
&\qquad A_2 = 0 \qquad A_1 = -v \qquad A_0 = -a_0 \qquad B = +a_0\\
&\qquad D_0 \to \infty \qquad B_0 = 0
\end{aligned} \tag{4.3.132}$$

Thus the optimal feedback control law is

$$u(t) = -E^{-1}a_0 \int_0^1\int_0^1 S(s, \rho, t)x(\rho, t)\, ds\, d\rho \tag{4.3.133}$$

where $S(r, s, t)$ is given by

$$S_t(r, s, t) = -S_s v - S_r v + 2Sa_0 + a_0^2 E^{-1}\left[\int_0^1 S(r, z, t)\, dz\right] \times \left[\int_0^1 S(\rho, s, t)\, d\rho\right] - F\,\delta(r - s) \tag{4.3.134}$$

with boundary conditions

$$S(r, 0, t) = S(0, s, t) = 0 \tag{4.3.135}$$

$$S(r, s, t_f) = S_f\,\delta(r - s) \tag{4.3.136}$$

Thus the optimal feedback control can be implemented if $x(z, t)$ can be estimated from the four measurements $x(0.25, t)$, $x(0.5, t)$, $x(0.75, t)$, and $x(1.0, t)$. More shall be said about this estimation problem in Chap. 5.

A fuller discussion of the linear-quadratic problem applied to first-order partial differential equations such as these may be found in [41, 42].

Example 4.3.5 Let us now consider the optimal feedback control of the heated rod problem of Example 4.2.4 with discrete spatial actuators. Recall that the model equations take the form

$$\frac{\partial x(z, t)}{\partial t} = \frac{\partial^2 x(z, t)}{\partial z^2} + \sum_{k=1}^{M} g_k(z)u_k(t) \tag{4.3.137}$$

$$\frac{\partial x}{\partial z} = 0 \qquad z = 0, 1 \tag{4.3.138}$$

and the linear-quadratic control problem (Example 4.3.3) is to minimize*

$$I = \frac{1}{2}\int_0^1 S_f x^2(z, t_f)\,dz + \frac{1}{2}\int_0^{t_f}\int_0^1 \left[Fx^2(z, t) + \sum_{k=1}^{M} E_k g_k(z)(u_k(t))^2 \right] dz\,dt \tag{4.3.139}$$

Thus by choosing

$$\mathbf{E}(r, s, t) = \begin{bmatrix} \int_0^1 E_1 g_1\,dz & & & \mathbf{0} \\ & \int_0^1 E_2 g_2\,dz & & \\ & & \ddots & \\ \mathbf{0} & & & \int_0^1 E_M g_M\,dz \end{bmatrix} = \hat{\mathbf{E}} \tag{4.3.140}$$

and applying the general results of Eq. (4.3.86), one obtains

$$\mathbf{E}^*(r, s, t) = \begin{bmatrix} \left(\int_0^1 E_1 g_1\,dz\right)^{-1} & & & \mathbf{0} \\ & \left(\int_0^1 E_2 g_2\,dz\right)^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \left(\int_0^1 E_M g_M\,dz\right)^{-1} \end{bmatrix} = \hat{\mathbf{E}}^{-1} \tag{4.3.141}$$

and the feedback control law

$$\mathbf{u}(t) = \hat{\mathbf{E}}^{-1}\int_0^1\int_0^1 \mathbf{g}(s) S(s, \rho, t) x(\rho, t)\,ds\,d\rho \tag{4.3.142}$$

where

$$\mathbf{u}(t) \equiv \begin{bmatrix} u_1(t) \\ \vdots \\ u_M(t) \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} g_1(z) \\ \vdots \\ g_M(z) \end{bmatrix} \tag{4.3.143}$$

The Riccati equation then becomes

$$S_t(r, s, t) = -S_{ss} - S_{rr} + \left[\int_0^1 S(r, z, t)\mathbf{g}^T(z)\,dz\right]\hat{\mathbf{E}}^{-1}\left[\int_0^1 S(\rho, s, t)\mathbf{g}(\rho)\,d\rho\right] - F\,\delta(r - s) \tag{4.3.144}$$

with boundary conditions given by Eqs. (4.3.126) and (4.3.127).

* Notice that we use only the first power of $g_k(x)$ in the objective. This is done to avoid mathematical complexities when delta functions such as in Eq. (4.2.152) are used.

To illustrate, the form of the control for discrete pointwise controllers, Eq. (4.2.152), is

$$u_k(t) = E_k^{-1}\int_0^1 S(z_k^*, \rho, t)x(\rho, t)\, d\rho \qquad k = 1, 2, \ldots, M \quad (4.3.145)$$

where $S(r, s, t)$ is the solution of

$$S_t(r, s, t) = -S_{ss} - S_{rr} + \sum_{k=1}^{M} S(r, z_k^*, t)E_k^{-1}S(z_k^*, s, t) - F\,\delta(r - s) \quad (4.3.146)$$

In the case of zone heating, Eq. (4.2.153), the control law is

$$u_k(t) = \left[E_k(z_{k+1}^* - z_k^*)\right]^{-1}\int_0^1\left[\int_{z_k^*}^{z_{k+1}^*} S(s, \rho, t)\, ds\right]x(\rho, t)\, d\rho \quad (4.3.147)$$

where $S(r, s, t)$ arises from

$$S_t(r, s, t) = -S_{ss} - S_{rr} + \sum_{k=1}^{M}\left\{\int_{z_k^*}^{z_{k+1}^*} S(r, z, t)\, dz\left[E_k(z_{k+1}^* - z_k^*)\right]^{-1} \times \int_{z_k^*}^{z_{k+1}^*} S(\rho, s, t)\, d\rho\right\} - F\,\delta(r - s) \quad (4.3.148)$$

In order to determine the *optimal* location and shape of the actuator signal $g_k(z)$, it is necessary to define a suitable objective functional. For example, the linear quadratic objective, Eq. (4.3.139), could be modified to

$$\min_{u_k(t),\, g_k(z)}\left\{I = \tfrac{1}{2}\int_0^1 S_f x^2(z, t_f)\, dz + \tfrac{1}{2}\int_0^{t_f}\int_0^1\left[Fx^2(z, t) + \sum_{k=1}^{M} E_k g_k(z)u_k^2(t)\right] dt\, dz\right\} \quad (4.3.149)$$

where one must optimize both with respect to $u_k(t)$ and to the function $g_k(z)$. Note that having *complete* freedom in the choice of $g_k(z)$ is equivalent to choosing the continuous control $u(z, t)$ optimally. Thus Eq. (4.3.149) is most useful when $g_k(z)$ is restricted in form. For a detailed discussion of the optimal shape and location of discrete controllers, see Refs. [25–29].

It is possible to extend the linear-quadratic optimal feedback control problem to *nonlinear* partial differential equation systems by linearizing about some nominal optimal open-loop trajectory, just as was done for lumped parameter systems in Chap. 3. However, this extension is straightforward and shall not be discussed further here.

Some more detailed examples of the application of optimal feedback control are given in Chap. 6.

4.4 FEEDBACK CONTROLLER DESIGN FOR NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

Just as for lumped parameter systems, nonlinear distributed parameter processes are difficult to analyze because most of the powerful tools of linear analysis fail to apply. In addition, many of the exact lumping methods of the last section also are not applicable. Thus one must resort to a host of ad hoc methods for controller design similar to those employed for lumped parameter systems, and some new lumping methods must be used. In this section we shall begin with a brief discussion of some nonlinear controller design methods and then proceed to a more detailed treatment of efficient lumping techniques for distributed parameter systems.

Controller Design Methods

The design methods available for nonlinear distributed parameter systems are essentially of the same type as for lumped parameter systems. They are principally:

1. Linearized linear-quadratic optimal feedback control
2. Feedback controller parameterization
3. Linearization and application of linear design methods (e.g., Sec. 4.2) to the linearized equations
4. Lumping the system to ODEs and application of the lumped parameter design methods of Sec. 3.2

Linearized linear-quadratic feedback control was discussed in Sec. 4.3 and is similar in approach and philosophy to the design procedure for lumped systems.

Feedback controller parameterization is conceptually the same procedure as discussed in Sec. 3.4 for lumped parameter systems. Basically one defines a feedback controller structure, e.g.,

$$\mathbf{u}(z, t) = \int_0^1 \mathbf{F}_B(z, r, t, \boldsymbol{\alpha}, \mathbf{x}(r, t), \mathbf{x}_d(r, t))\, dr \tag{4.4.1}$$

with parameters $\boldsymbol{\alpha}$, and then chooses the controller parameters to minimize some desired objective functional. Some examples of this design procedure may be found in [43, 44].

Linearization of the distributed parameter system about some steady state is possible in some instances. However, there can be some difficulties when the linearization is about a nonhomogeneous steady state because the coefficients become spatially dependent. Example 4.4.1 below illustrates these points.

Lumping of the distributed system and then applying lumped parameter design methods is the most straightforward approach. However, one must be aware of the potential loss of information in the lumping process, as discussed in Sec. 4.2.

Various approaches to this problem are illustrated in what follows.

Example 4.4.1 Let us consider the nonlinear problem of the control of a short, homogeneous chemical reactor in which a zero-order exothermic reaction is taking place. A mathematical model for such a system is the axial dispersion model

$$\rho_f C_{p_f} \frac{\partial T}{\partial t'} = -\rho_f C_{p_f} v \frac{\partial T}{\partial z'} + k \frac{\partial^2 T}{\partial z'^2} + (-\Delta H) k_0 e^{-E/RT} - hA_s(T - T_w) \qquad \begin{matrix} 0 < z' < l \\ t' > 0 \end{matrix} \tag{4.4.2}$$

with boundary conditions

$$z' = 0 \qquad \rho_f C_{p_f} v (T - T_f) = k \frac{\partial T}{\partial z'} \tag{4.4.3}$$

$$z' = l \qquad \frac{\partial T}{\partial z'} = 0 \tag{4.4.4}$$

where $T_f(t)$, $T_w(t)$ may be considered the manipulated variables. Let us now put the model in dimensionless form by defining

$$t = \frac{t'v}{l} \qquad z = \frac{z'}{l} \qquad \mathrm{Pe} = \frac{\rho_f C_{p_f} v_l}{k}$$

$$B = \frac{(-\Delta H) k_0 e^{-E/RT_0}}{\rho_f C_{p_f} T_0} \frac{l}{v} \qquad \gamma = \frac{E}{RT_0} \qquad \beta = \frac{hA_s l}{\rho_f C_{p_f} v}$$

$$x(z, t) = \frac{T - T_0}{T_0} \qquad u = \frac{T_w - T_0}{T_0} \qquad u_0 = \frac{T_f - T_0}{T_0} \tag{4.4.5}$$

where T_0 is a reference temperature. The system then becomes

$$\frac{\partial x(z, t)}{\partial t} = -\frac{\partial x}{\partial z} + \frac{1}{\mathrm{Pe}} \frac{\partial^2 x}{\partial z^2} + Be^{\gamma x/(1+x)} - \beta x + \beta u(t) \tag{4.4.6}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = \mathrm{Pe}\, x - \mathrm{Pe}\, u_0(t) \tag{4.4.7}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.4.8}$$

Now we shall indicate how one may linearize this system about some steady state $x_s(z)$, u_s, u_{0s}, satisfying

$$0 = -\frac{\partial x_s}{\partial z} + \frac{1}{\mathrm{Pe}} \frac{\partial^2 x_s}{\partial z^2} + Be^{\gamma x_s/(1+x_s)} - \beta x_s + \beta u_s \tag{4.4.9}$$

$$z = 0 \qquad \frac{\partial x_s}{\partial z} = \mathrm{Pe}\, x_s - \mathrm{Pe}\, u_{0s} \tag{4.4.10}$$

$$z = 1 \qquad \frac{\partial x_s}{\partial z} = 0 \tag{4.4.11}$$

Subtracting Eqs. (4.4.9) to (4.4.11) from (4.4.6) to (4.4.8) and linearizing, one may obtain the linearized equation in $\check{x}(z, t) = x(z, t) - x_s(z)$, $\check{u} = u(t) - u_s$, $\check{u}_0 = u_0(t) - u_{0s}$

$$\frac{\partial \check{x}(z, t)}{\partial t} = -\frac{\partial \check{x}(z, t)}{\partial z} + \frac{1}{\text{Pe}} \frac{\partial^2 \check{x}(z, t)}{\partial z^2} + J(z)\check{x} + \beta\check{u} \tag{4.4.12}$$

$$z = 0 \qquad \frac{\partial \check{x}}{\partial z} = \text{Pe}\,\check{x} - \text{Pe}\,\check{u}_0 \tag{4.4.13}$$

$$z = 1 \qquad \frac{\partial \check{x}}{\partial z} = 0 \tag{4.4.14}$$

where J is the Jacobian of the nonlinear term evaluated at $x_s(z)$

$$J(z) = \frac{B\gamma}{(1 + x_s)^2} e^{\gamma x_s/(1+x_s)} - \beta \tag{4.4.15}$$

Note that it is this nonlinear term which makes the analysis difficult—due to the fact that J depends on $x_s(z)$ and is thus a function of z. However, let us proceed to show how one, in some instances, may be able to use *modal* decomposition and control on these linearized equations. Let us first make the boundary condition, Eq. (4.4.13), homogeneous by inserting the homogeneous part into the differential equation with a delta function to generate the equivalent system of equations [and let us suppress the ($\check{\ }$) notation]:

$$\frac{\partial x(z, t)}{\partial t} = -\frac{\partial x(z, t)}{\partial z} + \frac{1}{\text{Pe}} \frac{\partial^2 x(z, t)}{\partial z^2} + J(z)x + \beta u + \delta(z)u_0 \tag{4.4.16}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = \text{Pe}\,x \tag{4.4.17}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.4.18}$$

This system of equations is now amenable to modal decomposition. Thus let us assume that a solution to Eqs. (4.4.16) to (4.4.18) is in the form

$$x(z, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(z) \tag{4.4.19}$$

$$u(t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(z) = u(t)\left[\sum_{n=1}^{\infty} b_n\phi_n(z)\right] \tag{4.4.20}$$

$$\delta(z)u_0(t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(z) = u_0(t)\left[\sum_{n=1}^{\infty} c_n\phi_n(z)\right] \tag{4.4.21}$$

which means

$$\sum_{n=1}^{\infty} b_n\phi_n(z) = 1 \qquad \sum_{n=1}^{\infty} c_n\phi_n(z) = \delta(z) \tag{4.4.22}$$

Applying separation of variables leads to

$$\frac{1}{a_n}\frac{da_n}{dt} = \frac{\beta b_n}{a_n}u(t) - \frac{c_n u_0(t)}{a_n}$$

$$= \frac{1}{\phi_n(z)}\left[-\frac{d\phi_n}{dz} + \frac{1}{\mathrm{Pe}}\frac{d^2\phi_n}{dz^2} + J(z)\phi_n(z)\right] = -\lambda_n \tag{4.4.23}$$

and the equations

$$\frac{da_n}{dt} + \lambda_n a_n = \beta b_n u(t) + c_n u_0(t) \tag{4.4.24}$$

$$\frac{1}{\mathrm{Pe}}\frac{d^2\phi_n}{dz^2} - \frac{d\phi_n}{dz} + [J(z) + \lambda_n]\phi_n(z) = 0 \tag{4.4.25}$$

$$\frac{d\phi_n(0)}{dz} = \mathrm{Pe}\;\phi_n(0) \tag{4.4.26}$$

$$\frac{d\phi_n(1)}{dz} = 0 \tag{4.4.27}$$

Notice that the presence of $J(z)$ in Eq. (4.4.25) is a problem because it prevents a general analytical solution to the eigenvalue problem of Eqs. (4.4.25) to (4.4.27). To surmount this problem, there are several ways to proceed.

1. If the steady-state temperature profiles of interest are essentially uniform, then one may linearize about a uniform temperature $x_s(z) = \text{const}$, and $J(z)$ becomes a constant. In this case, the separation of variables solution proceeds in a straightforward way.
2. A second approach is to assume that the $J(z)x(z, t)$ term in Eq. (4.4.16) may be expanded as follows:

$$J(z)x(z, t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(z) \tag{4.4.28}$$

   where $f_n(t)$ is to be determined.

In this instance the separation of variables procedure leads to

$$\frac{da_n}{dt} + \lambda_n a_n(t) = \beta b_n u(t) + c_n u_0(t) + f_n(t) \tag{4.4.29}$$

$$\frac{1}{\mathrm{Pe}}\frac{d^2\phi_n}{dz^2} - \frac{d\phi_n}{dz} + \lambda_n\phi_n(z) = 0 \tag{4.4.30}$$

with boundary conditions (4.4.26) and (4.4.27). Now this equation may be put into Sturm-Liouville form

$$L(\cdot) = \frac{1}{\rho(z)}\frac{d}{dz}\left[p(z)\frac{d(\cdot)}{dz}\right] + q(z)(\cdot) \tag{4.2.108}$$

by noting that

$$\frac{p(z)}{\rho(z)} = \frac{1}{\text{Pe}} \qquad \frac{1}{\rho}\frac{dp(z)}{dz} = -1 \tag{4.4.31}$$

$$q(z) = \lambda_n$$

which immediately leads to

$$p(z) = e^{-\text{Pe}\, z} \qquad \rho(z) = \frac{1}{\text{Pe}} e^{-\text{Pe}\, z} \qquad q = \lambda_n \tag{4.4.32}$$

and Eq. (4.4.30) becomes

$$\text{Pe}\, e^{\text{Pe}\, z} \frac{d}{dz}\left(e^{-\text{Pe}\, z}\frac{d\phi_n}{dz}\right) + \lambda_n \phi_n(z) = 0 \tag{4.4.33}$$

with the orthogonality relation

$$\int_0^1 e^{-\text{Pe}\, z}\phi_n(z)\phi_m(z)\, dz = 0 \qquad n \neq m \tag{4.4.34}$$

Thus $\phi_n(z)$ is the system eigenfunction and $\phi_n^*(z) = e^{-\text{Pe}\, z}\phi_n(z)$ is the adjoint eigenfunction.

To solve Eq. (4.4.30) for $\phi_n(z)$, let us make the substitution

$$\phi_n(z) = e^{\text{Pe}\, z/2}\, w_n(z) \tag{4.4.35}$$

to yield

$$\frac{1}{\text{Pe}}\ddot{w}_n + \left(\lambda_n - \frac{\text{Pe}}{4}\right) w_n = 0 \tag{4.4.36}$$

subject to

$$\frac{dw_n(0)}{dz} = \frac{\text{Pe}}{2} w_n(0) \tag{4.4.37}$$

$$\frac{dw_n(1)}{dz} = -\frac{\text{Pe}}{2} w_n(1) \tag{4.4.38}$$

Now this has the solution

$$w_n = A_n \sin \alpha_n z + B_n \cos \alpha_n z \tag{4.4.39}$$

where

$$\alpha_n^2 = \text{Pe}\left(\lambda_n - \frac{\text{Pe}}{4}\right) \tag{4.4.40}$$

Now application of Eq. (4.4.37) yields

$$\frac{A_n}{B_n} = \frac{\text{Pe}}{2\alpha_n} \tag{4.4.41}$$

and Eq. (4.4.38) gives

$$\alpha_n(A_n \cos \alpha_n - B_n \sin \alpha_n) = -\frac{\text{Pe}}{2}(A_n \sin \alpha_n + B_n \cos \alpha_n) \tag{4.4.42}$$

or substituting Eq. (4.4.41) and collecting terms,

$$\cos \alpha_n = \left[\alpha_n - \left(\frac{\text{Pe}}{2} \right)^2 \frac{1}{\alpha_n} \right] \sin \alpha_n \tag{4.4.43}$$

which may be simplified to the transcendental equation

$$\tan \alpha_n = \frac{\text{Pe}\,\alpha_n}{\alpha_n^2 - (\text{Pe}/2)^2} \qquad n = 1, 2, \ldots \tag{4.4.44}$$

Thus

$$w_n = B_n \left(\cos \alpha_n z + \frac{\text{Pe}}{2\alpha_n} \sin \alpha_n z \right) \qquad n = 1, 2, \ldots \tag{4.4.45}$$

where α_n is determined from the roots of Eq. (4.4.44). In terms of the system eigenfunctions, the solution to Eq. (4.4.30) is

$$\phi_n(z) = B_n e^{\text{Pe}\,z/2} \left(\cos \alpha_n z + \frac{\text{Pe}}{2\alpha_n} \sin \alpha_n z \right) \qquad n = 1, 2, \ldots \tag{4.4.46}$$

We shall choose B_n to make the eigenfunctions orthonormal to the adjoint eigenfunctions [see Eq. (4.4.34)]:

$$B_n = \left[\int_0^1 \left(\cos \alpha_n z + \frac{\text{Pe}}{2\alpha_n} \sin \alpha_n z \right)^2 dz \right]^{-1/2} \tag{4.4.47}$$

Thus we have accomplished the *exact modal* decomposition, where the lumped system equations are given by Eq. (4.4.29). This can be used directly in a control synthesis scheme. The coefficients c_n, d_n, $f_n(t)$ are given from the orthogonality relations

$$b_n = \int_0^1 \phi_n \, dz \qquad n = 1, 2, \ldots \tag{4.4.48}$$

$$c_n = \phi_n(0) = B_n \qquad n = 1, 2, \ldots \tag{4.4.49}$$

$$f_n(t) = \int_0^1 \left[J(z)\phi_n(z) \sum_{m=1}^{\infty} a_m(t)\phi_m(z) \right] dz \tag{4.4.50}$$

Note that in practice only N terms in the eigenfunction expansion will be retained, and this means the evaluation of N integrals for $f_n(t)$, i.e.,

$$f_n(t) = \sum_{m=1}^{N} a_m(t) I_{nm} \tag{4.4.51}$$

where

$$I_{nm} = \int_0^1 J(z)\phi_n(z)\phi_m(z) \, dz \tag{4.4.52}$$

Thus one can effectively linearize nonlinear PDEs and perform *exact* modal analysis. However, the equations for $a_n(t)$ are now coupled [due to the $f_n(t)$ term], so that the problem is multivariable in $a_n(t)$ with interactions.

Even more complex nonlinear distributed parameter systems have been analyzed through such modal decomposition using the eigenfunctions of the associated linear operator (e.g., [45, 46]). This shall be illustrated through an example.

Example 4.4.2 Let us consider a packed bed reactor with jacket temperature cooling. It is assumed that the exothermic gas phase reaction $A \rightarrow B$ is carried out in the reactor and that the reaction is zero order. Furthermore, the thermal time constants of the packing are dominant, so that we assume the gas temperature is always at quasi-steady state (i.e., the gas residence time is much shorter than the packing thermal time constant). Thus the gas temperature is given by

$$u\rho_f C_{pf} \frac{\partial T_g}{\partial z} = h_c S_c (T - T_g) - h_g S_h (T_g - T_w) \tag{4.4.53}$$

with boundary conditions

$$z' = 0 \qquad T_g = T_f \tag{4.4.54}$$

Now the catalyst packing has the equation

$$\rho_s C_{p_s} \frac{\partial T}{\partial t'} - k_e \frac{\partial^2 T}{\partial z'^2} = (-\Delta H) k_0 e^{-E/RT} - h_c S_c (T - T_g) - h_p S_h [T - T_w(t)] \tag{4.4.55}$$

where S_c is the pellet surface area/volume and S_h is the surface area for wall heat transport/unit volume, and with boundary conditions

$$z' = 0 \qquad \frac{\partial T}{\partial z'} = 0 \tag{4.4.56}$$

$$z' = l \qquad \frac{\partial T}{\partial z'} = 0 \tag{4.4.57}$$

Now if we put this problem in dimensionless form by defining

$$x_g = \frac{T_g - T_f}{T_f} \qquad x = \frac{T - T_f}{T_f} \qquad \alpha_c = \frac{h_c S_c l}{u\rho_f C_{pf}} \qquad \alpha_g = \frac{h_g S_h l}{u\rho_f C_{pf}}$$

$$z = \frac{z'}{l} \qquad t = \frac{t' k_e}{\rho_s C_{p_s} l^2} \qquad \gamma = \frac{E}{RT_f}$$

$$B = \frac{(-\Delta H) k_0 e^{-\gamma} l^2}{k_e T_f} \qquad \beta_c = \frac{h_c S_c l^2}{k_e} \qquad \beta_p = \frac{h_p S_h l^2}{k_e} \tag{4.4.58}$$

$$u(t) = \frac{T_w}{T_f} \qquad \frac{E}{RT} = \gamma \frac{1}{x+1} \qquad \frac{E}{RT_f} - \frac{E}{RT} = \gamma \frac{x}{x+1}$$

then we get

$$\frac{\partial x_g}{\partial z} = \alpha_c(x - x_g) - \alpha_g(x_g - u) \tag{4.4.59}$$

$$x_g(0, t) = 0 \tag{4.4.60}$$

$$\frac{\partial x(z, t)}{\partial t} - \frac{\partial^2 x(z, t)}{\partial z^2} = Be^{\gamma[x/(x+1)]} - \beta_c(x - x_g) - \beta_p(x - u) \tag{4.4.61}$$

$$z = 0 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.4.62}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.4.63}$$

Now the equation for x_g can be solved as

$$\begin{aligned} x_g(z, t) &= \int_0^1 e^{-(\alpha_c+\alpha_g)(z-r)}\left[\alpha_c x(r, t) + \alpha_g u(t)\right] dr \\ &= \frac{\alpha_g u(t)}{\alpha_c + \alpha_g}\left(1 - e^{-(\alpha_c+\alpha_g)z}\right) + \alpha_c \int_0^1 e^{-(\alpha_c+\alpha_g)(z-r)} x(r, t)\, dr \end{aligned} \tag{4.4.64}$$

If we assume that the solution can be found in terms of

$$\begin{aligned} x_g(z, t) &= \sum_{n=0}^{N} c_n(t)\phi_n(z) \\ x(z, t) &= \sum_{n=0}^{N} a_n(t)\phi_n(z) \end{aligned} \tag{4.4.65}$$

$$u(t) = \sum_{n=0}^{N} b_n(t)\phi_n(z) \tag{4.4.66}$$

and assume the nonlinear terms can be expanded in terms of the complete set of functions $\phi_n(z)$:

$$F(x, x_g) = Be^{\gamma[x/(x+1)]} - \beta_c(x - x_g) - \beta_p x = \sum_{n=0}^{N} f_n(t)\phi_n(z) \tag{4.4.67}$$

then Eq. (4.4.61) becomes

$$\phi_n(z)\frac{da_n(t)}{dt} - a_n\frac{d^2\phi_n}{dz^2} = \left[\beta_p b_n(t) + f_n(t)\right]\phi_n(z) \tag{4.4.68}$$

with the boundary conditions

$$z = 0 \qquad \frac{d\phi_n}{dz} = 0 \tag{4.4.69}$$

$$z = 1 \qquad \frac{d\phi_n}{dz} = 0 \tag{4.4.70}$$

By separating variables we are led to the equations

$$\frac{da_n(t)}{dt} + \lambda_n a_n(t) = \beta_p b_n(t) + f_n(t) \tag{4.4.71}$$

$$\frac{d^2\phi_n(z)}{dz^2} + \lambda_n \phi_n(z) = 0 \tag{4.4.72}$$

Now the eigenvalue problem [Eqs. (4.4.69) to (4.4.72)] is the same one treated in Sec. 4.2 and has the solution

$$\phi_n(z) = \begin{cases} 1 & n = 0 \\ \sqrt{2}\ \cos n\pi z & n = 1, 2, \ldots, N \end{cases} \tag{4.4.73}$$

and

$$\lambda_n = n^2\pi^2 \qquad n = 0, 1, 2, \ldots, N \tag{4.4.74}$$

From the orthogonality relations

$$a_n(t) = \int_0^1 \phi_n(z)x(z, t)\, dz \tag{4.4.75}$$

$$b_n(t) = \int_0^1 \phi_n(z)u(t)\, dz = \begin{cases} u(t) & n = 0 \\ 0 & n = 1, 2, \ldots, N \end{cases} \tag{4.4.76}$$

$$c_n(t) = \int_0^1 \phi_n(z)x_g(z, t)\, dz \tag{4.4.77}$$

$$f_n(t) = \int_0^1 \phi_n(z)F(x, x_g)\, dz \tag{4.4.78}$$

Thus the control scheme has the structure shown in Fig. 4.17. Notice that a nonlinear function generator is needed to determine the coefficients f_n and that the controller must deal with nonlinear behavior $f_n(\mathbf{a}, \mathbf{c})$. The resulting lumped system, Eq. (4.4.71), is a nonlinear multivariable lumped parameter system with interactions because Eq. (4.4.67) leads to a nonlinear relation between f_n and a_n, c_n; i.e., substituting Eq. (4.4.65) into Eq. (4.4.67) and applying Eq. (4.4.78) yields

$$\int_0^1 \phi_n(z)\left\{B \exp\left[\gamma\left(\frac{\sum a_n\phi_n}{1 + \sum a_n\phi_n}\right)\right] - \beta_c \sum (a_n - c_n)\phi_n - \beta_p \sum a_n\phi_n\right\} dz = f_n \tag{4.4.79}$$

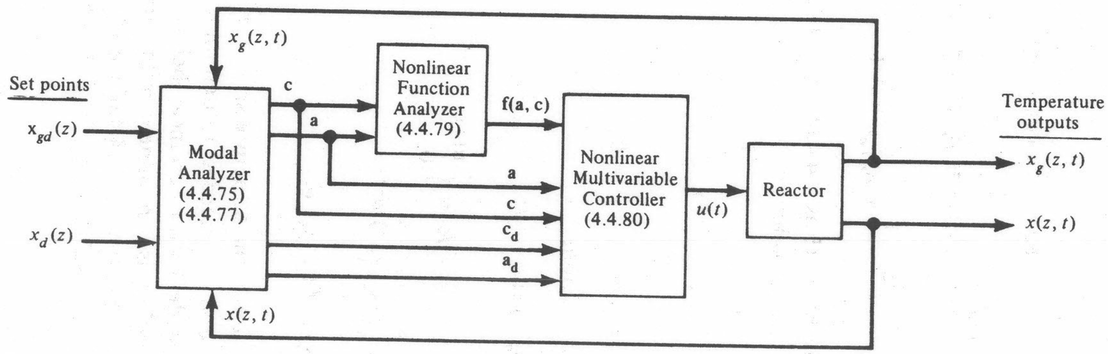

Figure 4.17 Nonlinear distributed parameter control scheme for packed bed reactor example.

Thus the lumped controller design equation takes the form

$$\frac{da_n(t)}{dt} = -n^2\pi^2 a_n(t) + f_n(a_0, a_1, \ldots, a_N, c_1, c_2, \ldots, c_N) + \beta_p b_n(t) \tag{4.4.80}$$

and we must appeal to the nonlinear design techniques of Sec. 3.4. This Galerkin modal decomposition procedure should converge for sufficient number of eigenfunctions. This method can be applied whenever the nonlinearity appears as a forcing function in the partial differential equation—a very frequent situation.

Lumping Methods

If one wishes to "lump" the partial differential equations (either before control system design or after the design in order to numerically solve the design equations), there are a large number of very efficient methods available. Therefore classical finite difference methods should be used only after several of the more efficient methods have been considered. The most efficient lumping methods may be viewed as *pseudo-modal methods*; that is, one expands the solution into a set of known basis functions $\phi_n(z)$; for example,

$$x_a(z, t) = \sum_{n=1}^{N} a_n(t)\phi_n(z) \tag{4.4.81}$$

Then one uses some goodness-of-fit criterion to determine the coefficients $a_n(t)$ which yield the best approximation to $x(z, t)$. Notice that this is simply an extension of the modal decomposition method with the difference that *pseudo-modal* methods may use any set of basis functions, while eigenfunction expansions use the eigenfunctions of the linear operator. It is possible to treat both linear and nonlinear partial differential equation systems by pseudo-modal methods such as the *method of weighted residuals* [47–51]. Given a nonlinear system, for example,

$$\frac{\partial x(z, t)}{\partial t} = A_2(z, t, x)\frac{\partial^2 x}{\partial z^2} + A_1(z, t, x)\frac{\partial x}{\partial z} + A_0(z, t, x)x + f(x, u, z, t)$$

$$\begin{aligned} &0 \le z \le 1 \\ &t > 0 \end{aligned} \tag{4.4.82}$$

with boundary conditions

$$z = 0 \qquad b_1(t, x)\frac{\partial x}{\partial z} + b_0(t, x)x = f_0(x, u_0, t) \tag{4.4.83}$$

$$z = 1 \qquad c_1(t, x)\frac{\partial x}{\partial z} + c_0(t, x)x = f_1(x, u_1, t) \tag{4.4.84}$$

one can proceed to reduce the system to a set of ODEs by pseudo-modal methods. To do this, one takes a set of *basis functions* $\phi_n(z)$, which are analogous to the eigenfunctions for the linear problem. These functions $\phi_n(z)$ should be *complete* and preferably orthogonal with some weighting function $\rho(z)$:

$$\int_0^1 \rho(z)\phi_n(z)\phi_m(z)\,dz = 0 \qquad \text{for } n \neq m \tag{4.4.85}$$

The choice of the $\phi_n(z)$ is arbitrary, but it is helpful to try the eigenfunctions of a related linear problem—particularly if Eqs. (4.4.82) to (4.4.84) are only slightly nonlinear.

The approximate solution to the problem is then expressed in terms of N basis functions, Eq. (4.4.81), where the coefficients $a_n(t)$ must be determined in such a way that $x_a(z, t)$ is a good approximation to the solution to Eqs. (4.4.82) to (4.4.84). Many criteria are possible for measuring a good approximation, and each criterion chosen leads to a different technique. *The method of weighted residuals* (MWR) is one class of methods. One often chooses the $\phi_n(z)$ so that the boundary conditions, Eqs. (4.4.83) and (4.4.84), are satisfied exactly* and the *residual* (let us assume u, u_0, $u_1 = 0$ for the moment)

$$R(z, t) = \frac{\partial \hat{x}}{\partial t} - \hat{A}_2\frac{\partial^2 \hat{x}}{\partial z^2} - \hat{A}_1\frac{\partial \hat{x}}{\partial z} - \hat{A}_0\hat{x} - \hat{f} \tag{4.4.86}$$

must be made small in the sense that

$$\int_0^1 w_i(z)R(z, t)\,dz = 0 \qquad i = 0, 1, 2, \ldots, N \tag{4.4.87}$$

where the $w_i(z)$ are a set of weighting functions to be chosen. The choice of weighting functions can lead to several different criteria. Let us discuss some of the types of criteria possible.

1. *Galerkin's method* If the weighting functions are chosen to be the basis functions themselves,

$$w_n(z) = \phi_n(z) \qquad n = 0, 1, 2, \ldots, N \tag{4.4.88}$$

then the technique is called Galerkin's method. This has the advantage that the residual is made orthogonal to each basis function and is, therefore, the best solution possible in the space made up of the $N + 1$ functions $\phi_n(z)$. Thus as $N \to \infty$, $R(z, t) \to 0$ because it will be orthogonal to every function in a complete set of functions.

2. *Method of subdomains* If we choose the w_n to be a set of Heaviside functions breaking the region $0 \leq z \leq 1$ into subdomains, i.e.,

$$w_n(z) = \begin{cases} 1 & z_n < z < z_{n+1} \\ 0 & \text{elsewhere} \end{cases} \tag{4.4.89}$$

then Eq. (4.4.87) becomes

$$\int_{z_n}^{z_{n+1}} R(z, t)\,dz = 0 \qquad n = 0, 1, 2, \ldots, N \tag{4.4.90}$$

* Cases where the boundary conditions are not satisfied exactly are also possible (e.g., [47], [51]), but usually it is more convenient to satisfy them a priori.

and the average value of the residual must vanish over each of $N + 1$ subdomains. For $N = 0$ (zeroth approximation), $z_0 = 0$, $z_1 = 1$, and we have

$$\int_0^1 R(z, t)\, dz = 0 \tag{4.4.91}$$

This special case is called the *integral method* and is used widely in boundary-layer problems.

3. *Method of moments*. If the w_n are chosen to be powers of z, then Eq. (4.4.87) becomes

$$\int_0^1 z^n R(z, t)\, dz = 0 \qquad n = 0, 1, 2, \ldots, N \tag{4.4.92}$$

and the first N moments of $R(z, t)$ are required to vanish.

4. *Collocation methods*. If the w_n are chosen to be delta functions $\delta(z - z_n)$, then Eq. (4.4.87) becomes

$$R(z_n, t) = 0 \qquad n = 0, 1, 2, \ldots, N \tag{4.4.93}$$

and the differential equation is required to be solved exactly at N points on the spatial domain. The collocation method has been refined greatly [47, 49, 51] and has been shown to be extremely powerful. The recent versions are called *orthogonal collocation* because orthogonal polynomials are used as the basis functions and the collocation points are specified automatically.

There are other pseudo-modal methods which can be considered as MWR techniques. For example, finite element methods [50], the use of spline functions [47], and other approaches may be shown to fall within this framework. Results on the convergence of pseudo-modal methods are not abundant. Galerkin methods may be shown to be uniformly convergent for a rather broad class of problems (see [47, 52]), but general convergence results have not been proved at present for most other methods.

Although pseudo-modal methods have been known for more than fifty years, it has only been recently that extensive computational experience has been available. The most popular approaches appear to be Galerkin's method (e.g., [47], [53–56]), collocation methods [47, 49, 51], and finite element methods [50]. We shall illustrate the application of some of these methods with examples.

Example 4.4.3 Let us consider the heating of a thin metal rod in a furnace as in Example 4.3.3. However, in this case, the temperature range of interest is so wide that ρC_p and k depend strongly on temperature. Thus modeling equations take the form

$$\rho C_p \frac{\partial T}{\partial t'} = \frac{\partial (k(T)\, \partial T/\partial z')}{\partial z'} + q(z', t') \tag{4.4.94}$$

with boundary conditions

$$\frac{\partial T}{\partial z'} = 0 \qquad \text{at } z = 0, l \tag{4.4.95}$$

In this instance the differential spatial operator is nonlinear and given by

$$\frac{\partial T}{\partial t'} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial z'^2} + \frac{1}{\rho C_p}\frac{\partial k}{\partial T}\left(\frac{\partial T}{\partial z'}\right)^2 + q(z', t) \tag{4.4.96}$$

Now let us suppose that it is possible to neglect the second term and represent the nonlinearity in the first term by the form

$$\alpha = \frac{k}{\rho C_p} = \alpha_0 + \alpha_1 T \tag{4.4.97}$$

Furthermore, let us put the problem in dimensionless form by setting

$$z = \frac{z'}{l} \qquad x = \frac{T}{T_0} \qquad t' = \frac{\alpha_0 t}{l^2} \qquad \beta = \frac{\alpha_1}{\alpha_0}T_0 \qquad u = \frac{ql^2}{\alpha_0 T_0} \tag{4.4.98}$$

then we get

$$\frac{\partial x(z, t)}{\partial t} = (1 + \beta x)\frac{\partial^2 x(z, t)}{\partial z^2} + u(z, t) \tag{4.4.99}$$

with boundary conditions

$$z = 0 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.4.100}$$

$$z = 1 \qquad \frac{\partial x}{\partial z} = 0 \tag{4.4.101}$$

Now if we choose as basis functions the eigenfunctions of the problem when $\beta = 0$, we obtain

$$x(z, t) = \sum_{n=0}^{N} a_n(t)\phi_n(z) \tag{4.4.102}$$

$$u(z, t) = \sum_{n=0}^{N} b_n(t)\phi_n(z) \tag{4.4.103}$$

where

$$\phi_n(z) = \begin{cases} 1 & n = 0 \\ \sqrt{2}\cos n\pi z & n = 1, 2, \ldots, N \end{cases} \tag{4.4.104}$$

Now if we substitute Eqs. (4.4.102) and (4.4.103) into Eqs. (4.4.99) to (4.4.101), we see that the boundary conditions are satisfied exactly and the residual becomes

$$R(z, t) = \sum_{n=0}^{N} \phi_n(z)\left\{\frac{da_n}{dt} - \sum_{m=0}^{N}\left[\beta a_n a_m (m\pi)^2 \phi_m(z)\right] - b_n(t) + a_n(t)(n\pi)^2\right\} \tag{4.4.105}$$

Now if we apply Galerkin's method to the problem, we must have

$$\int_0^1 \phi_n(z) R(z, t)\, dz = 0 \qquad n = 0, 1, 2, \ldots, N \tag{4.4.106}$$

which becomes

$$\frac{da_n}{dt} + (n\pi)^2 a_n - b_n(t) - \beta \int_0^1 \sum_{m=0}^{N} \sum_{k=0}^{N} \phi_n(z)\phi_m(z)\phi_k(z) a_m a_k (m\pi)^2 \, dz$$

or

$$\frac{da_n}{dt} + (n\pi)^2 a_n - b_n(t) - \beta \sum_{m=0}^{N} \sum_{k=0}^{N} a_m a_k (m\pi)^2 \int_0^1 \phi_n(z)\phi_m(z)\phi_k(z) \, dz = 0$$

$$n = 0, 1, 2, \ldots, N \qquad (4.4.107)$$

Now let us denote

$$I_{nmk} = \int_0^1 \phi_n \phi_m \phi_k \, dz \qquad (4.4.108)$$

Although it requires some algebra, this integral I_{nmk} can be evaluated analytically in the following way:

$$I_{n,m,k} = 2\sqrt{2} \int_0^1 \cos n\pi z \cos m\pi z \cos k\pi z \, dz = \sqrt{2} \int_0^1 [\cos (n+m)\pi z + \cos(n-m)\pi z] \cos k\pi z \, dz \qquad (4.4.109)$$

Now using integral tables (e.g., [57], p. 105), one obtains

$$I_{n,m,k} = 0 \qquad \text{if } n - m \neq k, m + n \neq k \qquad (4.4.110)$$

Also if $n - m = k$ or $n + m = k$ ([48], p. 101), then

$$I_{n,m,k} = \frac{\sqrt{2}}{2} \qquad (4.4.111)$$

and for $n = m = k$ ([48], p. 102),

$$I_{n,m,k} = 0 \qquad (4.4.112)$$

Thus in summary

$$I_{n,m,k} = \begin{cases} 0 & \text{if } n = m = k \text{ or } (n - m \neq k \text{ and } n + m \neq k) \\ \delta_{pq} & \text{if one of the indices } n, m, k = 0 \\ & \text{and } p, q \text{ are the remaining indices} \\ \dfrac{\sqrt{2}}{2} & \text{if } n + m = k \text{ or } n - m = k \end{cases} \qquad (4.4.113)$$

Therefore Eq. (4.4.107) becomes

$$\frac{da_n}{dt} + (n\pi)^2 a_n - b_n(t) - \beta \sum_{m=0}^{N} \sum_{k=0}^{N} a_m a_k (m\pi)^2 I_{n,m,k} = 0$$

$$n = 0, 1, 2, \ldots, N \qquad (4.4.114)$$

To illustrate this lumping, let us look at the first few terms

$$\frac{da_0}{dt} - b_0 - \beta(\pi^2 a_1^2 + 4\pi^2 a_2^2 + \cdots) = 0 \tag{4.4.115}$$

$$\frac{da_1}{dt} + \pi^2 a_1 - b_1 - \beta\left[a_1\pi^2\left(a_0 + \frac{a_2\sqrt{2}}{2}\right)\right.$$

$$\left. + a_2 4\pi^2\left(a_1\frac{\sqrt{2}}{2} + a_3\frac{\sqrt{2}}{2}\right) + \cdots\right] = 0 \tag{4.4.116}$$

$$\frac{da_2}{dt} + 4\pi^2 a_2 - b_2 - \beta\left[a_0 4\pi^2 a_2 + \frac{a_1^2\sqrt{2}}{2}\pi^2\right.$$

$$\left. + a_1 a_3\left(\frac{\sqrt{2}}{2}9\pi^2 + \frac{\sqrt{2}}{2}\pi^2\right) + \cdots\right] = 0$$

$$\text{etc.} \tag{4.4.117}$$

Thus the coefficients are coupled and nonlinear, and the controller design requires the use of nonlinear lumped parameter design techniques. The control structure will be very similar to that shown in Fig. 4.17.

It should be noted that Galerkin's method leads to *exact modal* analysis when the system becomes linear (e.g., when $\beta = 0$ here), and thus is to be recommended on that basis. Also if we choose an orthonormal basis, the orthogonality relations allow us to synthesize $u(z, t)$ from Eq. (4.4.102) and to obtain $a_n(t)$ from the data from

$$a_n(t) = \int_0^1 x(z, t)\phi_n(z)\, dz \tag{4.4.118}$$

Thus Galerkin's method when used with a set of orthonormal basis functions has all the properties of exact modal analysis *except*:

1. The solution to the system is only approximate, and the smallness of the residual $R(z, t)$ will depend on the type and number of basis functions chosen.
2. The lumped parameter controller design, Eq. (4.4.114), in the coefficients $a_n(t)$, $b_n(t)$ is a coupled, nonlinear multivariable design problem of order N so that simple PID single-loop controllers will not suffice, but some control design compensating for interaction must be used.

It is useful to compare the eigenfunction expansion method of Examples 4.4.1 and 4.4.2 with the approximate Galerkin method of Example 4.4.3. The principal differences in the first two examples were that the spatially dependent coefficients of the forcing function in Example 4.4.1 and the nonlinear forcing function of Example 4.4.2 are assumed to be exactly expandable in the eigenfunctions of the linear differential operator. In this case uniform convergence of the Galerkin aproximation can be shown.* In contrast, for Example 4.4.3, there is the requirement that the second spatial derivative be exactly expandable in the eigenfunctions of the linear operator for the separation of variables method to apply. However, the second space derivative of the solution will, in general, not be uniformly convergent in the basis functions (see [47], pp. 373–379), and thus the Galerkin method will not necessarily converge uniformly for this example.

It should be emphasized that Galerkin's method is not the only technique one may use for this example. In fact, collocation techniques would also seem to have some advantages, particularly if the control were applied at discrete points in space $u(z_i, t)$, $i = 1, 2, m$. If the collocation points were chosen at these points, then the control could be incorporated directly.

The detailed application of these methods to example nonlinear distributed systems is discussed in Chap. 6.

4.5 CONTROL OF SYSTEMS HAVING TIME DELAYS

An especially important class of distributed parameter systems is *hereditary systems*, or *systems having time delays*. This class of dynamic systems arises in a wide range of applications, including paper making, chemical reactors, and distillation. Example 4.1.2 serves to illustrate a very simple single-loop control problem with a transport time delay. The principal difficulty with time delays in the control loop is the increased phase lag, which leads to unstable control system behavior at relatively low controller gains. This limits the amount of control action possible in the presence of time delays.

In multivariable time-delay systems with multiple delays, these problems are even more complex. In these problems, the normal control difficulties due to loop interactions (see Chap. 3) are complicated by the additional effects of time delays. A good example of this type of problem is in *distillation column* control. To illustrate, let us consider the problem discussed in Example 3.2.8, where the column output compositions y_i are related to the sidestream flow rates u_j by a transfer function matrix

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) \tag{4.5.1}$$

Now in practice, the transfer function matrix often has elements in the form

$$g_{ij}(s) = \frac{K_{ij} e^{-\beta_{ij} s} \prod_{p=1}^{l} (a_{ijp} s + 1)}{\prod_{q=1}^{l} (b_{ijq} s + 1)} \tag{4.5.2}$$

or rewriting,

$$g_{ij}(s) = \frac{e^{-\beta_{ij} s}\left(e_{ijl} s^l + e_{ijl-1} s^{l-1} + \cdots + e_{ij0}\right)}{s^l + d_{ijl-1} s^{l-1} + \cdots + d_{ij0}} \tag{4.5.3}$$

* For convergence in Example 4.4.1, see [47], p. 371; for Example 4.4.2, see [47], pp. 373, 374.

Here the factor $e^{-\beta_{ij}s}$ denotes a time delay associated with the ijth element of $\mathbf{G}(s)$. Hence the transfer function for the distillation column of Example 3.2.8 would more often in practice have the form

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{0.7e^{-2s}}{1+9s} & 0 & 0 \\ \dfrac{2.0e^{-5s}}{1+8s} & \dfrac{0.4e^{-2s}}{1+6s} & 0 \\ \dfrac{2.3e^{-6s}}{1+10s} & \dfrac{2.3e^{-4s}}{1+8s} & \dfrac{2.1e^{-2s}}{1+7s} \end{bmatrix} \tag{4.5.4}$$

where the time delays in each element of the matrix will be different.

The time-domain realization of transfer functions such as Eqs. (4.5.1) to (4.5.3) will have a slightly different form from that discussed in Chap. 3. In terms of Eq. (4.5.3), the time-domain representation is given by

$$\frac{d\mathbf{z}_{ij}(t)}{dt} = \mathbf{A}_{ij}\mathbf{z}_{ij}(t) + \mathbf{h}_{ij}u_j(t-\beta_{ij}) \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{array} \tag{4.5.5}$$

where

$$x_i(t) = \sum_{j=1}^{m} \left[z_{ij1}(t) + h_{ij0}u_j(t-\beta_{ij}) \right] \tag{4.5.6}$$

$$\mathbf{z}_{ij} = \begin{bmatrix} z_{ij1} \\ z_{ij2} \\ \vdots \\ z_{ijl} \end{bmatrix} \qquad \mathbf{A}_{ij} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 \quad 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 \cdots & \cdots & & \cdots & 1 \\ -d_{ij0} & -d_{ij1} & & \cdots & -d_{ijl-1} \end{bmatrix} \tag{4.5.7}$$

$$\mathbf{h}_{ij} = \begin{bmatrix} h_{ij1} \\ h_{ij2} \\ \vdots \\ h_{ijl} \end{bmatrix} \tag{4.5.8}$$

and

$$h_{kj0} = e_{ijl}$$

$$h_{ijl-p} = e_{ijp} - \sum_{q=0}^{l-p-1} h_{ijq}d_{ijq+p} \qquad p = 0, 1, 2, \ldots, l-1 \tag{4.5.9}$$

Thus transfer functions of the form of Eqs. (4.5.1) to (4.5.3) when converted to state space take the form of ODEs with delays in the control. Other realizations of time delay systems are discussed by Ogunnaike [58].

In this section we shall first present a rather general formulation of time-delay control problems and then consider some relatively simple design procedures for this class of problem. Finally, optimal control theory and practice for time-delay systems shall be discussed.

A General Formulation

A very general representation of systems having delays in the control variables $\mathbf{u}(t)$, state variables $\mathbf{x}(t)$, or output variables $\mathbf{y}(t)$ is illustrated in Fig. 4.18. Each type of delay can be thought of as a transport lag in a pipe modeled by a first-order hyperbolic equation (see Example 4.1.2). The inlet to these pipes can be either the state variable $\mathbf{x}(t)$ or control $\mathbf{u}(t)$. If we further allow integral hereditary terms, the general formulation is given as follows:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\big(\mathbf{x}(t), \mathbf{w}_1(r_1, t), \mathbf{w}_1(r_2, t), \ldots \mathbf{w}_1(r_\delta, t), \mathbf{u}(t), \mathbf{w}_2(\hat{r}_1, t),$$

$$\mathbf{w}_2(\hat{r}_2, t), \ldots, \mathbf{w}_2(\hat{r}_\mu, t)\big) + \int_0^1 \mathbf{K}(\mathbf{w}_1(r, t), r, \mathbf{u}(t))\, dr \quad (4.5.10)$$

$$\frac{\partial \mathbf{w}_1(r, t)}{\partial t} = -\mathbf{M}_1(r, t)\frac{\partial \mathbf{w}_1(r, t)}{\partial r} + \mathbf{g}_1(\mathbf{w}_1(r, t), \mathbf{w}_2(r, t), \mathbf{x}(t), \mathbf{u}(t)) \quad (4.5.11)$$

$$\frac{\partial \mathbf{w}_2(r, t)}{\partial t} = -\mathbf{M}_2(r, t)\frac{\partial \mathbf{w}_2(r, t)}{\partial r} + \mathbf{g}_2(\mathbf{w}_1(r, t), \mathbf{w}_2(r, t), \mathbf{x}(t), \mathbf{u}(t)) \quad (4.5.12)$$

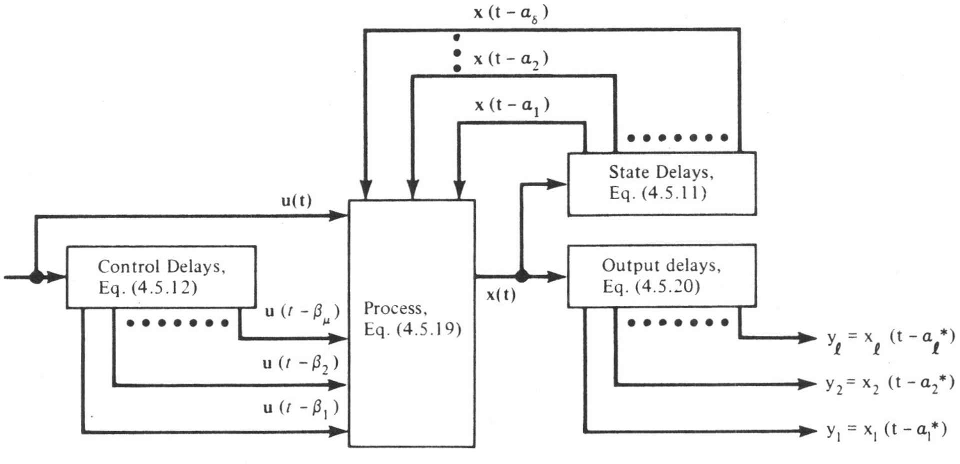

Figure 4.18 Example of transport lag models for systems having state, control, and output delays.

with initial and boundary conditions

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{4.5.13}$$

$$\mathbf{w}_1(r, 0) = \mathbf{w}_{10}(r) \tag{4.5.14}$$

$$\mathbf{w}_2(r, 0) = \mathbf{w}_{20}(r) \tag{4.5.15}$$

$$\mathbf{w}_1(0, t) = \mathbf{b}_1(\mathbf{x}(t)) \tag{4.5.16}$$

$$\mathbf{w}_2(0, t) = \mathbf{b}_2(\mathbf{u}(t)) \tag{4.5.17}$$

The outputs take the form

$$\mathbf{y}(t) = \mathbf{h}\big(\mathbf{x}(t), \mathbf{w}_1(r_1^*, t), \mathbf{w}_1(r_2^*, t), \ldots, \mathbf{w}_1(r_\gamma^*, t)\big) + \int_0^1 H(\mathbf{w}_1(r, t), r)\, dr \tag{4.5.18}$$

It is possible to show that this coupled set of ODEs and first-order hyperbolic PDEs have as special cases all the commonly encountered time-delay problems [59, 60].

As an example, if one allows that $\mathbf{K} = \mathbf{H} = \mathbf{g}_1 = \mathbf{g}_2 = 0$, $\mathbf{b}_1(\mathbf{x}(t)) = \mathbf{x}(t)$, $\mathbf{b}_2(\mathbf{u}(t)) = \mathbf{u}(t)$, $\mathbf{M}_1 = \mathbf{I}\alpha_{\max}^{-1}$, $\mathbf{M}_2 = \mathbf{I}\beta_\mu^{-1}$, $r_i = \alpha_i/\alpha_{\max}$, $r_j^* = \alpha_j^*/\alpha_{\max}$, $\hat{r}_i = \beta_i/\beta_\mu$, $\mathbf{w}_1(r, 0) = \mathbf{\Phi}(-r\beta_\mu)$,$\mathbf{w}_2(r, 0) = \phi(-r\alpha_{\max})$, then $\mathbf{w}_1(r_i, t) = \mathbf{x}(t - \alpha_i)$, $\mathbf{w}_1(r_i^*, t) = \mathbf{x}(t - \alpha_i^*)$, $\mathbf{w}_2(\hat{r}_i, t) = \mathbf{u}(t - \beta_i)$. Thus Eqs. (4.5.10) to (4.5.18) reduce to a *nonlinear ODE system having constant time delays*:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\big(\mathbf{x}(t), \mathbf{x}(t - \alpha_1), \ldots, \mathbf{x}(t - \alpha_\delta), \mathbf{u}(t), \mathbf{u}(t - \beta_1), \ldots, \mathbf{u}(t - \beta_\mu)\big) \tag{4.5.19}$$

$$\mathbf{y}(t) = \mathbf{h}\big(\mathbf{x}(t), \mathbf{x}(t - \alpha_1^*), \mathbf{x}(t - \alpha_2^*), \ldots, \mathbf{x}(t - \alpha_\gamma^*)\big) \tag{4.5.20}$$

$$\mathbf{x}(t) = \phi(t) \qquad -\alpha_{\max} \le t \le 0 \qquad \alpha_{\max} = \max(\alpha_\delta, \alpha_\gamma^*) \tag{4.5.21}$$

$$\mathbf{u}(t) = \mathbf{\Phi}(t) \qquad -\beta_\mu \le t \le 0 \tag{4.5.22}$$

Another important class of problems arises if one allows

$$\mathbf{K} = \mathbf{H} = \mathbf{g}_1 = \mathbf{g}_2 = \mathbf{0} \qquad \delta = \gamma = \lambda = 1 \qquad r_1 = r_1^* = \hat{r}_1 = 1$$

$$\mathbf{b}_1(\mathbf{x}(t)) = \big[\mathbf{x}^T(t), \mathbf{x}^T(t), \mathbf{x}^T(t), \ldots, \mathbf{x}^T(t)\big]^T$$

$$\mathbf{b}_2(\mathbf{u}(t)) = \big[\mathbf{u}^T(t), \mathbf{u}^T(t), \mathbf{u}^T(t), \ldots, \mathbf{u}^T(t)\big]^T$$

$$\mathbf{M}_1(r, t) = \big[M_{1ij}(r, t)\big] = \begin{cases} 0 & i \ne j \\ \dfrac{1 - r\dot{\alpha}_i}{\alpha_i}\mathbf{I} & i = 1, 2, \ldots, \rho \qquad i = j \\ \left(\dfrac{1 - r\dot{\alpha}_{i-\rho}^*}{\alpha_{i-\rho}^*}\right)\mathbf{I} & i = \rho + 1, \ldots, \rho + w \end{cases}$$

$$\mathbf{M}_2(r, t) = \big[M_{2ij}(r, t)\big] = \begin{cases} 0 & i \ne j \\ \dfrac{i - r\dot{\beta}_i}{\beta_i}\mathbf{I} & i = 1, 2, \ldots, \eta \end{cases}$$

Then
$$\mathbf{w}_1(r,t) = \left[\mathbf{w}_{11}^T(r,t), \mathbf{w}_{12}(r,t)^T, \ldots, \mathbf{w}_{1,\rho}(r,t)^T,\right.$$
$$\left.\mathbf{w}_{11}^*(r,t), \mathbf{w}_{12}^*(r,t), \ldots, \mathbf{w}_{1w}^*(r,t)\right]^T$$
$$\mathbf{w}_2(r,t) = \left[\mathbf{w}_{21}^T(r,t)\ \mathbf{w}_{22}^T(r,t), \ldots, \mathbf{w}_{2\eta}^T(r,t)\right]$$
$$\mathbf{w}_{1i}(r,0) = \phi(-r\alpha_i(0)), \mathbf{w}_{1j}^*(r,0) = \phi(-r\alpha_j^*(0))$$
$$\mathbf{w}_{2i}(r,0) = \Phi(-r\beta_i(0)),$$
then
$$\mathbf{w}_{1i}(1,t) = \mathbf{x}(t-\alpha_i(t)), \mathbf{w}_{1j}^*(1,t) = \mathbf{x}(t-\alpha_i^*(t))$$
$$\mathbf{w}_{2i}(1,t) = \mathbf{u}(t-\beta_i(t))$$

In this case we obtain a *nonlinear system of ODEs with time-varying time delays*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\alpha_1(t)), \ldots, \mathbf{x}(t-\alpha_\rho(t)),$$
$$\mathbf{u}(t), \mathbf{u}(t-\beta_1(t)), \ldots, \mathbf{u}(t-\beta_\eta(t))) \tag{4.5.23}$$
$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{x}(t-\alpha_1^*(t)), \ldots, \mathbf{x}(t-\alpha_\omega^*(t))) \tag{4.5.24}$$
$$\dot{\alpha}_i(t) < 1 \tag{4.5.25}$$
$$\dot{\alpha}_j^*(t) < 1 \tag{4.5.26}$$
$$\dot{\beta}_i(t) < 1 \tag{4.5.27}$$
$$\mathbf{x}(t) = \phi(t) \qquad -\alpha_{\max} \le t \le 0 \tag{4.5.28}$$
$$\mathbf{u}(t) = \Phi(t) \qquad -\beta_{\max} \le t \le 0 \tag{4.5.29}$$
$$\alpha_{\max} = \max\left[\alpha_1(0), \ldots, \alpha_\rho(0), \alpha_1^*(0), \ldots, \alpha_\omega^*(0)\right] \tag{4.5.30}$$
$$\beta_{\max} = \max\left[\beta_1(0), \ldots, \beta_\eta(0)\right] \tag{4.5.31}$$

The conditions of Eqs. (4.5.25) to (4.5.27) are necessary to ensure that the time delays do not increase faster than time itself.

Many other time-delay problems of interest may be extracted from the general formulation, Eqs. (4.5.10) to (4.5.18), but we shall not treat all of them here. The reader is referred to [59, 60] for a fuller discussion of these.

Time-Delay Compensation Methods

Aside from optimal control design methods to be discussed later in this section, controller design procedures for time-delay systems usually involve using a prediction device in the control loop to compensate for the time delays. If this is done, then often the standard ODE multivariable controller design procedures of Chap. 3 may be used. There are several methods which may be used to compensate for delays in the states, outputs, or controls. One may use one of the statistical state-estimation procedures of Chap. 5 or much simpler procedures such as the Smith predictor [61–63], which has been applied with success to processes having delays in both the outputs and the controls. Other methods which have been proposed involve the use of cascade control [64], feedforward control [65], or noninteracting control [66] to compensate for the time delays. Our emphasis in the discussion here will be on time-delay compensation procedures and will highlight recent results on multivariable, multidelay compensators [67].

Let us begin by considering the design of compensators for single-loop control problems with time delays. In the late 1950s Smith [61] developed a time-delay compensator for a single delay in a single control loop which eliminated the delay from the feedback loop, allowing higher controller gains to be used. This compensator, termed the Smith predictor, is shown in the block diagram in Fig. 4.19. Here the compensator

$$g_k(s) = h(s)g(s)(1 - e^{-\alpha s}) \tag{4.5.32}$$

acts to eliminate the time delay α from the system characteristic equation. To see this, note that when $g_k(s) = 0$, the closed-loop response of the system shown in Fig. 4.19 is given by

$$\bar{y}(s) = \left[1 + h(s)g(s)g_c(s)e^{-\alpha s}\right]^{-1}\left[g(s)e^{-\alpha s}g_c(s)\bar{y}_d(s) + g_d(s)\bar{d}(s)\right] \tag{4.5.33}$$

Thus the characteristic equation

$$1 + h(s)g(s)g_c(s)e^{-\alpha s} = 0 \tag{4.5.34}$$

contains the time delay α. When the Smith predictor, Eq. (4.5.32), is added to the loop as shown in Fig. 4.19, the closed-loop response becomes

$$\bar{y}(s) = \left[1 + h(s)g(s)g_c(s)\right]^{-1}\left[g(s)g_c(s)e^{-\alpha s}\bar{y}_d(s) + g_d(s)\bar{d}(s)\right] \tag{4.5.35}$$

and the time delay has been removed from the control loop characteristic equation,

$$1 + h(s)g(s)g_c(s) = 0 \tag{4.5.36}$$

so that higher controller gains are allowed before the system becomes unstable.

Moore et al. [68], working with a scalar state-space model, used the analytic solution of the modeling equation to predict the value of the state one delay time ahead. This analytical predictor was developed primarily for sampled data systems and hence included in its structure corrections for the effect of sam-

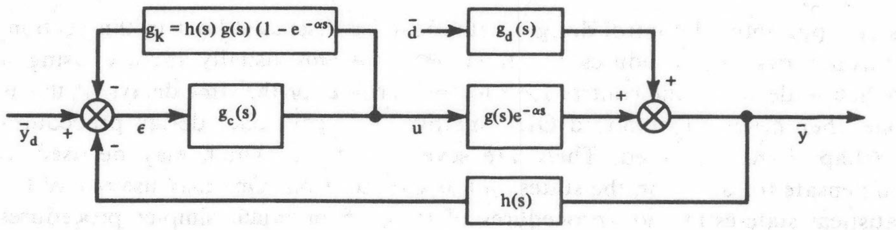

Figure 4.19 Single-loop control system including a Smith predictor for a system having a single delay.

pling, and the zero-order hold. It can be shown that the Smith and Moore predictors are equivalent [67].

Example 4.5.1 To illustrate these compensators, let us consider the control problem described in Example 4.1.2 and Fig. 4.2. Recall that the relation between the fraction of hot stream fed to the tank $u(t)$ and the tank outlet temperature $y(t)$ was given by

$$\theta \frac{dy}{dt} = (T_H - T_C)u(t - \alpha) - y \tag{4.1.14}$$

in the time domain and by the transfer function

$$\bar{y}(s) = \frac{(T_H - T_C)e^{-\alpha s}}{\theta s + 1}\bar{u}(s) \tag{4.1.15}$$

in the Laplace transform domain. If we assume a perfect measuring device $h = 1$ and a proportional feedback controller $g_c = k_c$, then the closed-loop transfer function, Eq. (4.5.33), takes the form

$$\bar{y}(s) = \left[1 + \frac{k_c(T_H - T_C)e^{-\alpha s}}{\theta s + 1}\right]^{-1}\left[\frac{k_c(T_H - T_C)e^{-\alpha s}}{\theta s + 1}\bar{y}_d(s)\right] \tag{4.5.37}$$

By application of the Smith predictor, Eq. (4.5.32),

$$g_k(s) = \frac{T_H - T_C}{\theta s + 1}(1 - e^{-\alpha s}) \tag{4.5.38}$$

the closed-loop response of the compensated system becomes

$$\bar{y}(s) = \left[1 + \frac{k_c(T_H - T_C)}{\theta s + 1}\right]^{-1}\left[\frac{k_c(T_H - T_C)e^{-\alpha s}}{\theta s + 1}\bar{y}_d(s)\right] \tag{4.5.39}$$

Note that the characteristic equation of the compensated system, Eq. (4.5.39),

$$1 + \frac{k_c(T_H - T_C)}{\theta s + 1} = 0 \tag{4.5.40}$$

has stable roots for all positive controller gains, so that stability is not a problem. By contrast, the characteristic equation for the uncompensated control system, Eq. (4.5.37),

$$1 + \frac{k_c(T_H - T_C)}{\theta s + 1}e^{-\alpha s} = 0 \tag{4.5.41}$$

has unstable roots for sufficiently large values of controller gain. For example, if $\theta = 10$, $\alpha = 2$, the uncompensated system becomes unstable for $k_c(T_H - T_C) > 7.04$.

The practical value of this compensator may be seen by comparing the control system response both with and without the compensator when Ziegler-Nichols controller tuning [i.e., $k_c(T_H - T_C) = 3.52$] is used with the

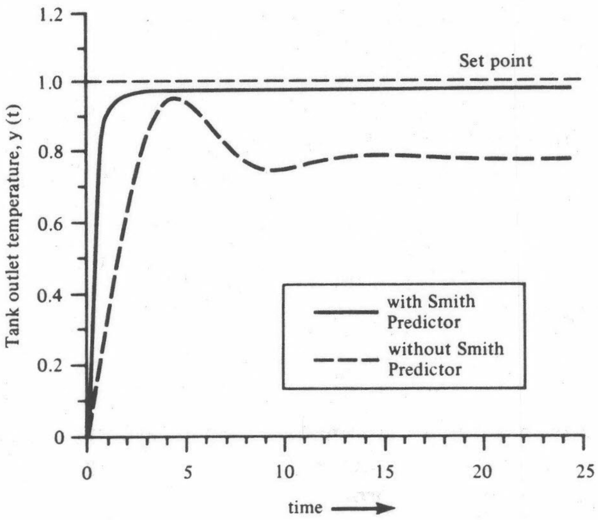

Figure 4.20 Response of heated mixing tank to set-point change in outlet temperature; dashed lines show the response without the Smith predictor, $k_c(T_H - T_C) = 3.52$; solid lines show the response with the Smith predictor, $k_c(T_H - T_C) = 30$.

uncompensated system and $k_c(T_H - T_C) = 30$ is used with the compensated system. The response of the system to a unit set point change may be seen in Fig. 4.20. Note the much slower response and much greater offset with the uncompensated system.

For the more general case of multivariable systems with multiple delays, the design of a compensator becomes more complex. Let us consider the general form of a multivariable transfer function such as described in Eqs. (4.5.1) to (4.5.3). Recall that the transfer function between outputs y and controls u is

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) \tag{4.5.1}$$

where $\mathbf{y}$ is an l vector of outputs and $\mathbf{u}$ is an m vector of controls. Similarly, the transfer function between the outputs and disturbances $\mathbf{d}$ is

$$\bar{\mathbf{y}}(s) = \mathbf{G}_d(s)\bar{\mathbf{d}}(s) \tag{4.5.42}$$

where $\mathbf{d}$ is a k vector of disturbances. The transfer functions $\mathbf{G}(s)$, $\mathbf{G}_d(s)$ are matrices of the form

$$\mathbf{G}(s) = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{l1} & \cdots & \cdots & g_{lm} \end{bmatrix} \quad \mathbf{G}_d(s) = \begin{bmatrix} g_{11}^d & g_{12}^d & \cdots & g_{1k}^d \\ g_{21}^d & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{l1}^d & \cdots & \cdots & g_{lk}^d \end{bmatrix} \tag{4.5.43}$$

In many practical applications where the transfer functions are fitted to experimental data, $g_{ij}(s)$ or $g_{ij}^d(s)$ have the rather simple form of Eq. (4.5.2). However, more complex transfer functions sometimes arise as illustrated below.

The block diagram for the system under conventional feedback control may be seen in Fig. 4.21, where $\mathbf{G}_c$ represents the feedback controller, $\mathbf{H}$ the output measurement device, and $\mathbf{y}_d$ the output set point. The closed-loop response for the conventional controller is then given by

$$\bar{\mathbf{y}}(s) = (\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})^{-1}\left[\mathbf{G}\mathbf{G}_c\bar{\mathbf{y}}_d(s) + \mathbf{G}_d\bar{\mathbf{d}}(s)\right] \tag{4.5.44}$$

in the Laplace domain. In the absence of time delays, there are many multivariable control design procedures available (see Chap. 3) for choosing the elements $\mathbf{G}_c$ in order to achieve good control system performance. However, when there are multiple delays in the transfer function, as in Eq. (4.5.2), the choice of design algorithm is more limited. Thus it is advantageous to use time-delay compensation methods in combination with the conventional controller design methods of Chap. 3. These compensation techniques can largely eliminate the effects of the time delays and allow conventional multivariable controller design procedures to be used for systems with multiple time delays. The multidelay compensator can be formulated so as to apply in either a continuous time or in a discrete time (DDC) mode [67].

There are many different forms that linear systems with time delays may take. For constant delays, a general formulation for linear multivariable systems in the time domain is

$$\dot{\mathbf{x}} = \sum_i \mathbf{A}_i\mathbf{x}(t - \rho_i) + \sum_j \mathbf{B}_j\mathbf{u}(t - \beta_j) + \sum_k \mathbf{D}_k\mathbf{d}(t - \delta_k) \tag{4.5.45}$$

$$\mathbf{y} = \sum_i \mathbf{C}_i\mathbf{x}(t - \gamma_i) + \sum_j \mathbf{E}_j\mathbf{u}(t - \varepsilon_j) \tag{4.5.46}$$

where $\mathbf{x}$ is an n vector of state variables and the ρ_i, β_j, δ_k, γ_i, ε_j are constant time delays. By taking Laplace transforms of Eqs. (4.5.45) and (4.5.46), transfer functions of the form

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) + \mathbf{G}_d\bar{\mathbf{d}}(s) \tag{4.5.47}$$

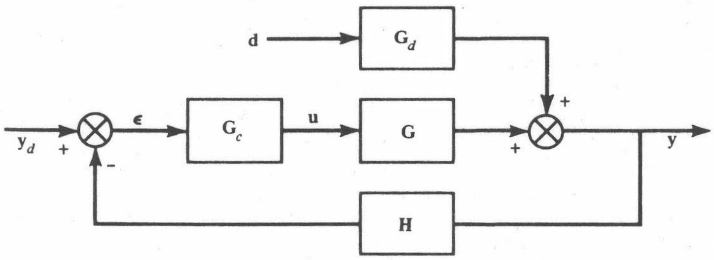

Figure 4.21 Block diagram for conventional feedback control of a multivariable system.

arise, where

$$\mathbf{G}(s) = \sum_j \mathbf{E}_j e^{-\varepsilon_j s} + \left[\left(\sum_i \mathbf{C}_i e^{-\gamma_i s}\right)\left(s\mathbf{I} - \sum_i \mathbf{A}_i e^{-\rho_i s}\right)^{-1}\right]\left(\sum_j \mathbf{B}_j e^{-\beta_j s}\right) \tag{4.5.48}$$

$$\mathbf{G}_d(s) = \left[\left(\sum_i \mathbf{C}_i e^{-\gamma_i s}\right)\left(s\mathbf{I} - \sum_i \mathbf{A}_i e^{-\rho_i s}\right)^{-1}\right]\left(\sum_k \mathbf{D}_k e^{-\delta_k s}\right) \tag{4.5.49}$$

Note that $\mathbf{G}(s)$ and $\mathbf{G}_d(s)$ in Eqs. (4.5.48) and (4.5.49) are very much more general than the more commonly encountered forms of $\mathbf{G}(s)$ and $\mathbf{G}_d(s)$ given by Eq. (4.5.2). However, in normal engineering practice, the usual modeling procedure is to carry out step, pulse, or frequency response measurements on the actual process to obtain an approximate transfer function model in the simpler form of Eqs. (4.5.2) and (4.5.42). The more complex forms [Eqs. (4.5.48) and (4.5.49)] usually arise when the model is formulated as differential equations and transformed to the Laplace domain.

For the case of multivariable systems in the general form [Eqs. (4.5.48) and (4.5.49)] that have multiple delays in the transfer functions $\mathbf{G}$, $\mathbf{G}_d$, $\mathbf{H}$, it is possible to design a compensator analogous to the Smith predictor which eliminates the time delays in the characteristic equation. As we show [67], this is *not equivalent* to predicting the output variable at some single time in the future but corresponds to the prediction of certain *state* variables at various specific times in the future. By analogy with the philosophy of the original Smith predictor, the corresponding multivariable multidelay compensator would have the structure shown in Fig. 4.22, where the compensator $\mathbf{G}_K$ could have many forms. Let us demonstrate that with the particular choice

$$\mathbf{G}_K = \mathbf{H}^*\mathbf{G}^* - \mathbf{H}\mathbf{G} \tag{4.5.50}$$

(where $\mathbf{H}^*$, $\mathbf{G}^*$ are the transfer functions $\mathbf{H}$, $\mathbf{G}$ without the delays), the compensator eliminates both the delays in the output variable signal sent to the

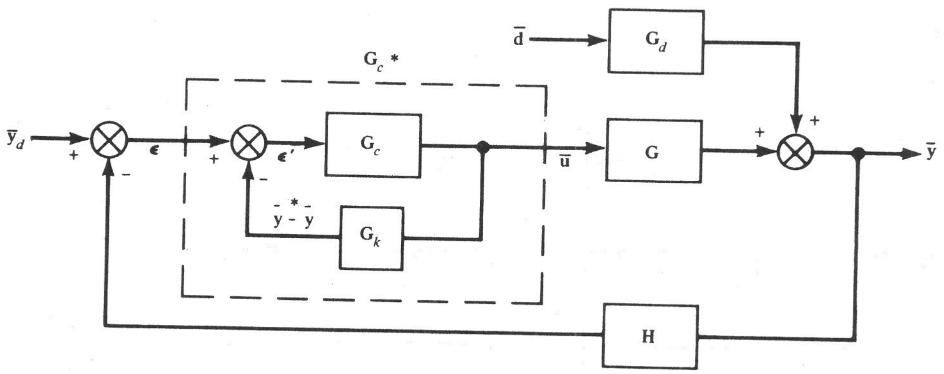

Figure 4.22 Block diagram of feedback control of a multivariable system with time-delay compensation.

controller and the delays in the characteristic equation. To see this, let us evaluate the inner loop $\mathbf{G}_c^*$ in Fig. 4.22. Here

$$\bar{\mathbf{u}} = \mathbf{G}_c^* \bar{\varepsilon}$$

or

$$\mathbf{G}_c^* = (\mathbf{I} + \mathbf{G}_c \mathbf{G}_k)^{-1} \mathbf{G}_c \tag{4.5.51}$$

Thus the entire transfer function including the multidelay compensator is

$$\bar{\mathbf{y}} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c^* \mathbf{H})^{-1} \left(\mathbf{G}\mathbf{G}_c^* \bar{\mathbf{y}}_d + \mathbf{G}_d \bar{\mathbf{d}}\right) \tag{4.5.52}$$

One of the principal goals of time delay compensation is to eliminate the time delay from the characteristic equation of the closed-loop transfer function so that higher controller gains and standard multivariable controller design algorithms may be used. Let us show that the multidelay compensator noted above achieves this goal. Substituting Eqs. (4.5.5) and (4.5.51) into (4.5.52) yields

$$\bar{\mathbf{y}} = \left(\mathbf{I} + \mathbf{G}\mathbf{R}^{-1}\mathbf{G}_c \mathbf{H}\right)^{-1} \left(\mathbf{G}\mathbf{R}^{-1}\mathbf{G}_c \bar{\mathbf{y}}_d + \mathbf{G}_d \bar{\mathbf{d}}\right) \tag{4.5.53}$$

where

$$\mathbf{R} = \mathbf{I} + \mathbf{G}_c(\mathbf{H}^*\mathbf{G}^* - \mathbf{H}\mathbf{G}) \tag{4.5.54}$$

Now it is easy to show that if $\mathbf{G}$ is square and nonsingular* then the following identity holds:

$$\left(\mathbf{I} + \mathbf{G}\mathbf{R}^{-1}\mathbf{G}_c \mathbf{H}\right)^{-1} = \mathbf{G}(\mathbf{R} + \mathbf{G}_c \mathbf{H}\mathbf{G})^{-1} \mathbf{R}\mathbf{G}^{-1}$$

Now, from Eq. (4.5.54),

$$\mathbf{R} + \mathbf{G}_c \mathbf{H}\mathbf{G} = \mathbf{I} + \mathbf{G}_c \mathbf{H}^* \mathbf{G}^*$$

Thus Eq. (4.5.53) becomes

$$\bar{\mathbf{y}} = \mathbf{G}(\mathbf{I} + \mathbf{G}_c \mathbf{H}^*\mathbf{G}^*)^{-1} \mathbf{G}_c \bar{\mathbf{y}}_d + \mathbf{G}(\mathbf{I} + \mathbf{G}_c \mathbf{H}^*\mathbf{G}^*)^{-1} \mathbf{R}\mathbf{G}^{-1}\mathbf{G}_d \bar{\mathbf{d}} \tag{4.5.55}$$

Hence the stability of the closed-loop system including the compensator (Fig. 4.22) is determined by the characteristic equation

$$|\mathbf{I} + \mathbf{G}_c \mathbf{H}^*\mathbf{G}^*| = 0 \tag{4.5.56}$$

and the compensator has indeed removed the time delays from the characteristic equation. Thus the delays do not influence the closed-loop stability if the model matches the plant exactly. In practice, modeling errors usually allow some delays to remain in the system, so that one should be conservative in controller tuning. However, even with very conservative gains chosen for the compensated system, the control system response will be much better than in the case with no compensation.

It is useful to note that the Smith predictor [61] and the Alevisakis and Seborg predictor [62, 63] all become special cases of this general multidelay

* This is by no means restrictive, since we can always construct pseudo-inverses or generalized inverses of the matrix **G**. We note, however, that in most cases the number of inputs and outputs used in a feedback control scheme are equal; hence **G** is square.

compensator. It is also important to realize that any type of delay can be dealt with through the use of this compensator, even the very complex types shown in Eqs. (4.5.48) and (4.5.49). In addition, it can be shown that the feedback law resulting from using the multidelay compensator has exactly the same structure as that obtained from the optimal feedback controller (to be discussed below), and can therefore be made an optimal controller with proper tuning.

Let us demonstrate the multidelay compensator with some illustrative examples.

Example 4.5.2 Some physical interpretation of the effective action of this compensator is useful and can be illustrated by the following 2×2 example system with $\mathbf{H} = \mathbf{H}^* = \mathbf{I}$ and

$$\mathbf{G} = \begin{bmatrix} a_{11}(s)e^{-\alpha_{11}s} & a_{12}(s)e^{-\alpha_{12}s} \\ a_{21}(s)e^{-\alpha_{21}s} & a_{22}(s)e^{-\alpha_{22}s} \end{bmatrix} \tag{4.5.57}$$

By definition,

$$\mathbf{G}^* = \begin{bmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{bmatrix} \tag{4.5.58}$$

and for $\mathbf{G}_c$ consisting of two proportional controllers,

$$\mathbf{G}_c = \begin{bmatrix} k_{c_{11}} & 0 \\ 0 & k_{c_{22}} \end{bmatrix}$$

then

$$\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^* = \begin{bmatrix} 1 + k_{c_{11}}a_{11} & k_{c_{11}}a_{12} \\ k_{c_{22}}a_{21} & 1 + k_{c_{22}}a_{22} \end{bmatrix} \tag{4.5.59}$$

and the characteristic equation is

$$|\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*| = 1 + k_{c_{11}}a_{11} + k_{c_{22}}a_{22} + k_{c_{11}}k_{c_{22}}(a_{11}a_{22} - a_{12}a_{21}) = 0 \tag{4.5.60}$$

which contains no time delays.

Further, because of the compensator (see Fig. 4.22), the error signal fed to the controller is

$$\bar{\boldsymbol{\varepsilon}}' = \bar{\mathbf{y}}_d - \bar{\mathbf{y}}^*$$

Here

$$\bar{\mathbf{y}}^* = \mathbf{G}^*\bar{\mathbf{u}} \tag{4.5.61}$$

is the output variable without delays in $\mathbf{G}$. Now it is interesting to note that $\bar{\mathbf{y}}^*$ does not correspond to the actual value of $\bar{\mathbf{y}}$ at any specific time, but is a totally fictitious value composed of certain predicted "state" variables. To illustrate, let us define variables x_{ij} by

$$\bar{x}_{ij}(s) = a_{ij}(s)e^{-\alpha_{ij}s}\bar{u}_j(s) \tag{4.5.62}$$

Thus the system with delays

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s)$$

may be written

$$\begin{aligned}\bar{y}_1(s) &= \bar{x}_{11}(s) + \bar{x}_{12}(s)\\ \bar{y}_2(s) &= \bar{x}_{21}(s) + \bar{x}_{22}(s)\end{aligned} \tag{4.5.63}$$

or in the time domain

$$\begin{aligned}y_1(t) &= x_{11}(t) + x_{12}(t)\\ y_2(t) &= x_{21}(t) + x_{22}(t)\end{aligned} \tag{4.5.64}$$

By adding the compensator to the loop, the controller receives $\bar{\mathbf{y}}^*(s)$ defined by Eq. (4.5.61), which may be written as

$$\begin{aligned}\bar{y}_1^*(s) &= e^{\alpha_{11}s}\bar{x}_{11}(s) + e^{\alpha_{12}s}\bar{x}_{12}(s)\\ \bar{y}_2^*(s) &= e^{\alpha_{21}s}\bar{x}_{21}(s) + e^{\alpha_{22}s}\bar{x}_{22}(s)\end{aligned} \tag{4.5.65}$$

where $\bar{x}_{ij}(s)$ is defined by Eq. (4.5.62). Thus in the time domain

$$\begin{aligned}y_1^*(t) &= x_{11}(t + \alpha_{11}) + x_{12}(t + \alpha_{12})\\ y_2^*(t) &= x_{21}(t + \alpha_{21}) + x_{22}(t + \alpha_{22})\end{aligned} \tag{4.5.66}$$

and the compensated output $\mathbf{y}^*(t)$ is composed of predictions of the "state" variables x_{ij}. Because the time delays in all the state variables are different, $\mathbf{y}^*(t)$ is a totally fictitious output never attained in reality. However, if the control system is stable, then $\mathbf{y}^* \to \mathbf{y}$ as $t \to \infty$ and the fictitious value $\mathbf{y}^*$ is a good "aiming point" for the controller.

Example 4.5.3 As a means of illustrating the case of state variable delays combined with output delays, consider the two-stage chemical reactor with recycle shown in Fig. 4.23. The irreversible reaction $A \to B$ with negligible heat effect is carried out in the two-stage reactor system. Reactor temperature is maintained constant so that only c_1 and c_2, the composition of product streams from the two reactors, need to be controlled. However, there is substantial analysis delay. The manipulated variables are the feed compositions to the two reactors c_{1f} and c_{2f}, and the process disturbance is an extra feed stream F_d whose composition c_d varies because it comes from another processing unit. The flow rates to the reactor system are fixed, and only the compositions vary. The state delay arises due to the transient lags in the recycle stream.

A material balance on the reactor train yields

$$\begin{aligned}V_1\frac{dc_1}{dt} &= F_1c_{1f} + Rc_2(t - \alpha) + F_dc_d - (F_1 + R + F_d)c_1 - V_1k_1c_1\\ V_2\frac{dc_2}{dt} &= (F_1 + R + F_d - F_{p_1})c_1 + F_2c_{2f} - (F_{p_2} + R)c_2 - V_2k_2c_2\end{aligned} \tag{4.5.67}$$

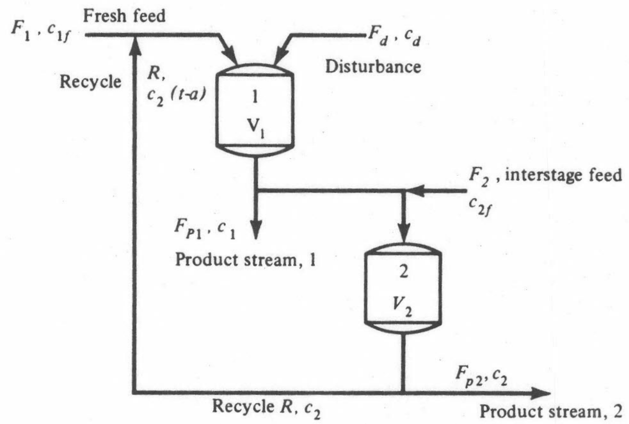

Figure 4.23 Two-stage chemical reactor train with delayed recycle.

where the second product stream F_{p_2} is given by

$$F_{p_2} = F_1 + F_d - F_{p_1} + F_2$$

Defining the variables

$$\theta_1 = \frac{V_1}{F_1 + R + F_d} \qquad \theta_2 = \frac{V_2}{F_{p_2} + R}$$

$$\lambda_R = \frac{R}{F_1 + R + F_d} \qquad \mu = \frac{F_{p_2} - F_2 + R}{F_{p_2} + R}$$

$$\lambda_d = \frac{F_d}{F_1 + R + F_d} \qquad \text{Da}_1 = k_1\theta \qquad \text{Da}_2 = k_2\theta_2$$

$$u_1 = c_{1f} - c_{1f_s} \qquad u_2 = c_{2f} - c_{2f_s}$$

$$x_1 = c_1 - c_{1s} \qquad x_2 = c_2 - c_{2s} \qquad d = c_d - c_{d_s}$$

(where c_{1f_s}, c_{2f_s}, c_{1s}, c_{2s}, c_{d_s} denote steady-state values) allows one to use vector-matrix notation, so that Eq. (4.5.67) becomes

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t - \alpha) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}d \tag{4.5.68}$$

$$\mathbf{y}(t) = \mathbf{x}(t)$$

$$\mathbf{y}_m(t) = \mathbf{H}\mathbf{x}(t) \tag{4.5.69}$$

where $y_m(t)$ is the measured output and

$$\mathbf{A}_0 = \begin{bmatrix} -\dfrac{1 + \mathrm{Da}_1}{\theta_1} & 0 \\[2ex] \dfrac{\mu}{\theta_2} & -\dfrac{1 + \mathrm{Da}_2}{\theta_2} \end{bmatrix} \qquad \mathbf{A}_1 = \begin{bmatrix} 0 & \dfrac{\lambda R}{\theta_1} \\[2ex] 0 & 0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} e^{-\tau_1 s} & 0 \\[2ex] 0 & e^{-\tau_2 s} \end{bmatrix} \tag{4.5.70}$$

$$\mathbf{B} = \begin{bmatrix} \dfrac{1 + \lambda_R - \lambda_d}{\theta_1} & 0 \\[2ex] 0 & \dfrac{1 - \mu}{\theta_2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \dfrac{\lambda_d}{\theta_1} \\[2ex] 0 \end{bmatrix}$$

Here τ_1 and τ_2 are the delays in analyzing c_1 and c_2, respectively. Taking Laplace transforms, one obtains a transfer function model of the form

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) + \mathbf{G}_d(s)\bar{\mathbf{d}}(s) \tag{4.5.47}$$

or, since what we observe is $\bar{\mathbf{y}}_m$,

$$\bar{\mathbf{y}}_m(s) = \mathbf{H}(s)\mathbf{G}(s)\bar{\mathbf{u}}(s) + \mathbf{H}(s)\mathbf{G}_d(s)\bar{\mathbf{d}}(s) \tag{4.5.71}$$

where

$$\mathbf{G}(s) = \frac{\begin{bmatrix} \dfrac{1 - \lambda_R - \lambda_d}{\theta_1}\left(s + \dfrac{1 + \mathrm{Da}_2}{\theta_2}\right) & \dfrac{\lambda_R(1 - \mu)}{\theta_1\theta_2} e^{-\alpha s} \\[2ex] \dfrac{1 - \lambda_R - \lambda_d}{\theta_1\theta_2}\mu & \dfrac{1 - \mu}{\theta_2}\left(s + \dfrac{1 + \mathrm{Da}_1}{\theta_1}\right) \end{bmatrix}}{s^2 + \left(\dfrac{1 + \mathrm{Da}_1}{\theta_1} + \dfrac{1 + \mathrm{Da}_2}{\theta_2}\right)s + \dfrac{(1 + \mathrm{Da}_1)(1 + \mathrm{Da}_2)}{\theta_1\theta_2} - \dfrac{\lambda_R \mu e^{-\alpha s}}{\theta_1\theta_2}} \tag{4.5.72}$$

$$\mathbf{G}_d(s) = \frac{\begin{bmatrix} \dfrac{\lambda_d}{\theta_1} \quad \left(s + \dfrac{1 + \mathrm{Da}_1}{\theta_2}\right) \\[2ex] \dfrac{\lambda_d}{\theta_1} \quad \dfrac{\mu}{\theta_2} \end{bmatrix}}{s^2 + \left(\dfrac{1 + \mathrm{Da}_1}{\theta_1} + \dfrac{1 + \mathrm{Da}_2}{\theta_2}\right)s + \dfrac{(1 + \mathrm{Da}_1)(1 + \mathrm{Da}_2)}{\theta_1\theta_2} - \dfrac{\lambda_R \mu e^{-\alpha s}}{\theta_1\theta_2}} \tag{4.5.73}$$

and if we let

$$\mathbf{G}_0(s) = \mathbf{H}(s)\mathbf{G}(s) \qquad \mathbf{G}_{d0}(s) = \mathbf{H}(s)\mathbf{G}_d(s)$$

then the working transfer functions are

$$\mathbf{G}_0(s) = \frac{\begin{bmatrix} \dfrac{1-\lambda_R-\lambda_d}{\theta_1}\left(s+\dfrac{1+\mathrm{Da}_2}{\theta_2}\right)e^{-\tau_1 s} & \dfrac{\lambda_R(1-\mu)}{\theta_1\theta_2}e^{-(\tau_1+\alpha)s} \\ \dfrac{(1-\lambda_R-\lambda_d)\mu e^{-\tau_2 s}}{\theta_1\theta_2} & \dfrac{1-\mu}{\theta_2}\left(s+\dfrac{1+\mathrm{Da}_1}{\theta_1}\right)e^{-\tau_2 s} \end{bmatrix}}{s^2+\left(\dfrac{1+\mathrm{Da}_1}{\theta_1}+\dfrac{1+\mathrm{Da}_2}{\theta_2}\right)s+\dfrac{(1+\mathrm{Da}_1)(1+\mathrm{Da}_2)}{\theta_1\theta_2}-\dfrac{\lambda_R\mu e^{-\alpha s}}{\theta_1\theta_2}} \tag{4.5.74}$$

$$\mathbf{G}_{d0}(s) = \frac{\begin{bmatrix} \dfrac{\lambda_d}{\theta_1}\left(s+\dfrac{1+\mathrm{Da}_2}{\theta_2}\right)e^{-\tau_1 s} \\ \dfrac{\lambda_d}{\theta_1}\dfrac{\mu}{\theta_2}e^{-\tau_2 s} \end{bmatrix}}{s^2+\left(\dfrac{1+\mathrm{Da}_1}{\theta_1}+\dfrac{1+\mathrm{Da}_2}{\theta_2}\right)s+\dfrac{(1+\mathrm{Da}_1)(1+\mathrm{Da}_2)}{\theta_1\theta_2}-\dfrac{\lambda_R\mu e^{-\alpha s}}{\theta_1\theta_2}} \tag{4.5.75}$$

Now, using the multidelay compensator shown in Fig. 4.22, $\mathbf{H}^*$ becomes $\mathbf{I}$ and

$$\mathbf{G}^* = \mathbf{G}_0^* = \frac{\begin{bmatrix} \dfrac{1-\lambda_R-\lambda_d}{\theta_1}\left(s+\dfrac{1+\mathrm{Da}_2}{\theta_2}\right) & \dfrac{\lambda_R(1-\mu)}{\theta_1\theta_2} \\ \dfrac{(1-\lambda_R-\lambda_d)\mu}{\theta_1\theta_2} & \dfrac{1-\mu}{\theta_2}\left(s+\dfrac{1+\mathrm{Da}_1}{\theta_1}\right) \end{bmatrix}}{s^2+\left(\dfrac{1+\mathrm{Da}_1}{\theta_1}+\dfrac{1+\mathrm{Da}_2}{\theta_2}\right)s+\dfrac{(1+\mathrm{Da}_1)(1+\mathrm{Da}_2)}{\theta_1\theta_2}-\dfrac{\lambda_R\mu}{\theta_1\theta_2}} \tag{4.5.76}$$

and by choosing two single-loop proportional controllers for $\mathbf{G}_c$, i.e.,

$$\mathbf{G}_c = \begin{bmatrix} k_{c_{11}} & 0 \\ 0 & k_{c_{22}} \end{bmatrix} \tag{4.5.77}$$

we may compare the control system performance both with and without the compensator. To illustrate, let us choose

$$\theta_1 = 1 \qquad \theta_2 = 1 \qquad \mathrm{Da}_1 = 1 \qquad \mathrm{Da}_2 = 1 \qquad \lambda_R = 0.5 \qquad \lambda_d = 0.1$$
$$\mu = 0.5$$

with time delays $\alpha = 1$, $\tau_1 = 3$, $\tau_2 = 2$.

In this case

$$\mathbf{G}_0(s) = \mathbf{H}(s)\mathbf{G}(s) = \begin{bmatrix} \dfrac{0.4(s+2)e^{-3s}}{(s+2)^2 - 0.25e^{-s}} & \dfrac{0.25e^{-4s}}{(s+2)^2 - 0.25e^{-s}} \\ \dfrac{0.2e^{-2s}}{(s+2)^2 - 0.25e^{-s}} & \dfrac{0.5(s+2)e^{-2s}}{(s+2)^2 - 0.25e^{-s}} \end{bmatrix}$$

$$\mathbf{G}_0^*(s) = \mathbf{H}^*(s)\mathbf{G}^*(s) = \begin{bmatrix} \dfrac{0.4(s+2)}{(s+2.5)(s+1.5)} & \dfrac{0.25}{(s+2.5)(s+1.5)} \\ \dfrac{0.2}{(s+2.5)(s+1.5)} & \dfrac{0.5(s+2)}{(s+2.5)(s+1.5)} \end{bmatrix}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} \dfrac{0.1(s+2)e^{-3s}}{(s+2)^2 - 0.25e^{-s}} \\ \dfrac{0.05e^{-2s}}{(s+2)^2 - 0.25e^{-s}} \end{bmatrix} \tag{4.5.78}$$

so that the closed-loop system *without the compensator* is

$$\mathbf{y} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})^{-1}\mathbf{G}\mathbf{G}_c\mathbf{y}_d + (\mathbf{I} + \mathbf{G}\mathbf{G}_c\mathbf{H})^{-1}\mathbf{G}_d\mathbf{d} \tag{4.5.79}$$

or equivalently, since $\mathbf{y}_m$ is observed and $\mathbf{y}_m = \mathbf{H}\mathbf{y}$,

$$\mathbf{y}_m = (\mathbf{I} + \mathbf{H}\mathbf{G}\mathbf{G}_c)^{-1}\mathbf{H}\mathbf{G}\mathbf{G}_c\mathbf{y}_d + (\mathbf{I} + \mathbf{H}\mathbf{G}\mathbf{G}_c)^{-1}\mathbf{H}\mathbf{G}_d\mathbf{d}$$

or in terms of $\mathbf{G}_0$, $\mathbf{G}_{d0}$,

$$\mathbf{y}_m = (\mathbf{I} + \mathbf{G}_0\mathbf{G}_c)^{-1}\mathbf{G}_0\mathbf{G}_c\mathbf{y}_d + (\mathbf{I} + \mathbf{G}_0\mathbf{G}_c)^{-1}\mathbf{G}_{d0}\mathbf{d} \tag{4.5.80}$$

Now *with the compensator* the closed-loop expressions are

$$\mathbf{y} = \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\mathbf{G}_c\mathbf{y}_d + \mathbf{G}(\mathbf{I} + \mathbf{G}_c\mathbf{H}^*\mathbf{G}^*)^{-1}\mathbf{R}\mathbf{G}^{-1}\mathbf{G}_d\mathbf{d}$$

or in terms of $\mathbf{G}_0$, $\mathbf{G}_{d0}$,

$$\mathbf{y}_m = \mathbf{G}_0(\mathbf{I} + \mathbf{G}_c\mathbf{G}_0^*)^{-1}\mathbf{G}_c\mathbf{y}_d + \mathbf{G}_0(\mathbf{I} + \mathbf{G}_c\mathbf{G}_0^*)\mathbf{R}\mathbf{G}^{-1}\mathbf{G}_d\mathbf{d} \tag{4.5.81}$$

where

$$\mathbf{R} = \mathbf{I} + \mathbf{G}_c(\mathbf{G}_0^* - \mathbf{G}_0)$$

The control system performance for set-point changes $y_{1d} = 0.5$, $y_{2d} = 1.0$ is shown in Fig. 4.24. The dashed lines represent the performance without the compensator for controller gains $k_{c_{11}} = 3.0$, $k_{c_{22}} = 3.5$. In the neighborhood of $k_{c_{11}} = 5.0$ serious instabilities set in due to the presence of the time delays.

The application of the multidelay compensator permits the use of higher controller gains $k_{c_{11}} = k_{c_{22}} = 20.0$, and as seen in Fig. 4.24 (solid lines), greatly improved performance is obtained.

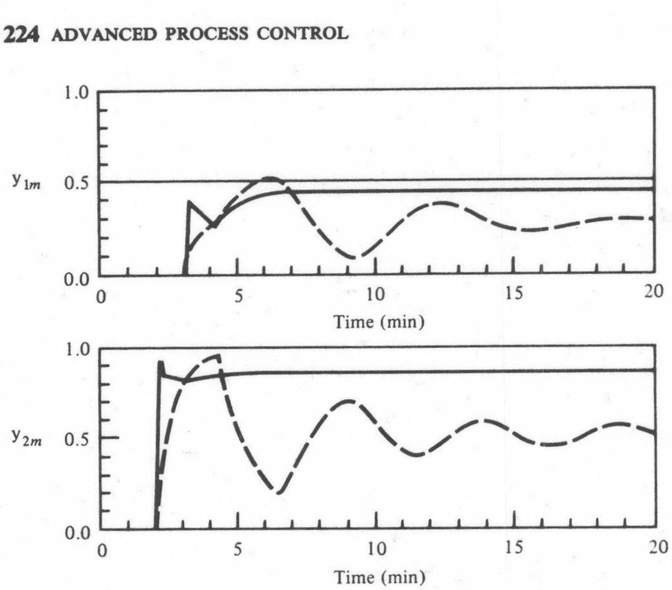

Figure 4.24 Chemical reactor response to set-point changes. Dashed lines: proportional control without compensator; solid lines: proportional control with compensator.

In Fig. 4.25 the response to a step input in disturbance $d = 5.0$ is shown. The dashed lines show the response without the compensator. Again, controller gains $k_{c_{11}} = 3.0$, $k_{c_{22}} = 3.5$ are used, these being the largest before the onset of serious instabilities. The continuous lines show the performance using the compensator with gains $k_{c_{11}} = 45.0$, $k_{c_{22}} = 20.0$.

This example serves to illustrate the improvements in control with the multidelay compensator for a problem having both state and measurement delays.

Example 4.5.4 To illustrate the effects of multiple delays in the control and output variables, let us consider the binary distillation column studied by Wood and Berry [69], Shah and Fisher [70], and Meyer et al. [71, 72]. The column, shown in schematic in Fig. 4.26, was used for methanol-water separation and was found to be well modeled by the transfer function model

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) + \mathbf{G}_d(s)\bar{d}(s) \tag{4.5.47}$$

where, in terms of deviation variables,

y_1 = overhead mole fraction methanol

y_2 = bottoms mole fraction methanol

u_1 = overhead reflux flow rate

u_2 = bottom steam flow rate

d = column feed flow rate

After pulse testing of the column, the transfer functions determined from

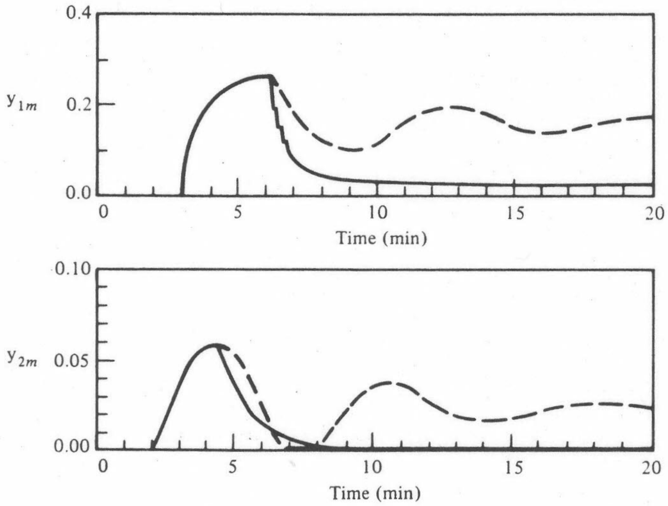

Figure 4.25 Chemical reactor response to a step change disturbance. Dashed lines: proportional control without compensator; solid lines: proportional control with compensator.

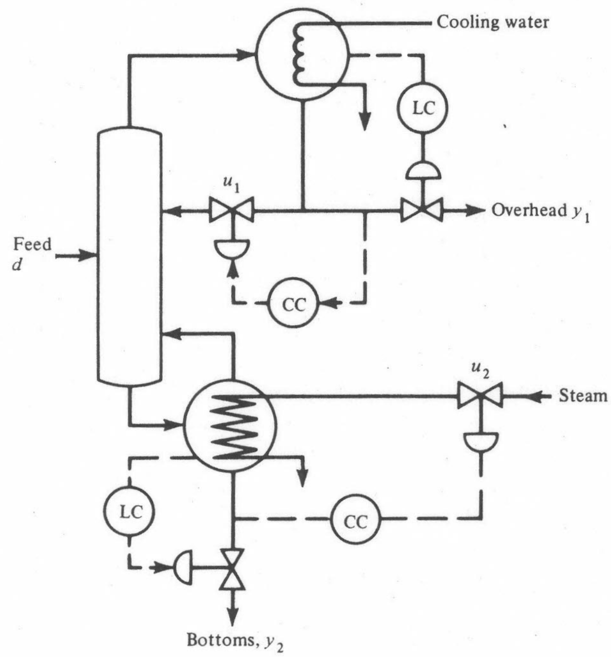

Figure 4.26 Schematic diagram of the methanol distillation column with conventional two-point column control system, Wood and Berry [69].

the data were [69]

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s + 1} & \dfrac{-18.9e^{-3s}}{21.0s + 1} \\ \dfrac{6.6e^{-7s}}{10.9s + 1} & \dfrac{-19.4e^{-3s}}{14.4s + 1} \end{bmatrix}$$

$$G_d(s) = \begin{bmatrix} \dfrac{3.8e^{-8.1s}}{14.9s + 1} \\ \dfrac{4.9e^{-3.4s}}{13.2s + 1} \end{bmatrix} \tag{4.5.82}$$

where the time constants and time delays are given in minutes.* Here we take $\mathbf{H} = \mathbf{H}^* = \mathbf{I}$ because any measurement delays may be included in $\mathbf{G}$ for this problem.

We shall illustrate the performance of the system under conventional PI control (Fig. 4.26) both with and without the compensator. The steady-state values for the overhead and bottoms compositions are taken to be 96.25 percent and 0.5 percent methanol, respectively, (see [69]) for this simulation. With the "tuned" conventional controller settings which were originally used experimentally by Wood and Berry [69], reported in their Table 1,† i.e.,

Overhead		Bottoms	
K_p	K_I	K_p	K_I
0.20	0.045	−0.040	−0.015

the control system response is shown in dashed lines in Figs. 4.27 and 4.28. Figure 4.27 is the response to the same positive disturbance, 0.34 lb/min in feed flow rate, as that used in the experimental study. Figure 4.28 is the response to the negative disturbance, −0.36 lb/min in feed flow rate.

This simulated response is seen to be essentially identical to the experimental response reported by Wood and Berry [69] for conventional control. Larger controller gains cannot be taken because the characteristic equation

$$\begin{aligned} |\mathbf{I} + \mathbf{G}\mathbf{G}_c| = {} & 55{,}045s^4 + 14{,}698s^3 + 1219s^2 + 62s + 1 \\ & + (228.9s^2 + 31.9s + 1)\big[g_{c_1}(12 + 172.8s)e^{-s} \\ & - g_{c_2}(19.4 + 323.8s)e^{-3s} - 232.8 g_{c_1} g_{c_2} e^{-4s} \big] \\ & + 124.7(240.5s^2 + 31.1s + 1) g_{c_1} g_{c_2} e^{-10s} = 0 \end{aligned}$$

* The time unit of minutes was used here in place of the appropriate SI unit (seconds) to avoid confusing the reader who may refer to the original articles from which this example was taken.

† Private communications with Professor R. K. Wood confirmed that the signs of the controller gains reported in the original publication should be corrected as shown here.

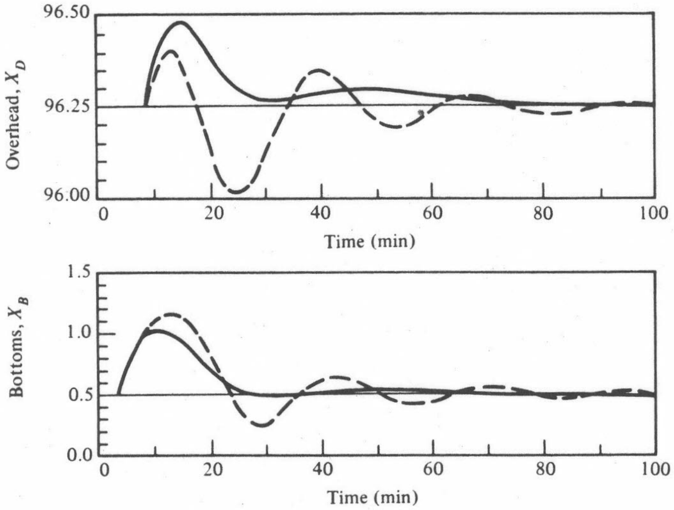

Figure 4.27 Comparison of column performance with and without the multidelay compensator (positive feed-rate disturbance). Dashed line: without compensator; solid line: with compensator.

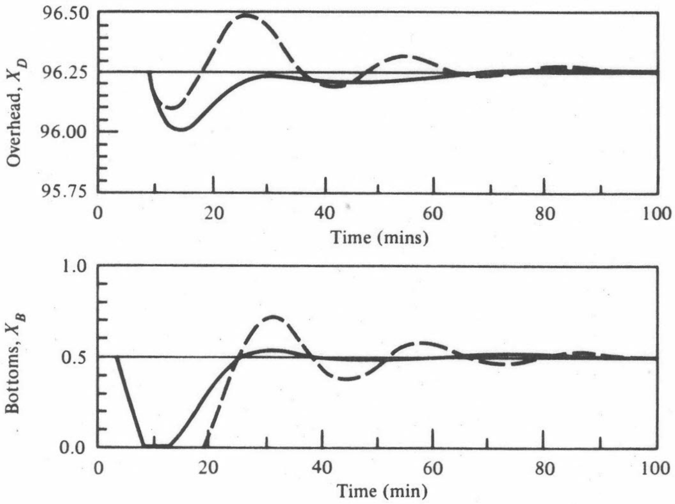

Figure 4.28 Comparison of column performance with and without the multidelay compensator (negative feed-rate disturbance). Dashed line: without compensator; solid line: with compensator.

contains time delays which cause stability problems. Here we have taken

$$\mathbf{G}_c = \begin{bmatrix} g_{c_{11}} & 0 \\ 0 & g_{c_{22}} \end{bmatrix} \tag{4.5.83}$$

with

$$g_{c_{ii}} = K_{P_i} + \frac{K_{I_i}}{s}$$

in keeping with the notation of Wood and Berry.

When the multidelay compensator is applied to the control loop as in Fig. 4.22, with

$$\mathbf{G}^* = \begin{bmatrix} \dfrac{12.8}{16.7s + 1} & \dfrac{-18.9}{21.0s + 1} \\ \dfrac{6.6}{10.9s + 1} & \dfrac{-19.4}{14.4s + 1} \end{bmatrix}$$

and $\mathbf{G}_c$ as in Eq. (4.5.83), the characteristic equation becomes

$$\begin{aligned} |\mathbf{I} + \mathbf{G}_c\mathbf{G}^*| = {} & 55{,}045s^4 + (14{,}698s + 39{,}553g_{c_{11}} - 74{,}117g_{c_{22}})s^3 \\ & + (1219 + 8259g_{c_{11}} - 14{,}769g_{c_{22}} - 23{,}290g_{c_{11}}g_{c_{22}})s^2 \\ & + (62 + 555g_{c_{11}} - 942g_{c_{22}} - 3546g_{c_{11}}g_{c_{22}})s \\ & + (1 - 108g_{c_{11}}g_{c_{22}}) = 0 \end{aligned}$$

which contains no time delays. The improved response obtained with the compensator is shown as continuous lines in Figs. 4.27 and 4.28 for precisely the same controller settings specified in the table. Apart from noting that there are no serious oscillations, observe that the bottoms composition benefits more from the use of the multidelay compensator. That this should indeed be so can be readily seen by merely inspecting the transfer function $\mathbf{G}(s)$ and noting that the time delays associated with the bottoms are substantially larger than those associated with the overhead.

One interesting feature of this distillation column is the appreciable amount of interactions existing between the system variables. This is due to the presence of off-diagonal elements in $\mathbf{G}(s)$, with the result that y_1 is affected by u_2, and y_2 by u_1. The system performance is most affected by this coupling when set-point changes are made. For example, when a set-point change from 96.25 to 97.0 is made in the overhead composition, the multidelay compensated system responds as shown in dashed lines in Fig. 4.29. The interesting point to note is the resulting effect on the bottoms composition. Were there no coupling between the overheads and bottoms compositions, such a set-point change would not have perturbed the bottoms composition.

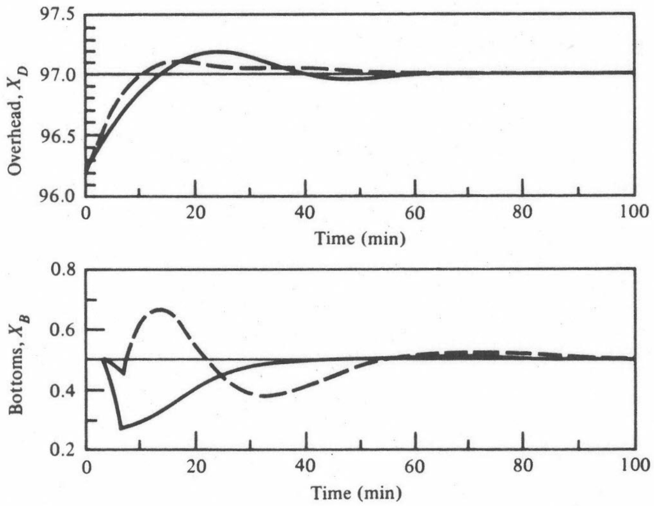

Figure 4.29 Column response to a positive set-point change in overhead composition using the multidelay compensator. Dashed line: without steady-state decoupling; solid line: with steady-state decoupling.

To illustrate how the multidelay compensator may be used in conjunction with the conventional multivariable control design techniques of Chap. 3, we shall attempt to eliminate some of the interaction effects through the use of steady-state decoupling along with the multidelay compensator. This combined control scheme is illustrated in Fig. 4.29 by the solid lines. Note the improvements in the bottoms response resulting from this very simple additional design change.

Optimal Control of Time-Delay Systems

Optimal control is one approach to controller design which can be used either in an open-loop fashion (such as in start-up) or as a closed-loop feedback control system. Here we shall outline the essence of optimal control theory and present some examples to demonstrate how the control schemes might be implemented. General fundamental results on the optimal control of time-delay systems are discussed in [60, 73]; however, here we shall discuss only the case for *constant time delays*, i.e., for the system

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\big(\mathbf{x}(t), \mathbf{x}(t-\alpha_1), \ldots, \mathbf{x}(t-\alpha_\delta), \mathbf{u}(t), \mathbf{u}(t-\beta_1), \ldots, \mathbf{u}(t-\beta_\mu)\big) \tag{4.5.19}$$

$$\begin{aligned} \mathbf{x}(t) &= \phi(t) \qquad -\alpha_\delta \le t \le 0 \\ \mathbf{u}(t) &= \Phi(t) \qquad -\beta_\mu \le t \le 0 \end{aligned} \tag{4.5.22}$$

where the α_i, $i = 1, 2, \ldots, \delta$, and β_j, $j = 1, 2, \ldots, \mu$ are constant state and control delays, respectively. The maximum principle, which provides necessary conditions for optimality, takes the following form:

Theorem For the optimal control problem given by the system equations (4.5.19) to (4.5.22) with the objective

$$I = G(\mathbf{x}(t_f)) + \int_0^{t_f} F(\mathbf{x}(t), \mathbf{x}(t-\alpha_1), \ldots, \mathbf{x}(t-\alpha_\delta), \mathbf{u}(t),$$
$$\mathbf{u}(t-\beta_1), \ldots, \mathbf{u}(t-\beta_\mu))\, dt \tag{4.5.84}$$

to be maximized, and where $\mathbf{u}(t)$ belongs to a constraint set Ω, the optimal control $\bar{\mathbf{u}}(t)$ must satisfy the conditions

$$\frac{\partial H(t)}{\partial \mathbf{u}(t)} + \sum_{j=1}^{\mu} \left[\frac{\partial H(\tau)}{\partial \mathbf{u}(\tau - \beta_j)} \right]_{\tau = t + \beta_j} = \mathbf{0} \qquad 0 < t < t_f - \beta_\mu$$

$$\frac{\partial H(t)}{\partial \mathbf{u}(t)} + \sum_{j=1}^{\mu-k} \left[\frac{\partial H(\tau)}{\partial \mathbf{u}(\tau - \beta_j)} \right]_{\tau = t + \beta_j} = \mathbf{0} \qquad t_f - \beta_{\mu+1-k} < t < t_f - \beta_{\mu-k}$$

$$\frac{\partial H(t)}{\partial \mathbf{u}(t)} = \mathbf{0} \qquad t_f - \beta_1 < t < t_f \tag{4.5.85}$$

for $\bar{\mathbf{u}}(t)$ in the interior of Ω (**u** unconstrained), while at constraints the quantities

$$H(t) + \sum_{j=1}^{\mu} H(t + \beta_j) \qquad 0 < t < t_f - \beta_\mu$$

$$H(t) + \sum_{j=1}^{\mu-k} H(t + \beta_j) \qquad \begin{array}{l} t_f - \beta_{\mu+1-k} < t < t_f - \beta_{\mu-k} \\ k = 1, 2, \ldots, \mu - 1 \end{array}$$

$$H(t) \qquad t_f - \beta_1 < t < t_f \tag{4.5.86}$$

must be a maximum with respect to $\mathbf{u}(t)$. In addition, it is necessary that this last maximum condition hold even on unconstrained portions of the control trajectory. The Hamiltonian is given by

$$H(t) = F + \boldsymbol{\lambda}^T(t)\mathbf{f} \tag{4.5.87}$$

and the adjoint variables $\boldsymbol{\lambda}$ must satisfy:

$$\frac{d\boldsymbol{\lambda}^T(t)}{dt} = \begin{cases} -\left\{\dfrac{\partial H(t)}{\partial \mathbf{x}(t)} + \displaystyle\sum_{i=1}^{\delta}\left[\dfrac{\partial H(\tau)}{\partial \mathbf{x}(\tau - \alpha_i)}\right]_{\tau = t+\alpha_i}\right\} & 0 < t < t_f - \alpha_\delta \\ -\left\{\dfrac{\partial H(t)}{\partial \mathbf{x}(t)} + \displaystyle\sum_{i=1}^{\delta-k}\left[\dfrac{\partial H(\tau)}{\partial \mathbf{x}(\tau - \alpha_i)}\right]_{\tau = t+\alpha_i}\right\} & \begin{array}{l} t_f - \alpha_{\delta+1-k} \\ \quad < t < t_f - \alpha_{\delta-k} \\ k = 1, 2, \ldots, \delta - 1 \end{array} \\ -\dfrac{\partial H}{\partial \mathbf{x}(t)} & t_f - \alpha_1 < t < t_f \end{cases} \tag{4.5.88}$$

$$\boldsymbol{\lambda}(t_f - \alpha_i^+) = \boldsymbol{\lambda}(t_f - \alpha_i^-) \qquad i = 1, 2, \ldots, \delta \tag{4.5.89}$$

$$\boldsymbol{\lambda}(t_f - \beta_j^+) = \boldsymbol{\lambda}(t_f - \beta_j^-) \qquad j = 1, 2, \ldots, \mu \tag{4.5.90}$$

and for $\mathbf{x}(t_f)$ unspecified,

$$\boldsymbol{\lambda}^T(t_f) = \frac{\partial G}{\partial \mathbf{x}(t_f)} \tag{4.5.91}$$

The proof of this maximum principle and more general results may be found in [73]. Let us now proceed to illustrate the application of control vector iteration computational procedures through an example problem.

Example 4.5.5* To demonstrate the application of the theory we have developed, we shall treat an example of the open-loop control of a continuous stirred tank reactor (CSTR). Suppose we would like to optimally move from one steady state to another in the CSTR shown in Fig. 4.30. The describing equations are

$$V\frac{dc}{dt'} = u_3'(c_f - c) - K_0 V e^{-E/RT}\hat{c}c \qquad c(0) = c_0 \tag{4.5.92}$$

$$V\frac{d\hat{c}}{dt'} = (1 - \gamma)u_2'(t' - \beta_1') + \gamma u_2'(t') - \hat{c}u_3' \qquad \hat{c}(0) = \hat{c}_0 \tag{4.5.93}$$

$$\begin{aligned} \rho C_p V\frac{dT}{dt'} &= \rho C_p u_3'(T_f - T) + (-\Delta H)K_0 V e^{-E/RT}\hat{c}c \\ &\quad - \{hA + u_1'(t')[T(t' - \alpha_1') - T_s]\}(T - T_c) \\ T(0) &= T_0 \end{aligned} \tag{4.5.94}$$

where it is assumed that:

1. The chemical reaction $A_1 \rightarrow A_2$ is first-order in both the catalyst $\hat{c}$ and reactant c.

* This example is taken from [73] with permission of Pergamon Press Ltd.

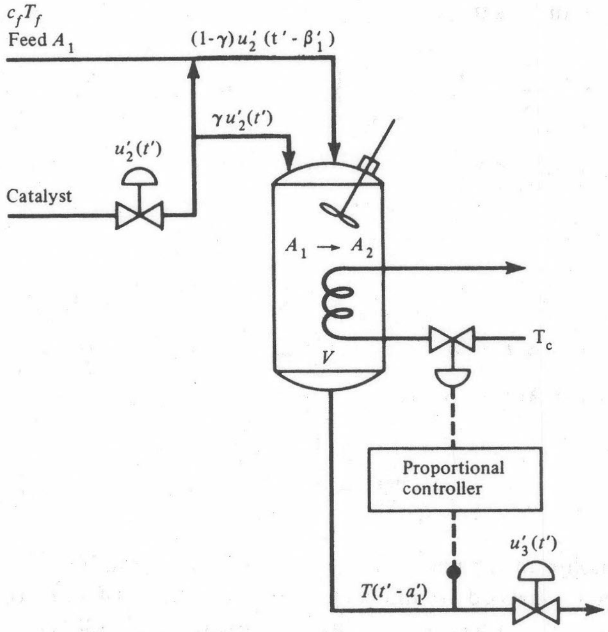

Figure 4.30 The CSTR system [73]. (*Reproduced by permission of Pergamon Press Ltd.*)

2. The catalyst feed u_2' is made up of two parts (see Fig. 4.30): a fraction γ entering the reactor directly, and $(1-\gamma)$ mixed with the feed a time β_1' upstream of the reactor.
3. The temperature is controlled by a feedback controller which uses continuous temperature measurements at time $t' - \alpha_1'$ to adjust the coolant flow rate, where $u_1'(t')$ is a time-variable proportional gain.
4. The molar feed rate of catalyst u_2' is negligible compared with the reactants feed rate $u_3' c_f$, so that the physical properties are unaffected by catalyst addition.

If we wish to move from the state $(c_0, \hat{c}_0, T_0)$ to $(c_s, \hat{c}_s, T_s)$, then it is convenient to define the dimensionless variables

$$
\begin{gathered}
x_1 = \frac{c - c_s}{c_s} \qquad x_2 = \frac{\hat{c} - \hat{c}_s}{\hat{c}_s} \qquad x_3 = \frac{T - T_s}{T_s} \qquad \theta_h = \frac{V}{F_s} \\
t = \frac{t'}{\theta_h} \qquad u_1 = \frac{u_1' T_s}{F_s} \qquad u_2 = \frac{u_2'}{\hat{c}_s F_s} \qquad u_3 = \frac{u_3'}{F_s} \\
\Theta = \frac{E}{RT_s} \qquad \beta_1 = \frac{\beta_1'}{\theta_h} \qquad \alpha_1 = \frac{\alpha_1'}{\theta_h} \qquad Q = \frac{hA}{\rho C_p F_s} \\
P = K_0 \theta_h e^{-\Theta} \hat{c}_s \qquad J = \frac{(-\Delta H)\hat{c}_s}{\rho C_p T_s} \qquad x_{3c} = \frac{T_c - T_s}{T_s}
\end{gathered}
\tag{4.5.95}
$$

so that

$$\frac{dx_1}{dt} = -u_3 x_1 - P\left[(1 + x_1)(1 + x_2)e^{\Theta x_3/(x_3+1)} - u_3\right] \qquad x_1(0) = x_{10} \tag{4.5.96}$$

$$\frac{dx_2}{dt} = (1 - \gamma)u_2(t - \beta_1) + \gamma u_2(t) - u_3(x_2 + 1) \qquad x_2(0) = x_{20} \tag{4.5.97}$$

$$\begin{aligned}\frac{dx_3}{dt} = &-u_3 x_3 + JP\left[(1 + x_1)(1 + x_2)e^{x_3\theta/(1+x_3)} - u_3\right] \\ &- Q\left[x_3(t) - x_{3c}(1 - u_3)\right] - u_1(t)x_3(t - \alpha_1)\left[x_3(t) - x_{3c}\right]\end{aligned}$$

$$x_3(0) = x_{30} \tag{4.5.98}$$

We shall apply our maximum principle to this system so that we find the controls which minimize the functional

$$I = \int_0^{t_f}\left[x_1^2 + x_2^2 + x_3^2 + \eta_2(u_2 - 1)^2 + \eta_3(u_3 - 1)^2\right] dt \tag{4.5.99}$$

subject to the constraints

$$u_{i*} \le u_i \le u_i^* \qquad i = 1, 2, 3 \tag{4.5.100}$$

We will choose the set of parameters

$$\begin{array}{llll} \Theta = 25 & Q = 1 & x_{3c} = -0.125 & P = 1 \quad J = 0.25 \\ \beta_{1s} = 0.020 & \theta = 0.4 & \gamma = 0.1 & u_{1*} = -500 \\ u_1^* = 500 & u_{2*} = 0 & u_{3*} = 0.01 & u_2^* = u_3^* = 2 \\ x_{10} = 0.49 & x_{20} = 0.0002 & x_{30} = -0.02 & \alpha_{1s} = 0.015 \end{array} \tag{4.5.101}$$

and the initial functions

$$\begin{array}{ll} x_3(t) = -0.02 & -\alpha_1 \le t \le 0 \\ u_2(t) = 1 & -\beta_1 \le t < 0 \\ u_3(t) = 1 & -\max(\alpha_{1s}, \beta_{1s}) \le t < 0 \end{array} \tag{4.5.102}$$

and apply a control vector iteration method to the problem to find the optimum. The detailed algorithm is described in [73].

We shall assume that the flow rate u_3 is not a control but is held constant at the steady-state value. Hence, our time delays are constants. The results of applying the control vector iteration procedure from two initial guesses of u_1, u_2 are shown in Figs. 4.31 and 4.32. We see that $\eta_2 = 1$ is too large to allow any control action in u_2 and all the control is being done by u_1. Note that the same optimal state trajectories $x_1(t)$, $x_2(t)$, $x_3(t)$ are found from both starting guesses, which suggests adequate convergence of the algorithm. The differences in u_1 for large t are due to the fact that $x_3 \cong 0$, so

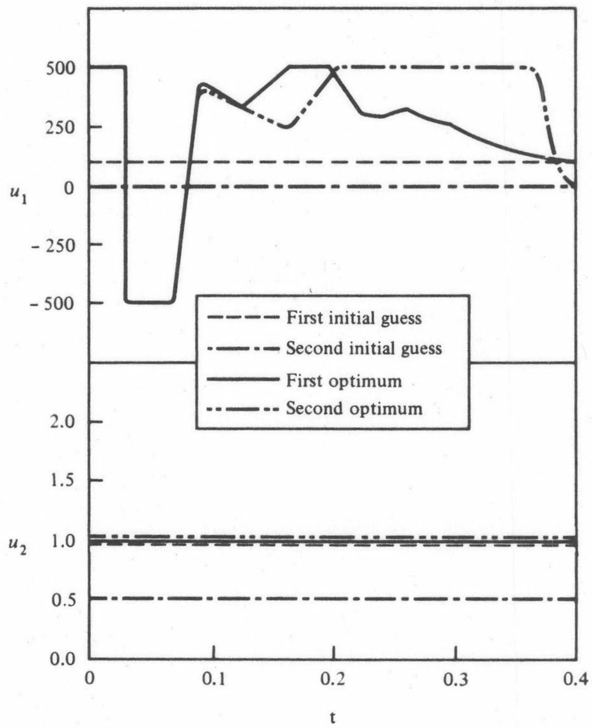

Figure 4.31 Optimal open-loop control policy for Example 4.5.5 when $\eta_2 = 1.0$ [73]. (*Reproduced by permission of Pergamon Press, Ltd.*)

that any gain $u_1(t)$ gives good results. If we remove the penalty on the control u_2 from the objective by setting $\eta_2 = 0$, the u_2 does exercise some control, but gives only a slight improvement in the state trajectories and the objective functional.

Linear-Quadratic Feedback Control

As in other distributed parameter systems, it is possible to design an optimal feedback controller by considering the linear problem with constant time delays:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}_0\mathbf{x}(t) + \sum_{i=1}^{\delta} \mathbf{A}_i\mathbf{x}(t - \alpha_i) + \mathbf{B}_0\mathbf{u}(t) + \sum_{j=1}^{\mu} \mathbf{B}_j\mathbf{u}(t - \beta_j) \qquad (4.5.103)$$

$$\mathbf{x}(t) = \phi(t) \qquad -\alpha_\delta \le t \le 0 \qquad (4.5.104)$$

$$\mathbf{u}(t) = \Phi(t) \qquad -\beta_\mu \le t \le 0 \qquad (4.5.105)$$

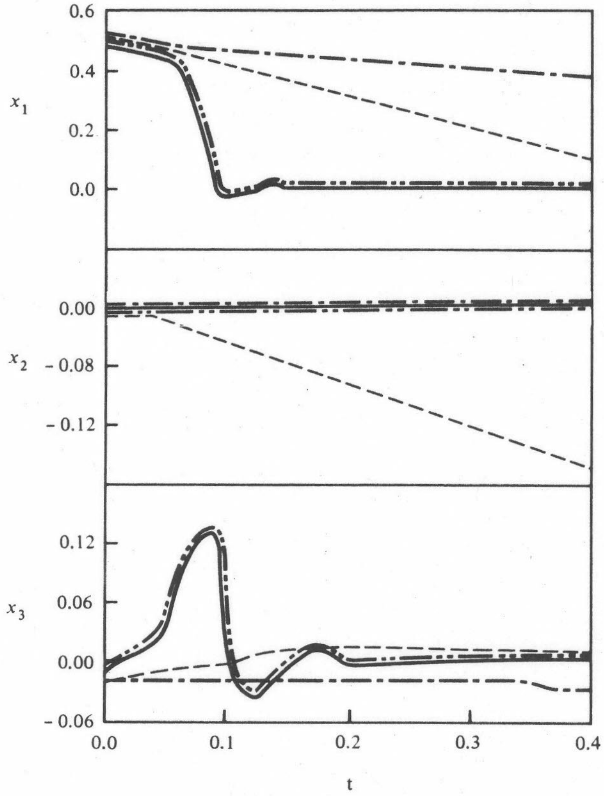

Figure 4.32 Optimal open-loop state trajectories for Example 4.5.5 when $\eta_2 = 1.0$ [73]. (*Reproduced by permission of Pergamon Press, Ltd.*)

with the quadratic objective

$$I = \tfrac{1}{2}\mathbf{x}^T(t_f)\mathbf{S}_f\mathbf{x}(t_f) + \tfrac{1}{2}\int_0^{t_f}\left[\mathbf{x}^T(t)\mathbf{F}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{E}\mathbf{u}(t)\right] dt \qquad (4.5.106)$$

The optimal feedback control law has been derived by many routes [60, 74, 75] and takes the form

$$\begin{aligned}\mathbf{u}(t) = -\mathbf{E}^{-1}\Big\{&\left[\mathbf{B}_0^T\mathbf{E}_0(t) + \mathbf{E}_3^T(t, 0)\right]\mathbf{x}(t)\\ &+ \int_{-\alpha_g}^{0}\left[\mathbf{B}_0^T\mathbf{E}_1(t, s) + \mathbf{E}_5^T(t, s, 0)\right]\mathbf{x}(t + s)\, ds\\ &+ \int_{-\beta_\mu}^{0}\left[\mathbf{B}_0^T\mathbf{E}_3(t, s) + \mathbf{E}_4(t, 0, s)\right]\mathbf{u}(t + s)\, ds\Big\}\end{aligned} \qquad (4.5.107)$$

where the controller parameters may be precomputed from

$$\frac{d\mathbf{E}_0(t)}{dt} = -\mathbf{E}_0(t)\mathbf{A}_0 - \mathbf{A}_0^T\mathbf{E}_0(t) - \mathbf{E}_1^T(t,0) - \mathbf{E}_1(t,0)$$

$$+\left[\mathbf{E}_0(t)\mathbf{B}_0 + \mathbf{E}_3(t,0)\right]\mathbf{E}^{-1}\left[\mathbf{B}_0^T\mathbf{E}_0(t) + \mathbf{E}_3^T(t,0)\right] - \mathbf{F} \tag{4.5.108}$$

$$\frac{\partial \mathbf{E}_1(t,s)}{\partial t} = \frac{\partial \mathbf{E}_1(t,s)}{\partial s} - \mathbf{A}_0^T\mathbf{E}_1(t,s) - \mathbf{E}_2(t,0,s) + \left[\mathbf{E}_0(t)\mathbf{B}_0 + \mathbf{E}_3(t,0)\right]$$
$$\times\, \mathbf{E}^{-1}\left[\mathbf{B}_0^T\mathbf{E}_1(t,s) + \mathbf{E}_5^T(t,s,0)\right] \qquad -\alpha_8 \le s \le 0 \tag{4.5.109}$$

$$\frac{\partial \mathbf{E}_2(t,r,s)}{\partial t} = \frac{\partial \mathbf{E}_2(t,r,s)}{\partial r} + \frac{\partial \mathbf{E}_2(t,r,s)}{\partial s} + \left[\mathbf{E}_1^T(t,r)\mathbf{B}_0 + \mathbf{E}_5(t,r,0)\right]$$
$$\times\, \mathbf{E}^{-1}\left[\mathbf{B}_0^T\mathbf{E}_1(t,s) + \mathbf{E}_5^T(t,s,0)\right] \qquad \begin{array}{l} -\alpha_8 \le r \le 0 \\ -\alpha_8 \le s \le 0 \end{array} \tag{4.5.110}$$

$$\frac{\partial \mathbf{E}_3(t,s)}{\partial t} = \frac{\partial \mathbf{E}_3(t,s)}{\partial s} - \mathbf{A}_0^T\mathbf{E}_3(t,s) - \mathbf{E}_5(t,0,s) + \left[\mathbf{E}_0(t)\mathbf{B}_0 + \mathbf{E}_3(t,0)\right]$$
$$\times\, \mathbf{E}^{-1}\left[\mathbf{B}_0^T\mathbf{E}_3(t,s) + \mathbf{E}_4(t,0,s)\right] \qquad -\beta_\mu \le s \le 0 \tag{4.5.111}$$

$$\frac{\partial \mathbf{E}_4(t,r,s)}{\partial t} = \frac{\partial \mathbf{E}_4(t,r,s)}{\partial r} + \frac{\partial \mathbf{E}_4(t,r,s)}{\partial s} + \left[\mathbf{E}_3^T(t,r)\mathbf{B}_0 + \mathbf{E}_4(t,r,0)\right]$$
$$\times\, \mathbf{E}^{-1}\left[\mathbf{B}_0^T\mathbf{E}_3(t,s) + \mathbf{E}_4(t,0,s)\right] \qquad \begin{array}{l} -\beta_\mu \le r \le 0 \\ -\beta_\mu \le s \le 0 \end{array} \tag{4.5.112}$$

$$\frac{\partial \mathbf{E}_5(t,r,s)}{\partial t} = \frac{\partial \mathbf{E}_5(t,r,s)}{\partial r} + \frac{\partial \mathbf{E}_5(t,r,s)}{\partial s} + \left[\mathbf{E}_1^T(t,r)\mathbf{B}_0 + \mathbf{E}_5(t,r,0)\right]$$
$$\times\, \mathbf{E}^{-1}\left[\mathbf{B}_0^T\mathbf{E}_3(t,s) + \mathbf{E}_4(t,0,s)\right] \qquad \begin{array}{l} -\beta_\mu \le s \le 0 \\ -\alpha_8 \le r \le 0 \end{array} \tag{4.5.113}$$

where $\mathbf{E}_0(t)$, $\mathbf{E}_1(t,s)$, and $\mathbf{E}_2(t,r,s)$ are $n \times n$ matrix functions; $\mathbf{E}_3(t,s)$ and $\mathbf{E}_5(t,r,s)$ are $n \times m$ matrix functions; and $\mathbf{E}_4(t,r,s)$ is an $m \times m$ matrix function. The boundary conditions on the delayed time variables, s and r are given by

$$\begin{aligned}
\mathbf{E}_1(t,-\alpha_8) &= \mathbf{E}_0(t)\mathbf{A}_1 \\
\mathbf{E}_3(t,-\beta_\mu) &= \mathbf{E}_0(t)\mathbf{B}_1 \\
\mathbf{E}_2(t,-\alpha_8,s) &= \mathbf{A}_1^T\mathbf{E}_1(t,s) \\
\mathbf{E}_4(t,-\beta_\mu,s) &= \mathbf{B}_1^T\mathbf{E}_3(t,s) \\
\mathbf{E}_5(t,-\alpha_8,s) &= \mathbf{A}_1^T\mathbf{E}_3(t,s) \\
\mathbf{E}_5(t,r,-\beta_\mu) &= \mathbf{E}_1^T(t,r)\mathbf{B}_1
\end{aligned} \tag{4.5.114}$$

The matrices $\mathbf{E}_2(t, r, s)$, $\mathbf{E}_4(t, r, s)$ have the property

$$\begin{aligned} \mathbf{E}_2^T(t, r, s) &= \mathbf{E}_2(t, s, r) \\ \mathbf{E}_4^T(t, r, s) &= \mathbf{E}_4(t, s, r) \end{aligned} \tag{4.5.115}$$

while the terminal conditions are

$$\begin{aligned} \mathbf{E}_0(t_f) &= \mathbf{S} \\ \mathbf{E}_1(t_f, s) &= \mathbf{0} \quad -\alpha_\delta \le s \le 0 \\ \mathbf{E}_2(t_f, r, s) &= \mathbf{0} \quad -\alpha_\delta \le s \le 0 \quad -\alpha_\delta \le r \le 0 \\ \mathbf{E}_3(t_f, s) &= \mathbf{0} \quad -\beta_\mu \le s \le 0 \\ \mathbf{E}_4(t_f, r, s) &= \mathbf{0} \quad -\beta_\mu \le s \le 0 \quad -\beta_\mu \le r \le 0 \\ \mathbf{E}_5(t_f, r, s) &= \mathbf{0} \quad -\beta_\mu \le s \le 0 \quad -\alpha_\delta \le r \le 0 \end{aligned} \tag{4.5.116}$$

In addition there are certain discontinuities which must be satisfied at α_i, β_j:

$$\begin{aligned} \mathbf{E}_1(t, -\alpha_a) &= \mathbf{E}_0(t)\mathbf{A}_a \\ \mathbf{E}_1(t, -\alpha_i^+) &= \mathbf{E}_1(t, -\alpha_i^-) + \mathbf{E}_0(t)\mathbf{A}_i \qquad i = 1, 2, \ldots, a-1 \\ \mathbf{E}_3(t, -\beta_b) &= \mathbf{E}_0(t)\mathbf{B}_b \\ \mathbf{E}_3(t, -\beta_j^+) &= \mathbf{E}_3(t, -\beta_j^-) + \mathbf{E}_0(t)\mathbf{B}_j \qquad j = 1, 2, \ldots, b-1 \\ \mathbf{E}_2(t, -\alpha_a, s) &= \mathbf{A}_a^T\mathbf{E}_1(t, s) \\ \mathbf{E}_2(t, -\alpha_i^+, s) &= \mathbf{E}_2(t, -\alpha_i^-, s) + \mathbf{A}_i^T\mathbf{E}_1(t, s) \qquad i = 1, 2, \ldots, a-1 \\ \mathbf{E}_4(t, -\beta_b, s) &= \mathbf{B}_b^T\mathbf{E}_3(t, s) \\ \mathbf{E}_4(t, -\beta_j^+, s) &= \mathbf{E}_4(t, -\beta_j^-, s) + \mathbf{B}_j^T\mathbf{E}_3(t, s) \qquad j = 1, 2, \ldots, b-1 \\ \mathbf{E}_5(t, -\alpha_a, s) &= \mathbf{A}_a^T\mathbf{E}_3(t, s) \\ \mathbf{E}_5(t, -\alpha_i^+, s) &= \mathbf{E}_5(t, -\alpha_i^-, s) + \mathbf{A}_i^T\mathbf{E}_3(t, s) \qquad i = 1, 2, \ldots, a-1 \\ \mathbf{E}_5(t, r, -\beta_b) &= \mathbf{E}_1^T(t, r)\mathbf{B}_b \\ \mathbf{E}_5(t, r, -\beta_j^+) &= \mathbf{E}_5(t, r, -\beta_j^-) + \mathbf{E}_1^T(t, r)\mathbf{B}_j \qquad j = 1, 2, \ldots, b-1 \end{aligned} \tag{4.5.117}$$

Notice that the first term in the feedback control law, Eq. (4.5.107), is analogous to the familiar form for systems without delay, while the additional terms account for the delays in the state and in the control. As would be expected, elimination of both state and control delays reduces the control law to the conventional type (Chap. 3), with Eq. (4.5.108) taking the form of the well-known Riccati equations for such systems.

Example 4.5.6* Let us now illustrate the application of linear quadratic controller design by recalling the class of problems represented by Eqs. (4.5.1) and (4.5.2), i.e., linear transfer function models having time delays. As an example, the exit temperature from a tubular heat exchanger $x(t)$ can be shown [76] to be related to the jacket temperature $u(t)$ by the transfer function

$$\bar{x}(s) = \frac{K_1[1 - K_2 e^{-\beta s}]}{(as + 1)(bs + 1)} \bar{u}(s) \tag{4.5.118}$$

Let us suppose we wish to control the exit temperature by adjusting the jacket temperature while minimizing the quadratic objective:

$$I = \tfrac{1}{2} \int_0^{t_f} \left[F(x - x_s)^2 + E(u - u_s)^2 \right] dt \tag{4.5.119}$$

For this problem the differential-difference equation representation becomes

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}\mathbf{z} + \mathbf{B}_0 u(t) + \mathbf{B}_1 u(t - \beta) \qquad \mathbf{z}(0) = \mathbf{0} \\ x &= z_1 \\ u(t) &= 0 \qquad -\beta \le t < 0 \end{aligned} \tag{4.5.120}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{ab} & -\dfrac{(a+b)}{ab} \end{bmatrix} \qquad \mathbf{B}_0 = \begin{bmatrix} 0 \\ \dfrac{K_1}{ab} \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ \dfrac{-K_1 K_2}{ab} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{4.5.121}$$

For the numerical computation we choose the parameters $F(t) = 2t^2$, $E = 20$, $K_1 = 1$, $K_2 = 0.5$, $\beta = 10$, $a = 40$, $b = 15$, $t_f = 80$, $x_s = 0.2$, $u_s = 0.4$ and apply the optimal feedback control law developed above. This becomes

$$\begin{aligned} u = u_s - (E)^{-1} \Big(&[\mathbf{B}_0^T \mathbf{E}_0(t) + \mathbf{E}_1^T(t, 0)]\mathbf{y} \\ &+ \int_{-\beta}^{0} \{[\mathbf{B}_0^T \mathbf{E}_1(t, s) + E_2(t, 0, s)][u(t + s) - u_s]\}\, ds \Big) \end{aligned} \tag{4.5.122}$$

where

$$\mathbf{y} = \begin{bmatrix} x - x_s \\ \dot{x} \end{bmatrix} = \begin{bmatrix} z_1 - x_s \\ z_2 \end{bmatrix} \tag{4.5.123}$$

* This example is taken from [76] with permission of Pergamon Press Ltd.

and feedback controller parameters

$$\mathbf{E}_0 = \begin{bmatrix} E_{011} & E_{012} \\ E_{021} & E_{022} \end{bmatrix} \qquad \mathbf{E}_1 = \begin{bmatrix} E_{11} \\ E_{12} \end{bmatrix}$$

and E_2 are determined from the following Riccati differential equations:

$$\frac{d\mathbf{E}_0(t)}{dt} = -\mathbf{E}_0(t)\mathbf{A} - \mathbf{A}^T\mathbf{E}_0(t) + [\mathbf{E}_0(t)\mathbf{B}_0 + \mathbf{E}_1(t, 0)]E^{-1} \times [\mathbf{B}_0^T\mathbf{E}_0(t) + \mathbf{E}_1^T(t, 0)] - \mathbf{F}_0 \tag{4.5.124}$$

$$\frac{\partial \mathbf{E}_1(t, s)}{\partial t} = \frac{\partial \mathbf{E}_1(t, s)}{\partial s} - \mathbf{A}^T\mathbf{E}_1(t, s) + [\mathbf{E}_0(t)\mathbf{B}_0 + \mathbf{E}_1(t, 0)]E^{-1} \times [\mathbf{B}_0^T\mathbf{E}_1(t, s) + \mathbf{E}_2(t, 0, s)] \qquad -\beta \le s \le 0 \tag{4.5.125}$$

$$\frac{\partial E_2(t, r, s)}{\partial t} = \frac{\partial E_2(t, r, s)}{\partial r} + \frac{\partial E_2(t, r, s)}{\partial s} + [\mathbf{E}_1^T(t, r)\mathbf{B}_0 + E_2(t, r, 0)]E^{-1} \times [\mathbf{B}_0^T\mathbf{E}_1(t, s) + E_2(t, 0, s)] \qquad \begin{matrix} -\beta \le r \le 0 \\ -\beta \le s \le 0 \end{matrix} \tag{4.5.126}$$

where $\mathbf{F}_0$ is defined as

$$\mathbf{F}_0 = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$$

The boundary conditions on the variables r, s are

$$\begin{aligned} \mathbf{E}_1(t, -\beta) &= \mathbf{E}_0(t)\mathbf{B}_1 \\ E_2(t, -\beta, s) &= \mathbf{B}_1^T\mathbf{E}_1(t, s) \end{aligned} \tag{4.5.127}$$

and the terminal conditions are

$$\begin{aligned} \mathbf{E}_0(t_f) &= \mathbf{0} \\ \mathbf{E}_1(t_f, s) &= \mathbf{0} \qquad -\beta \le s \le 0 \\ E_2(t_f, r, s) &= 0 \qquad -\beta \le r \le 0 - \beta \le s \le 0 \end{aligned} \tag{4.5.128}$$

Even though these equations appear quite formidable, they can be precomputed off-line and the optimal feedback controller of the form

$$u = u_s - P_1(t)(x - x_s) - P_2(t)\dot{x} + \int_{-\beta}^{0} P_3(t, s)[u(t + s) - u_s]\, ds \tag{4.5.129}$$

implemented, where $P_1(t)$, $P_2(t)$, $P_3(t, s)$ are the obvious combination of terms in Eq. (4.5.122). For this case it is seen that the optimal feedback controller is a proportional-derivative controller with a memory of the control required over the interval $(t - \beta, t)$.

The feedback controller may be implemented by precomputing Eqs. (4.5.124) to (4.5.126) and forming the variable gains $P_1(t)$, $P_2(t)$, $P_3(t, s)$. The

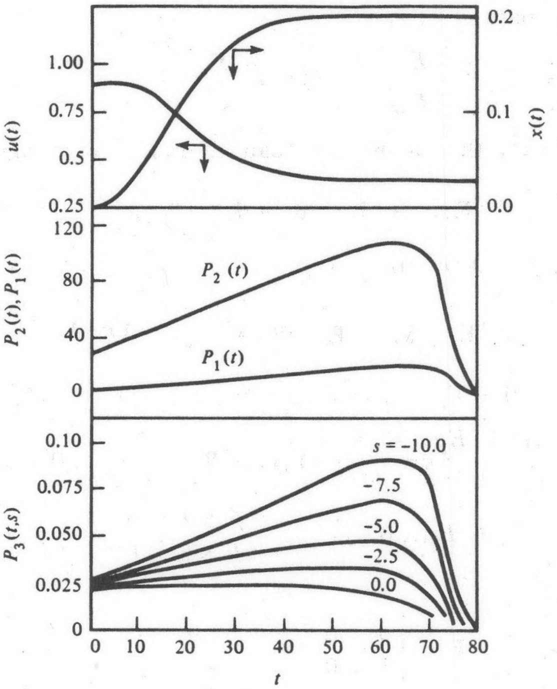

Figure 4.33 Optimal linear-quadratic feedback control of a tubular heat exchanger [76]. (*Reproduced with permission of Pergamon Press, Ltd.*)

results are plotted in Fig. 4.33 and show the controller gains as well as the state and control variables under optimal feedback control.

The linear quadratic formulation may be extended to nonlinear problems by linearization about a nominal open-loop trajectory. The feedback control law in this case and an example problem may be found in Ref. [77].

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PROBLEMS

4.1 Consider the chemical reaction $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ described in Example 3.3.2, except that it is now carried out in a homogeneous tubular reactor. If the reactor has length L, mean velocity v, and a flat velocity profile, a possible model is

$$\frac{\partial c_1}{\partial t} = -v\frac{\partial c_1}{\partial z'} - k_1(T)c_1^2 \qquad \begin{array}{l} 0 \le z' \le L \\ t > 0 \end{array}$$

$$\frac{\partial c_2}{\partial t} = -v\frac{\partial c_2}{\partial z'} + k_1(T)c_1^2 - k_2(T)c_2 \qquad \begin{array}{l} 0 \le z' \le L \\ t > 0 \end{array}$$

$$z' = 0,\ c_1 = 1.0,\ c_2 = 0 \qquad \text{for all } t > 0$$

$$t = 0,\ c_1 = 1.0,\ c_2 = 0 \qquad \text{for all } z' \epsilon (0, L)$$

Suppose that the reactor tube is immersed in a well-agitated bath and the heat of reaction is sufficiently small that isothermal operation is possible at the bath temperature $T(t)$.

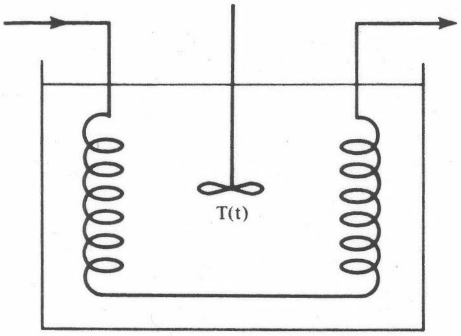
By defining $\theta = L/v$, the mean residence time, and $z = z'/L$, one obtains the model

$$\frac{\partial c_1(z, t)}{\partial t} = -\frac{1}{\theta}\frac{\partial c_1}{\partial z} - k_1(T)c_1^2 \qquad \begin{array}{l} 0 \le z \le 1 \\ t > 0 \end{array}$$

$$\frac{\partial c_2(z, t)}{\partial t} = -\frac{1}{\theta}\frac{\partial c_2}{\partial z} + k_1(T)c_1^2 - k_2(T)c_2$$

Now it is desired to start-up the reactor in such a way that we maximize the production of species B, i.e.,

$$\max_{T(t)} \left\{ I = \int_0^{t_f} c_2(1, t)\, dt \right\}$$



where the bath temperature is constrained by

$$T_* \leq T \leq T^*$$

(*a*) Derive the necessary conditions for optimality of $T(t)$.

(*b*) Describe in detail the control vector iteration procedure you would use to determine $T(t)$. List all the equations (including boundary conditions) needing solution. Describe your proposed numerical procedure.

(*c*) Carry out this control vector iteration procedure for the parameters given in Example 3.3.2 and with the additional parameters $\theta = 1$ h, $t_f = 2$ h.

4.2 In Sec. 4.3 the linear-quadratic optimal feedback control law has been derived for $\mathbf{A}_2$, $\mathbf{A}_1$, $\mathbf{A}_0$, $\mathbf{B}$ only functions of t. Rederive the equations for the case where $\mathbf{A}_2$, $\mathbf{A}_1$, $\mathbf{A}_0$, $\mathbf{B}$ all depend on the spatial variable z as well.

4.3 For the heated rod problem of Example 4.3.3, use modal analysis to decompose both the state and Riccati equations.

(*a*) Describe in detail the modal computational algorithm for $S(r, s, t)$ to be solved off-line.

(*b*) Describe how you would simulate the controlled system and how you would implement the feedback control law.

(*c*) Carry out the computations and show the response of the feedback controller for $F = 1$, $E = 0.25$, $x(z, 0) = \sqrt{2}\cos \pi z$, $t_f = 4$.

4.4 Consider the boundary control of a thin metal rod which has one end in a water bath at 25°C and the other end inserted into a steam chest. Air at 25°C is blowing transversely across the rod. The temperature of the right-hand end is assumed fixed at 25°C, while the temperature of the left-hand end may be controlled by adjusting the steam pressure. Thus the system may be modeled by

$$\rho C_p \frac{\partial T}{\partial t'}(z', t') = k\frac{\partial^2 T}{\partial z'^2} - h'(T - 25) \qquad \begin{array}{l} 0 < z' < L \\ t' > 0 \end{array}$$

$$z' = 0 \qquad T(0, t') = T_s(t')$$

$$z' = L \qquad T(L, t') = 25$$

where $T_s(t)$ is adjustable.

By using the dimensionless variables

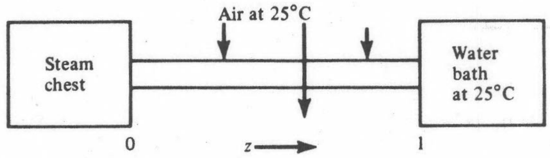
$$x = \frac{T - 25}{25} \qquad u_0 = \frac{T_s - 25}{25} \qquad t = \frac{t'k}{\rho C_p L^2} \qquad \beta = \frac{h'L^2}{k} \qquad z = \frac{z'}{L}$$

we obtain the model

$$\frac{\partial x(z, t)}{\partial t} = \frac{\partial^2 x}{\partial z^2} - \beta x$$

$$z = 0 \qquad x(0, t) = u_0(t)$$

$$z = 1 \qquad x(1, t) = 0$$



Now it may be more convenient for you to consider the equivalent form

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} - \beta x - \dot{\delta}(z) u_0(t)$$

with boundary conditions

$$z = 0, 1 \qquad x = 0$$

Note that $\dot{\delta}(z) \equiv d(\delta(z))/dz$ where $\delta(z)$ is the dirac delta function. Also note that for any continuous function $\phi_n(z)$,

$$\int_0^1 \dot{\delta}(z)\phi_n(z)\,dz \equiv -\int_0^1 \dot{\phi}_n(z)\delta(z)\,dz$$

Given this information:

(*a*) Solve the modeling equations through an eigenfunction expansion of the form

$$x(z, t) = \sum_{n=1}^{N} a_n(t)\phi_n(z)$$

and determine an orthonormal set $\phi_n(z)$ and the equations for $a_n(t)$.

(*b*) Determine if the system is *approximately controllable*.

(*c*) Develop the optimal linear-quadratic feedback control law, Riccati equations, and implementation scheme for this boundary control problem.

(*d*) Carry out the computations and show the response of the feedback controller for $F = 1$, $E_0 = 0.25$, $\beta = 3$, $x(z, 0) = \sqrt{2}\,\sin \pi z$, $t_f = 10$.

4.5 For the nonlinear system described in Example 4.4.3, develop a pseudo-modal feedback controller. In particular,

(*a*) Outline how one would implement such a controller.

(*b*) Simulate the system under feedback control using the described Galerkin procedure and computationally determine the number of "eigenfunctions" necessary to obtain reasonable convergence.

(*c*) Discuss the multivariable controller design problem arising from this pseudo-modal analysis.

4.6 Consider the heat exchanger control problem posed in Example 4.3.4. Show that this system in the Laplace transform domain has the same form as Example 4.5.6. Discuss the relationship between first-order hyperbolic partial differential equations and differential-difference equations. Compare the optimal feedback control law formulations for both forms of the problem.

4.7 For the distillation column discussed in Example 3.2.8 but for the case where $\mathbf{G}(s)$ contains time delays [i.e., $\mathbf{G}(s)$ given by Eq. (4.5.4)]:

(*a*) Determine the equation for the time-domain realization of $y_1(t)$, $y_2(t)$, $y_3(t)$ in response to $u_1(t)$, $u_2(t)$, $u_3(t)$.

(*b*) Calculate the open-loop response for step changes in the controls, $u_1 = 1.0$, $u_2 = 2.0$, $u_3 = 3.0$.

(*c*) Design a multidelay compensator to be included in a feedback control system where proportional controllers are used and the loop pairings are $y_1 \leftrightarrow u_1$, $y_2 \leftrightarrow u_2$, $y_3 \leftrightarrow u_3$. Develop all the necessary design equations.

(*d*) Simulate the performance of your multidelay compensator using the realization equations similar to those developed in part (*a*).