CHAPTER

# THREE

# CONTROL OF LUMPED PARAMETER SYSTEMS

## 3.1 INTRODUCTION

In this chapter we shall consider the control of *lumped parameter processes*, i.e., processes described by ordinary differential equations. These include both single-variable and multivariable dynamic systems. We shall begin with a discussion of some of the key concepts and practical difficulties encountered in lumped parameter control system design. Some of the control system design procedures available for linear multivariable systems shall be presented and illustrated with examples. Following this we shall introduce *optimal control theory*, which represents one approach to control system design, and which provides a theoretical foundation helpful in understanding material to be covered later in the book. Finally, we shall discuss special techniques which may be used for nonlinear dynamic systems.

To provide a very simple example of the type of problem we shall be attacking and to clarify a number of concepts, let us consider the simple, well-stirred, steam-heated mixing tank shown in Fig. 3.1. It is used to mix and preheat a recipe of reactants before passing them to a chemical reactor. Now let us suppose that we wish to design a scheme for controlling the effluent temperature $T(t)$ to a desired value $T_d$ by manipulating the steam rate $Q(t)$ with valve $v$.

An *open-loop control scheme* for such a process would involve programming the steam valve position over time without benefit of *feedback* information, such as a tank effluent temperature measurement. Such a scheme might be useful if a very good *mathematical model* of the process were available, e.g.,

$$\frac{dT}{dt} = \frac{F(T_f - T)}{V} + \frac{Q(t)}{V\rho C_p} \tag{3.1.1}$$

$$T(0) = T_0 \tag{3.1.2}$$

and one had the start-up problem of driving $T(t)$ from the initial condition $T_0$ to

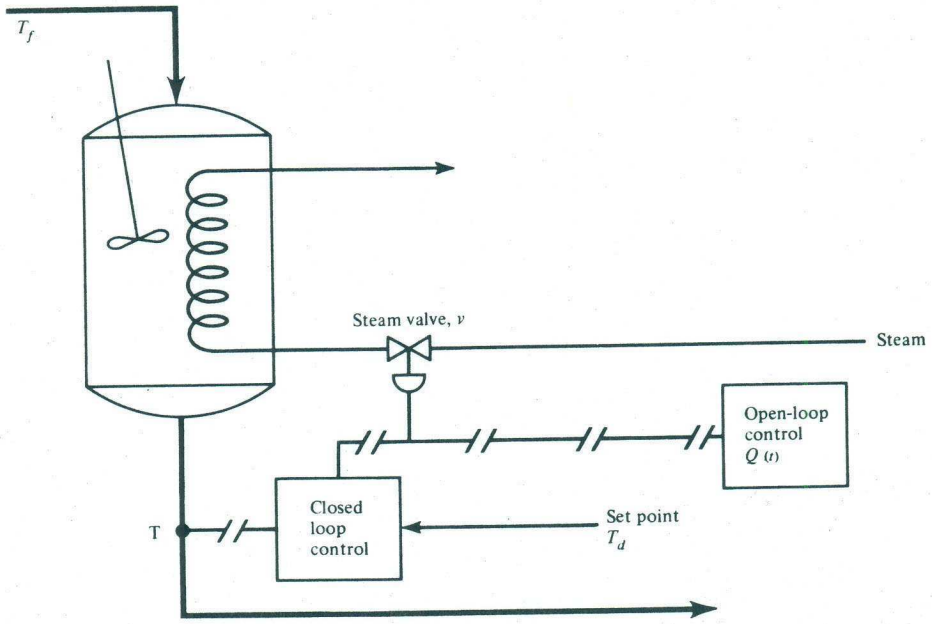


**Figure 3.1** A steam-heated mixing tank.

the desired set point $T_d$. An open-loop control policy $Q(t)$ could be calculated from the model and used to program the steam valve without making use of tank temperature measurements.

In practice, however, most process models contain some error, and *closed-loop control schemes*, which involve process measurements (tank temperature, in this case), have been found necessary for satisfactory controller performance. The choice of the feedback controller structure is left to the designer; for example, a common structure is the three-mode proportional-integral-derivative (PID) controller given by

$$Q(t) = Q_s + K_c\left[(T_d - T) + \frac{1}{\tau_I}\int (T_d - T)\,dt' + \tau_D\frac{d(T_d - T)}{dt}\right] \quad (3.1.3)$$

where $K_c$, $\tau_I$, and $\tau_D$ are the controller parameters. However, in what follows we shall not restrict ourselves to this limited class of controllers, but shall show, in a more general way, how one may design both *open-loop* and *closed-loop* control schemes for processes described by ordinary differential equations.

## 3.2 LINEAR MULTIVARIABLE CONTROL SYSTEMS

Perhaps the most commonly encountered control system design problem in the process industries is the design of multivariable control systems. If the process under study is linear, a very general model in the time domain (so-called *state*

*variable* notation) is

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{\Gamma}\mathbf{d} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.2.1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{3.2.2}$$

where

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix}$$

is a $k$ vector of disturbances,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is an $n$ vector of states,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

is an $m$ vector of controls (manipulated variables), and

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{bmatrix}$$

is an $l$ vector of outputs (those states or combination of states which can be measured). The matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & \cdots & & b_{nm} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{l1} & \cdots & & c_{ln} \end{bmatrix} \qquad \mathbf{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1k} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{n1} & \cdots & & \gamma_{nk} \end{bmatrix}$$

can be either constant or time-varying. This very general model shall be used as a basis for much of the discussion which follows.

It is important to note that Eqs. (3.2.1) and (3.2.2) have analytical solutions which are useful to know. In the case of an *autonomous system* (i.e., where **A**, **B**, **C**, and **Γ** are constant matrices), one may take the Laplace transform of Eq. (3.2.1) to yield

$$s\mathbf{I}\bar{\mathbf{x}}(s) - \mathbf{x}_0 = \mathbf{A}\bar{\mathbf{x}}(s) + \mathbf{B}\bar{\mathbf{u}}(s) + \boldsymbol{\Gamma}\bar{\mathbf{d}}(s)$$

$$\bar{\mathbf{y}}(s) = \mathbf{C}\bar{\mathbf{x}}(s)$$

Solving for $\bar{\mathbf{x}}(s)$, we obtain

$$\bar{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\bar{\mathbf{u}}(s) + \boldsymbol{\Gamma}\bar{\mathbf{d}}(s)\right] \tag{3.2.3}$$

Now, using the convolution theorem of Laplace transforms to invert this expression, the analytical solution takes the form

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-r)}\left[\mathbf{B}\mathbf{u}(r) + \boldsymbol{\Gamma}\mathbf{d}(r)\right] dr \tag{3.2.4}$$

where the exponential matrix $e^{\mathbf{A}\tau}$ must be evaluated. Let us note that $\mathbf{X} = e^{\mathbf{A}\tau}$ is the solution to the simple homogeneous matrix differential equation

$$\frac{d\mathbf{X}}{d\tau} = \mathbf{A}\mathbf{X} \qquad \mathbf{X}(0) = \mathbf{I} \tag{3.2.5}$$

where **X** is an $n \times n$ matrix. If the eigenvalues of **A** are distinct, then there is a canonical transformation [1]

$$\mathbf{A} = \mathbf{M}\boldsymbol{\Lambda}\mathbf{M}^{-1} \tag{3.2.6}$$

relating the matrix **A** to its diagonal matrix of eigenvalues

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix} \tag{3.2.7}$$

and matrix of eigenvectors **M**. Substituting Eq. (3.2.6) into Eq. (3.2.5) yields

$$\frac{d(\mathbf{M}^{-1}\mathbf{X})}{d\tau} = \boldsymbol{\Lambda}(\mathbf{M}^{-1}\mathbf{X}) \qquad \mathbf{M}^{-1}\mathbf{X}(0) = \mathbf{M}^{-1} \tag{3.2.8}$$

which has the solution

$$\mathbf{M}^{-1}\mathbf{X} = e^{\boldsymbol{\Lambda}\tau}\mathbf{M}^{-1}$$

or

$$\mathbf{X} = e^{\mathbf{A}\tau} = \mathbf{M}e^{\boldsymbol{\Lambda}\tau}\mathbf{M}^{-1} \tag{3.2.9}$$

where $e^{\mathbf{\Lambda}\tau}$ is the diagonal $n \times n$ matrix

$$e^{\mathbf{\Lambda}\tau} = \begin{bmatrix} e^{\lambda_1\tau} & & & \mathbf{0} \\ & e^{\lambda_2\tau} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda_n\tau} \end{bmatrix} \tag{3.2.10}$$

Thus one direct way to evaluate $e^{\mathbf{A}\tau}$ is through Eq. (3.2.9).

In the event that the matrices **A**, **B**, **C**, and **Γ** are time-varying, the system represented by Eqs. (3.2.1) and (3.2.2) is *nonautonomous*, and an alternative solution must be used. In this case, the analytical solution takes the form [2]

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \mathbf{\Phi}(t, t_0)\int_{t_0}^{t} \mathbf{\Phi}(r, t_0)^{-1}\left[\mathbf{B}(r)\mathbf{u}(r) + \mathbf{\Gamma}(r)\mathbf{d}(r)\right] dr \tag{3.2.11}$$

where $\mathbf{\Phi}(t, t_0)$ is an $n \times n$ time-varying matrix known as *the fundamental matrix solution*. The fundamental matrix solution arises from the solution of the equation

$$\frac{d\mathbf{\Phi}(t, t_0)}{dt} = \mathbf{A}(t)\mathbf{\Phi}(t, t_0) \qquad \mathbf{\Phi}(t_0, t_0) = \mathbf{I} \tag{3.2.12}$$

**Example 3.2.1** To illustrate how these general results apply to a particular multivariable system, let us consider the stirred mixing tank in Fig. 3.2 with cross-sectional area $A_c$. There are three streams entering the tank: (1) a hot

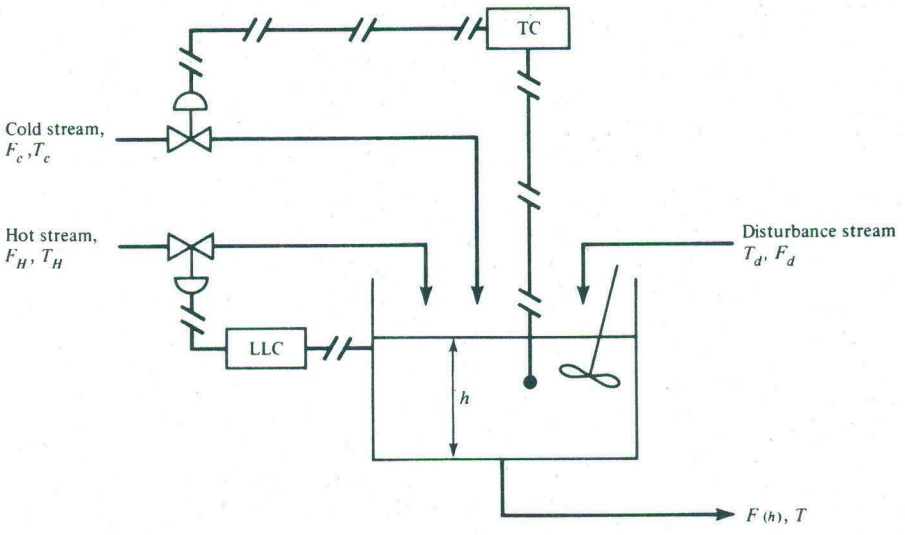


**Figure 3.2** Stirred mixing tank requiring level and temperature control.

stream at temperature $T_H$ with adjustable flow $F_H$, (2) a cold stream at temperature $T_C$ with adjustable flow $F_C$, and (3) a disturbance stream from another process unit with variable temperature $T_d$ and flow $F_d$. The tank is well stirred, with liquid outflow determined by the liquid height in the tank, $h$, i.e., $F(h) = Kh^{1/2}$. The modeling equations arise from material and energy balances:

$$A_c \frac{dh}{dt} = F_H + F_C + F_d - F(h)$$

$$\rho C_p A_c \frac{d(hT)}{dt} = \rho C_p [F_H T_H + F_C T_C + F_d T_d - F(h)T] \quad (3.2.13)$$

Clearly the tank model is nonlinear; however, it is possible to linearize the equations about the desired operating point $h_s$, $T_s$ by using Taylor series expansions truncated after the first-order terms:

$$F(h) \cong F(h_s) + \frac{1}{2}\frac{K}{h_s^{1/2}}(h - h_s) + \cdots$$

$$F(h)T \cong F(h_s)T_s + F(h_s)(T - T_s) + \frac{1}{2}\frac{T_s K(h - h_s)}{h_s^{1/2}} + \cdots$$

$$hT \cong h_s T_s + h_s(T - T_s) + T_s(h - h_s) + \cdots \quad (3.2.14)$$

$$F_d T_d \cong F_{ds} T_{ds} + F_{ds}(T_d - T_{ds}) + T_{ds}(F_d - F_{ds}) + \cdots$$

If we then define $F_{Hs}$, $F_{Cs}$, $F_{ds}$, $T_{ds}$ as the steady-state values of $F_H$, $F_C$, $F_d$, $T_d$ corresponding to $h = h_s$, $T = T_s$, that is, satisfying

$$0 = F_{Hs} + F_{Cs} + F_{ds} - F(h_s)$$
$$0 = F_{Hs} T_H + F_{Cs} T_C + F_{ds} T_{ds} - F(h_s)T_s \quad (3.2.15)$$

then one can use the deviation variables

$$x_1 = h - h_s \qquad x_2 = T - T_s \qquad u_1 = F_H - F_{Hs}$$
$$u_2 = F_C - F_{Cs} \qquad d_1 = F_d - F_{ds} \qquad d_2 = T_d - T_{ds} \quad (3.2.16)$$

to obtain the linearized modeling equations

$$A_c \frac{dx_1}{dt} = u_1 + u_2 + d_1 - \frac{1}{2}\frac{K}{h_s^{1/2}} x_1$$

$$A_c \left[ h_s \frac{dx_2}{dt} + T_s \frac{dx_1}{dt} \right] = T_H u_1 + T_C u_2 + T_{ds} d_1 + F_{ds} d_2$$
$$- \frac{T_s}{2}\frac{K}{h_s^{1/2}} x_1 - F(h_s) x_2 \quad (3.2.17)$$

or rearranging,

$$\frac{dx_1}{dt} = \frac{1}{A_c}\left[u_1 + u_2 + d_1 - \frac{1}{2}\frac{F(h_s)}{h_s}x_1\right]$$

$$\frac{dx_2}{dt} = \frac{1}{A_c h_s}[(T_H - T_s)u_1 + (T_C - T_s)u_2 + (T_{ds} - T_s)d_1 + F_{ds}d_2 - F(h_s)x_2] \tag{3.2.18}$$

Thus, if we define the vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

and the matrices

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2}\frac{F(h_s)}{A_c h_s} & 0 \\ 0 & -\frac{F(h_s)}{A_c h_s} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{A_c} & \frac{1}{A_c} \\ \frac{T_H - T_s}{A_c h_s} & \frac{T_C - T_s}{A_c h_s} \end{bmatrix}$$

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{1}{A_c} & 0 \\ \frac{T_{ds} - T_s}{A_c h_s} & \frac{F_{ds}}{A_c h_s} \end{bmatrix} \tag{3.2.19}$$

then the model for this system is in the form of Eq. (3.2.1). If both the state variables are measured, then

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To use the analytical solution, Eq. (3.2.4), for this example problem, let us note that the matrix **A** is diagonal and that the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

yields the eigenvalues

$$\lambda_1 = -\frac{1}{2}\frac{F(h_s)}{A_c h_s} \qquad \lambda_2 = -\frac{F(h_s)}{A_c h_s} \tag{3.2.20}$$

Hence **A** is already in canonical form and

$$e^{\mathbf{A}\tau} = \begin{bmatrix} e^{\lambda_1\tau} & 0 \\ 0 & e^{\lambda_2\tau} \end{bmatrix} \tag{3.2.21}$$

Let us now evaluate the response of the system to a unit step change in $u_1$ and $u_2$ at $t = 0$, that is $u_1 = 1$, $u_2 = 1$. Further, let us assume

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \quad \text{and} \quad d_1 = d_2 = 0$$

In this case Eq. (3.2.4) becomes

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{u}(s)\, ds \tag{3.2.22}$$

or

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}x_{10} + \int_0^t e^{\lambda_1(t-s)}\dfrac{2}{A_c}\, ds \\ e^{\lambda_2 t}x_{20} + \int_0^t e^{\lambda_2(t-s)}\left(\dfrac{T_H + T_C - 2T_s}{A_c h_s}\right) ds \end{bmatrix}$$

Evaluation of the integrals yields the analytical solution:

$$x_1(t) = \exp\left[-\frac{1}{2}\frac{F(h_s)}{A_c h_s}t\right]x_{10} + \frac{4h_s}{F(h_s)}\left\{1 - \exp\left[-\frac{1}{2}\frac{F(h_s)}{A_c h_s}t\right]\right\}$$

$$x_2(t) = \exp\left[-\frac{F(h_s)}{A_c h_s}t\right]x_{20} + \frac{T_H + T_C - 2T_s}{F(h_s)}\left\{1 - \exp\left[\frac{-F(h_s)t}{A_c h_s}\right]\right\} \tag{3.2.23}$$

## Time-Domain versus Transform-Domain Representation

In considering the dynamics of multivariable linear systems with constant coefficients, one may choose to do the analysis either in the time domain, with models of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \Gamma\mathbf{d} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.2.24}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{3.2.25}$$

or in the Laplace transform domain, involving transfer functions in the form

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) + \mathbf{G}_d(s)\bar{\mathbf{d}}(s) \tag{3.2.26}$$

Here the overbar denotes the Laplace-transformed variable and the matrices $\mathbf{G}(s)$, $\mathbf{G}_d(s)$ are the multivariable transfer functions relating the system control

variables $\bar{\mathbf{u}}(s)$ and disturbances $\bar{\mathbf{d}}(s)$ to the system output, i.e.,

$$\mathbf{G}(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1m}(s) \\ g_{21}(s) & g_{22}(s) & \cdots & g_{2m}(s) \\ \cdots & \cdots & \cdots & \cdots \\ g_{l1}(s) & \cdots & & g_{lm}(s) \end{bmatrix}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} g^d_{11}(s) & g^d_{12}(s) & \cdots & g^d_{1k}(s) \\ g^d_{21}(s) & g^d_{22}(s) & \cdots & g^d_{2k}(s) \\ \cdots & \cdots & \cdots & \cdots \\ g^d_{l1}(s) & \cdots & & g_{lk}(s) \end{bmatrix} \tag{3.2.27}$$

Note that the state variables $\mathbf{x}$ are not used in the Laplace-domain representation.

Some control system design procedures are easiest in the first representation, while other algorithms are more convenient using the second. Thus the control system designer should have the capability to quickly switch back and forth between the two formulations. Let us discuss how this may be easily done.

The transformation from the state space [Eqs. (3.2.1) and (3.2.2)] to the transform space is unique and simply done by taking the Laplace transform of Eqs. (3.2.1) and (3.2.2) with $\mathbf{x}_0 = 0$, to yield

$$\bar{\mathbf{y}} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\bar{\mathbf{u}}(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\Gamma}\bar{\mathbf{d}}(s) \tag{3.2.28}$$

Hence

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{3.2.29}$$

$$\mathbf{G}_d(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\Gamma} \tag{3.2.30}$$

There exist standard computer programs (see Appendix A) which can efficiently perform the operations in Eq. (3.2.28).

The reverse transformation from the transform domain to the time domain is more difficult for several reasons:

1. Because there is more information contained in the time-domain representation than in the simple input/output transfer function, the transformation from the transform domain is not unique. There are many sets of Eqs. (3.2.1), (3.2.2) equivalent to Eq. (3.2.28).
2. Among the many possible transformations, one wishes to choose one which provides the *minimal realization* of the transfer function. This means the smallest set of state variables necessary to match the transfer-function relationship. Even so, there are many possible choices of a minimal realization.

Among the several different algorithms for finding the minimal realization of a transfer function (see [3-5] for a more detailed discussion) is a most useful technique which works when all the poles of the transfer function matrix are real and distinct. The procedure may be outlined in the following way.

Let us consider the transfer function matrix $\mathbf{G}(s)$ which can be expanded in a partial fraction expansion of the form

$$\mathbf{G}(s) = \sum_{i=1}^{p} \frac{\mathbf{M}_i}{s + \lambda_i} \tag{3.2.31}$$

where the $-\lambda_i$, $i = 1, 2, \ldots p$, are the poles and the $l \times m$ matrix $\mathbf{M}_i$ is the matrix of residues for the pole $-\lambda_i$, defined by

$$\mathbf{M}_i = \lim_{s \to -\lambda_i} [(s + \lambda_i)\mathbf{G}(s)] \tag{3.2.32}$$

If $n_i$ is the rank of the $i$th matrix of residues, then the minimal state space realization must consist of $n$ state variables, where

$$n = \sum_{i=1}^{p} n_i \tag{3.2.33}$$

The construction of the matrices **A**, **B**, and **C** can be done in many different ways, and, as noted above, the selection is not unique. One procedure is to make **A** diagonal and of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & & & \mathbf{0} \\ & \mathbf{A}_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{A}_{pp} \end{bmatrix} \tag{3.2.34}$$

where the diagonal submatrices are given by

$$\mathbf{A}_{ii} = -\lambda_i \mathbf{I}_{n_i}$$

and $\mathbf{I}_{n_i}$ is an identity matrix of dimension $n_i$. To determine the elements of **B** and **C**, note that with **A** in diagonal form, the quantity $(s\mathbf{I} - \mathbf{A})^{-1}$ is also diagonal, so that Eq. (3.2.29) may be rewritten in the form of Eq. (3.2.31), where $\mathbf{M}_i$ will be

composed only of the elements of **C** and **B**:

$$\mathbf{M}_i = \sum_{j=1}^{n_i} \mathbf{c}_{ij}\mathbf{b}_{ij}^T \qquad i = 1, 2, \ldots, p \tag{3.2.35}$$

where $\mathbf{c}_{ij}$ is an $l$ vector and $\mathbf{b}_{ij}$ is an $m$ vector. Then the vectors $\mathbf{c}_{ij}$, $\mathbf{b}_{ij}$ are one possible choice for the columns of **C** and $\mathbf{B}^T$, respectively, and these would take the form

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_{11}^T \\ \vdots \\ \mathbf{b}_{1n_1}^T \\ \hline \mathbf{b}_{21}^T \\ \vdots \\ \mathbf{b}_{2n_2}^T \\ \hline \vdots \\ \mathbf{b}_{p1}^T \\ \vdots \\ \mathbf{b}_{pn_p}^T \end{bmatrix} \qquad \mathbf{C} = \left[ \mathbf{c}_{11} \cdots \mathbf{c}_{1n_1} \vdots \mathbf{c}_{21} \cdots \mathbf{c}_{2n_2} \vdots \cdots \vdots \mathbf{c}_{p1} \cdots \mathbf{c}_{pn_p} \right] \tag{3.2.36}$$

where, of course, **B** is an $n \times m$ matrix and **C** is an $l \times n$ matrix.

If in addition we must invert the disturbance transfer function

$$\mathbf{G}_d = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\Gamma} \tag{3.2.30}$$

we must now choose the $n \times k$ matrix $\boldsymbol{\Gamma}$ to obtain the proper transformation. This can be done by expanding $\mathbf{G}_d$ in the form of Eq. (3.2.29), where the poles of **G** and $\mathbf{G}_d$ must be the same:*

$$\mathbf{G}_d = \sum_{i=1}^{p} \frac{\mathbf{M}_i^d}{s + \lambda_i} \tag{3.2.37}$$

Here the matrix of residues $\mathbf{M}_i^d$ may be expanded as

$$\mathbf{M}_i^d = \sum_{j=1}^{n_i} \mathbf{c}_{ij}\boldsymbol{\gamma}_{ij}^T \tag{3.2.38}$$

* This is not really a restriction because a common denominator may be defined which contains all the poles of **G** and $\mathbf{G}_d$.

where the elements $\mathbf{c}_{ij}$ have already been determined and the $k$ vector $\boldsymbol{\gamma}_{ij}$ must be selected to form the matrix

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\gamma}_{11}^T \\ \boldsymbol{\gamma}_{12}^T \\ \vdots \\ \boldsymbol{\gamma}_{1n_1}^T \\ \hline \boldsymbol{\gamma}_{21}^T \\ \vdots \\ \boldsymbol{\gamma}_{2n_2}^T \\ \hline \vdots \\ \boldsymbol{\gamma}_{1n_p}^T \\ \vdots \\ \boldsymbol{\gamma}_{pn_p}^T \end{bmatrix} \tag{3.2.39}$$

Let us illustrate these procedures by examples.

**Example 3.2.2** Let us consider the dynamic modeling of a liquid-phase chemical reaction for reacting species $A$ to products. Assume that the rate of reaction is independent of temperature and that the reactor chosen is an adiabatic stirred tank. Composition and energy balances yield the modeling equations

$$\frac{dc_A}{dt} = \frac{1}{\theta}(c_{Af} - c_A) - kc_A$$

$$\frac{dT}{dt} = \frac{1}{\theta}(T_f - T) + Jkc_A$$

Here we assume $T_f$ is the manipulated variable and $T$ the output, so that by letting $x_1 = c_A$, $x_2 = T$, $u_1 = c_{Af}$, $u_2 = T_f$, and $y = T$, one obtains

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}u$$

$$y = \mathbf{C}\mathbf{x} \tag{3.2.40}$$

where

$$\mathbf{A} = \begin{bmatrix} -\left(\frac{1}{\theta} + k\right) & 0 \\ Jk & -\frac{1}{\theta} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{\theta} & 0 \\ 0 & \frac{1}{\theta} \end{bmatrix} \qquad \mathbf{C} = [0 \quad 1]$$

Then, by application of Eq. (3.2.28), one obtains

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

where

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s + \frac{1}{\theta} + k & 0 \\ -Jk & s + \frac{1}{\theta} \end{bmatrix}$$

and

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s + 1/\theta + k)(s + 1/\theta)} \begin{bmatrix} s + \frac{1}{\theta} & 0 \\ Jk & s + \frac{1}{\theta} + k \end{bmatrix}$$

Thus

$$\mathbf{G}(s) = \frac{1}{(s + 1/\theta + k)(s + 1/\theta)} \begin{bmatrix} \frac{Jk}{\theta} & \frac{s + \frac{1}{\theta} + k}{\theta} \end{bmatrix} \tag{3.2.41}$$

is the unique transfer function required.

**Example 3.2.3** Let us now determine a minimal state space realization for the transfer function, Eq. (3.2.41), derived in the previous example. Clearly the poles are $-(1/\theta + k)$ and $-1/\theta$. Thus a partial fraction expansion of the form of Eq. (3.2.31) gives

$$\mathbf{G}(s) = \frac{1}{s + 1/\theta + k} \begin{bmatrix} -\frac{J}{\theta} & 0 \end{bmatrix} + \frac{1}{s + \frac{1}{\theta}} \begin{bmatrix} \frac{J}{\theta} & \frac{1}{\theta} \end{bmatrix}$$

Since the rank of each of the $\mathbf{M}_i$, $i = 1, 2$, is 1, a minimal realization requires two state variables. Let us now determine a set of **A**, **B**, **C** for this realization using the algorithm suggested above. In diagonal form,

$$\mathbf{A} = \begin{bmatrix} -\left(\frac{1}{\theta} + k\right) & 0 \\ 0 & -\frac{1}{\theta} \end{bmatrix} \tag{3.2.42}$$

and we can choose $\mathbf{M}_1$ and $\mathbf{M}_2$ according to Eq. (3.2.35), i.e.,

$$\mathbf{M}_1 = \begin{bmatrix} -\frac{J}{\theta} & 0 \end{bmatrix} = [1] \begin{bmatrix} -\frac{J}{\theta} & 0 \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} \frac{J}{\theta} & \frac{1}{\theta} \end{bmatrix} = [1] \begin{bmatrix} \frac{J}{\theta} & \frac{1}{\theta} \end{bmatrix}$$

Thus

$$\mathbf{B} = \begin{bmatrix} -\frac{j}{\theta} & 0 \\ \frac{J}{\theta} & \frac{1}{\theta} \end{bmatrix} \qquad \mathbf{C} = [1 \quad 1] \tag{3.2.43}$$

is an acceptable choice. Note that this leads to the state space model

$$\frac{dx_1}{dt} = -\left(\frac{1}{\theta} + k\right)x_1 - \frac{J}{\theta}u_1$$

$$\frac{dx_2}{dt} = -\frac{1}{\theta}x_2 + \frac{J}{\theta}u_1 + \frac{1}{\theta}u_2$$

$$y = x_1 + x_2$$

Clearly $y$, $u_1$, and $u_2$ have the same physical meaning as in the previous example, but $x_1$ and $x_2$ here are not the same. In fact, $x_1 = Jc_A + T$, $x_2 = -Jc_A$ here.

This example clearly illustrates the nonuniqueness of the minimal state space realization, since the transfer function of Eq. (3.2.41) has been shown to be equivalent to two different state space models, Eqs. (3.2.40) and (3.2.42), (3.2.43).

More complicated transformations will be illustrated in the next example problem.

**Example 3.2.4** In this example we illustrate the state space representation of a more complicated transfer function relationship,

$$\bar{\mathbf{y}} = \mathbf{G}\bar{\mathbf{u}}(s) + \mathbf{G}_d\bar{d}(s) \tag{3.2.44}$$

where

$$\mathbf{G}(s) = \begin{bmatrix} \frac{0.7}{1+9s} & 0 \\ \frac{2.0}{1+8s} & \frac{0.4}{1+9s} \end{bmatrix}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} 0 & 0 \\ \frac{0.5}{1+8s} & \frac{1.0}{1+9s} \end{bmatrix}$$

Making use of Eq. (3.2.31), one obtains

$$\mathbf{G}(s) = \frac{1}{s+\frac{1}{9}}\begin{bmatrix} 0.078 & 0 \\ 0 & 0.044 \end{bmatrix} + \frac{1}{s+\frac{1}{8}}\begin{bmatrix} 0 & 0 \\ 0.25 & 0 \end{bmatrix}$$

where $\lambda_1 = -\frac{1}{9}$, $\lambda_2 = -\frac{1}{8}$ are the poles. Evaluating the ranks of the residue matrices, one obtains $n_1 = 2$, $n_2 = 1$, so that the minimal realization must be

of order 3. By using the diagonal form,

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{9} & 0 & 0 \\ 0 & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{1}{8} \end{bmatrix}$$

and expanding $\mathbf{M}_1$, $\mathbf{M}_2$ in the form of Eq. (3.2.35), then

$$\mathbf{M}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.078 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0.044 \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \end{bmatrix}$$

Thus $\mathbf{B}$ and $\mathbf{C}$ can take the form

$$\mathbf{B} = \begin{bmatrix} 0.078 & 0 \\ 0 & 0.044 \\ 0.25 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Now expanding $\mathbf{G}_d(s)$, one obtains

$$\mathbf{G}_d(s) = \frac{1}{s + \frac{1}{9}} \begin{bmatrix} 0 & 0 \\ 0 & 0.111 \end{bmatrix} + \frac{1}{s + \frac{1}{8}} \begin{bmatrix} 0 & 0 \\ 0.0625 & 0 \end{bmatrix}$$

and expanding the matrix of residues $\mathbf{M}_i^d$ according to Eq. (3.2.38), one obtains

$$\mathbf{M}_1^d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0.111 \end{bmatrix}$$

$$\mathbf{M}_2^d = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0.0625 & 0 \end{bmatrix}$$

Thus

$$\mathbf{\Gamma} = \begin{bmatrix} 0 & 0 \\ 0 & 0.111 \\ 0.0625 & 0 \end{bmatrix}$$

Summarizing, the state space model takes the form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} & 0 & 0 \\ 0 & -\frac{1}{9} & 0 \\ 0 & 0 & -\frac{1}{8} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0.078 \\ 0 & 0.044 \\ 0.25 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0.111 \\ 0.0625 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and is an equivalent minimal realization of the transfer function relationship, Eq. (3.2.44).

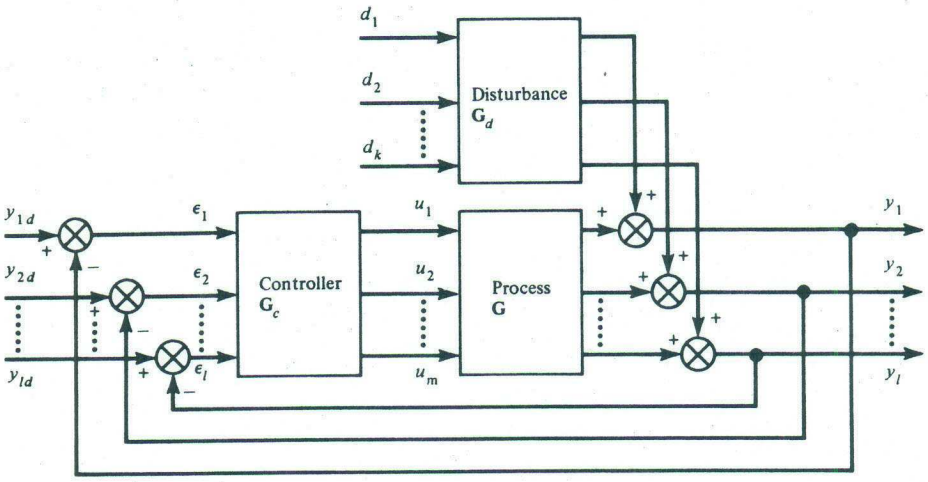


**Figure 3.3** Multivariable block diagram.

## Multivariable Block Diagrams

Having developed the facility to readily convert linear constant-coefficient systems between the time domain and the transform domain, it is now useful to consider multivariable feedback control loops. As a preliminary, let us discuss the meaning of the simple multivariable block diagram shown in Fig. 3.3. In the transform domain, $\mathbf{G}(s)$ and $\mathbf{G}_d(s)$ are the plant transfer functions relating the control variables and disturbances to the process outputs,

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s) + \mathbf{G}_d(s)\bar{\mathbf{d}}(s) \tag{3.2.45}$$

Here $\mathbf{G}(s)$ and $\mathbf{G}_d(s)$ are defined by Eq. (3.2.27), where the elements take the form*

$$g_{ij}(s) = \frac{K_{ij}\prod_p (1 + h_{ij}^p s)}{\prod_p (1 + e_{ij}^p s)} \tag{3.2.46}$$

for lumped parameter systems. The parameters are

| | |
|---|---|
| $K_{ij}$—transfer-function gain | $(-h_{ij}^p)^{-1}$ — transfer-function zeros |
| | $(-e_{ij}^p)^{-1}$ — transfer-function poles |

It is interesting to realize that these block diagrams may also be represented in the *time domain*. In this case, $\mathbf{G}$ and $\mathbf{G}_d$ may be thought of as integral operators. For the linear system equation (3.2.1) with initial condition $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{G}$ is defined as

$$\mathbf{y}(t) = \langle \mathbf{G}(t), \mathbf{u}(t) \rangle \tag{3.2.47}$$

* As described in Chap. 4, sometimes time delays also appear in the transfer function.

where

$$\mathbf{G}(t) = \mathbf{C}\boldsymbol{\Phi}(t)\int_0^t \boldsymbol{\Phi}(r)^{-1}\mathbf{B}(r)(\cdot)\,dr \tag{3.2.48}$$

and the brackets $\langle \mathbf{L}, \mathbf{w} \rangle$ represent a general inner product operation of the operator $\mathbf{L}$ operating on the variable $\mathbf{w}$. This result comes directly from the analytical solution to Eq. (3.2.1):

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, t_0)\mathbf{x}_0 + \boldsymbol{\Phi}(t, t_0)\int_{t_0}^t \boldsymbol{\Phi}(r, t_0)^{-1}[\mathbf{B}(r)\mathbf{u}(r)]\,dr \tag{3.2.49}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{3.2.2}$$

where we recall that $\boldsymbol{\Phi}(t, t_0)$ is the $n \times n$ *fundamental matrix solution* defined by

$$\dot{\boldsymbol{\Phi}} = \mathbf{A}(t)\boldsymbol{\Phi} \qquad \boldsymbol{\Phi}(t_0, t_0) = \mathbf{I} \tag{3.2.12}$$

The block diagram in Fig. 3.3 also includes a multivariable controller $\mathbf{G}_c$ which can be written in either the transform or state space domain. Although $\mathbf{G}_c$ can have entries in all elements, it is very common to have multiple single-loop controllers where $\mathbf{G}_c$ takes the diagonal form

$$\mathbf{G}_c = \begin{bmatrix} g_{11_c} & & & \mathbf{0} \\ & g_{22_c} & & \\ & & \ddots & \\ \mathbf{0} & & & g_{ll_c} \end{bmatrix} \tag{3.2.50}$$

Here the single-loop controllers could be of the simple linear proportional-integral-derivative type. In the Laplace transform domain, these take the form

$$g_{ii_c}(s) = K_{c_i}\left(1 + \frac{1}{\tau_{I_i}s} + \tau_{D_i}s\right) \qquad i = 1, 2, \ldots, l \tag{3.2.51}$$

In the time domain, the single-loop controller action $u_i(t)$ is

$$u_i(t) = \langle \mathbf{G}_c, \boldsymbol{\varepsilon} \rangle_i = K_{c_i}\left[\varepsilon_i + \frac{1}{\tau_{I_i}}\int \varepsilon_i(t)\,dt + \tau_{D_i}\frac{d\varepsilon_i}{dt}\right] \tag{3.2.52}$$

where the output error signal is

$$\varepsilon_i(t) = y_{d_i}(t) - y_i(t) \tag{3.2.53}$$

Thus block diagrams such as Fig. 3.3 can represent linear systems in either the transform or state space domain.

The closed-loop operator equations for the block diagram shown in Fig. 3.3 can be found in the standard way by writing

$$\mathbf{y} = \mathbf{G}\mathbf{G}_c\boldsymbol{\varepsilon} + \mathbf{G}_d\mathbf{d} \tag{3.2.54}$$

$$\boldsymbol{\varepsilon} = \mathbf{y}_d - \mathbf{y} \tag{3.2.55}$$

where $\boldsymbol{\varepsilon}$ is the $l$ vector of deviations from the set-point vector $\mathbf{y}_d$ and $\mathbf{d}$ is the process disturbance. Substituting Eq. (3.2.55) into Eq. (3.2.54) yields the closed-loop expression

$$\mathbf{y} = (\mathbf{I} + \mathbf{GG}_c)^{-1}(\mathbf{GG}_c\mathbf{y}_d + \mathbf{G}_d\mathbf{d}) \tag{3.2.56}$$

where for the transform domain, the inverse operation denotes matrix inversion. This expression may be simplified to

$$\mathbf{y} = \mathbf{Ty}_d + \mathbf{T}_d\mathbf{d} \tag{3.2.57}$$

where the closed-loop transfer functions $\mathbf{T}$ and $\mathbf{T}_d$ are defined by

$$\mathbf{T} = (\mathbf{I} + \mathbf{GG}_c)^{-1}\mathbf{GG}_c \tag{3.2.58}$$

$$\mathbf{T}_d = (\mathbf{I} + \mathbf{GG}_c)^{-1}\mathbf{G}_d \tag{3.2.59}$$

We shall discuss methods of designing these multivariable controllers in the latter part of this chapter. However, we must first introduce some important concepts.

Let us recall the time-domain representation of our system,

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu} + \boldsymbol{\Gamma}\mathbf{d} \qquad x(t_0) = x_0 \tag{3.2.1}$$

$$\mathbf{y} = \mathbf{Cx} \tag{3.2.2}$$

and consider some important properties of this system.

## Controllability

One notion which is very useful in analyzing control systems is *controllability* [2, 6, 7]. Loosely speaking, a system may be said to be *controllable* if there exists a control policy $\mathbf{u}(t)$ which will steer the system from *any* given initial state $\mathbf{x}_0$ to *any other* desired state $\mathbf{x}_d$ in finite time. A more precise definition may be given as follows:

*If every initial state $\mathbf{x}_0(t_0)$ can be taken to any other state $\mathbf{x}_d(t)$ in some finite time $t > t_0$, then the system is completely controllable.* It is also possible to have systems which are only partially controllable, i.e., in which there are some subsets of initial states $\mathbf{x}_0(t_0)$ which cannot reach every other state in finite time.

It is possible to define conditions of controllability for specific classes of systems [2]. For example, if the matrices $\mathbf{A}$ and $\mathbf{B}$ are constant, then it can be shown [2, 7] that the system of Eqs. (3.2.1) and (3.2.2) is *completely controllable* if and only if the rank of an $n \times nm$ "controllability matrix" $\mathbf{L}_c$ is $n$, where

$$\mathbf{L}_c \equiv \left[\mathbf{B}\,\vdots\,\mathbf{AB}\,\vdots\,\mathbf{A}^2\mathbf{B}\,\vdots\,\cdots\,\vdots\,\mathbf{A}^{n-1}\mathbf{B}\right] \tag{3.2.60}$$

An informal derivation of this result may be seen by considering the analytical solution to Eqs. (3.2.1) and (3.2.2).*

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{Bu}(s)\,ds \tag{3.2.61}$$

* We neglect the disturbances $\mathbf{d}(t)$ because *controllability* is a property of the system itself.

where the exponential matrix may be written [1]

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \tfrac{1}{2}\mathbf{A}^2t^2 + \cdots \tag{3.2.62}$$

which when combined with the Hamilton-Cayley theorem* [1] leads to the finite series representation

$$e^{\mathbf{A}t} = c_0\mathbf{I} + c_1\mathbf{A}t + c_2(\mathbf{A}t)^2 + \cdots + c_{n-1}(\mathbf{A}t)^{n-1} \tag{3.2.63}$$

Upon substitution of this into Eq. (3.2.61), one obtains

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \big[\, c_0\mathbf{B} + c_1(t-s)\mathbf{AB} + \cdots + c_{n-1}(t-s)^{n-1}\mathbf{A}^{n-1}\mathbf{B}\,\big]\mathbf{u}(s)\,ds \tag{3.2.64}$$

or

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \big[\, \mathbf{B} \,|\, \mathbf{AB} \,|\, \cdots \,|\, \mathbf{A}^{n-1}\mathbf{B}\,\big] \begin{bmatrix} c_0\mathbf{u}(s) \\ \cdots\cdots\cdots \\ c_1(t-s)\mathbf{u}(s) \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ c_{n-1}(t-s)^{n-1}\mathbf{u}(s) \end{bmatrix} ds \tag{3.2.65}$$

Now the concept of controllability means that the control $\mathbf{u}$ is capable of influencing all the states $\mathbf{x}$ through the integral in Eq. (3.2.65). Therefore the system is controllable if and only if the integrand in Eq. (3.2.65) allows the influence of $\mathbf{u}(t)$ to reach all the states $\mathbf{x}(t)$. This requires the $n \times nm$ matrix transformation

$$\big[\, \mathbf{B} \,|\, \mathbf{AB} \,|\, \cdots \,|\, \mathbf{A}^{n-1}\mathbf{B}\,\big]$$

to have rank $n$.

Note that *output controllability* conditions can be found by multiplying Eq. (3.2.64) by $\mathbf{C}$ to yield

$$\mathbf{y}(t) = \mathbf{Cx} = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \big[\, c_0\mathbf{CB} + c_1(t-s)\mathbf{CAB} + \cdots + c_{n-1}(t-s)^{n-1}\mathbf{CA}^{n-1}\mathbf{B}\,\big]\mathbf{u}(s)\,ds \tag{3.2.66}$$

Thus, by the same arguments, the controls $\mathbf{u}(t)$ must influence all the $l$ outputs $\mathbf{y}(t)$ for *output controllability*. This means that the outputs $\mathbf{y}$ are *completely controllable* if and only if the rank of the $l \times nm$ controllability matrix $\mathbf{L}_c^\circ$ is $l$, where

$$\mathbf{L}_c^\circ = \big[\, \mathbf{CB} \,|\, \mathbf{CAB} \,|\, \cdots \,|\, \mathbf{CA}^{n-1}\mathbf{B}\,\big] \tag{3.2.67}$$

* This theorem states that every matrix satisfies its own characteristic equation; thus every infinite series can be reduced to an $n$-term series.

Controllability conditions for the case of a *linear nonautonomous system* of the form of Eq. (3.2.1), when $\mathbf{A}(t)$, $\mathbf{B}(t)$ are known functions of time, specify that the nonsingularity of the $n \times n$ matrix

$$\mathbf{M}(t_0, t_f) = \int_{t_0}^{t_f} \mathbf{\Phi}(t, t_0)^{-1} \mathbf{B}(t) \mathbf{B}^T(t) \left[ \mathbf{\Phi}(t, t_0)^T \right]^{-1} dt \tag{3.2.68}$$

is necessary and sufficient for controllability. Here the $n \times n$ matrix $\mathbf{\Phi}(t, t_0)$ is the *fundamental matrix solution* defined by

$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0) \qquad \mathbf{\Phi}(t_0, t_0) = \mathbf{I} \tag{3.2.12}$$

The proof is straightforward [2], but shall not be given here. Note that since the integrand of Eq. (3.2.68) is positive semidefinite, it is sufficient for controllability that the integrand be nonsingular for any instant of time $t_0 < t < t_f$.

Let us illustrate these points with a few examples.

**Example 3.2.5** Let us consider an isothermal continuous-stirred tank reactor CSTR with the irreversible first-order reactions $A \xrightarrow{k_1} B \xrightarrow{k_3} C$ taking place.

The rates of reactions are given by

$$r_1 = k_1 c_A$$
$$r_2 = k_3 c_B$$

where $k_1$ and $k_3$ are constants.

The modeling equations for this system take the form

$$V \frac{dc_A}{dt'} = F(c_{Af} - c_A) - V(k_1 c_A) \qquad c_A(0) = c_{A0}$$

$$V \frac{dc_B}{dt'} = F(c_{Bf} - c_B) + V(k_1 c_A - k_3 c_B) \qquad c_B(0) = c_{B0}$$

It is required to control $c_A$, $c_B$ as close as possible to a desired set point $c_{Ad}$, $c_{Bd}$ by adjusting $c_{Af}$ and $c_{Bf}$, the feed concentrations of $A$ and $B$. Let us now define

$$\mathrm{Da}_1 = \frac{k_1 V}{F} \qquad \mathrm{Da}_3 = \frac{k_3 V}{F} \qquad t = \frac{t' F}{V}$$

$$x_1 = \frac{c_A}{c_{A\,\mathrm{ref}}} \qquad x_2 = \frac{c_B}{c_{A\,\mathrm{ref}}} \qquad u_1 = \frac{c_{Af}}{c_{A\,\mathrm{ref}}} \qquad u_2 = \frac{c_{Bf}}{c_{A\,\mathrm{ref}}}$$

where $c_{A\,\mathrm{ref}}$ is an arbitrary reference concentration of $A$. In this instance, the modeling equations take the form

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 + u_1 \qquad x_1(t_0) = x_{10}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + u_2 \qquad x_2(t_0) = x_{20}$$

Thus one wishes to control the reactor outlet concentration $x_1$, $x_2$ by

adjusting the feed concentrations $u_1$, $u_2$. By putting these equations into the general form of Eq. (3.2.1), the controllability criterion may be tested by noting that

$$\mathbf{A} = \begin{bmatrix} -(1 + \mathrm{Da}_1) & 0 \\ \mathrm{Da}_1 & -(1 + \mathrm{Da}_3) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the "controllability matrix" is

$$\mathbf{L}_c = \left[\begin{array}{cc:cc} 1 & 0 & -(1 + \mathrm{Da}_1) & 0 \\ 0 & 1 & \mathrm{Da}_1 & -(1 + \mathrm{Da}_3) \end{array}\right]$$

Clearly the rank of $\mathbf{L}_c$ is two for this second-order system, so that the system is completely controllable.

**Example 3.2.6** Let us now consider the question of *controllability* of the reactor in Example 3.2.5 with the modification that $u_2 = c_{Bf}/c_{A\,\mathrm{ref}} \equiv 0$ and only $u_1$, the feed concentration of $A$, may be manipulated to control the reactor. In this case $\mathbf{A}$ is given as before, but $\mathbf{B}$ and $\mathbf{L}_c$ take the form

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{L}_c = \left[\begin{array}{c:c} 1 & -(1 + \mathrm{Da}_1) \\ 0 & \mathrm{Da}_1 \end{array}\right]$$

Again the rank of $\mathbf{L}_c$ is two, so that the reactor is completely controllable with only control variable $u_1$. However, the controller performance would surely be poorer than for the case when both $u_1$ and $u_2$ are available.

### Stabilizability

A much weaker condition than *controllability* for a system is the property of *stabilizability* [2]. *Stabilizability is the property that all the unstable modes of the system Eq.* (3.2.1), *can be made stable by controller action.* This means that any positive eigenvalues of $\mathbf{A}$ may be made negative by controller action. Clearly then, any system with $\mathbf{A}$ having all negative eigenvalues is stabilizable (in fact, even without controller action). In addition, any system which is controllable is automatically stabilizable.

In the case of the constant-gain proportional feedback controller on the state variables

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) \tag{3.2.69}$$

where $\mathbf{K}$ is an $m \times n$ feedback matrix of controller gains, the system of Eq. (3.2.1) becomes

$$\frac{d\mathbf{x}}{dt} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{3.2.70}$$

The system would then be *stabilizable* with such a feedback controller if and only if there existed a combination of feedback gains $k_{ij}$ which would cause the real parts of all the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K})$ to be negative.

With a proportional controller on the output variables

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{y}(t) \tag{3.2.71}$$

the system of Eq. (3.2.1) takes the form

$$\frac{d\mathbf{x}}{dt} = (\mathbf{A} - \mathbf{BKC})\mathbf{x} \tag{3.2.72}$$

In this case, stabilizability requires that the real parts of all the eigenvalues of $(\mathbf{A} - \mathbf{BKC})$ be negative for some selection of the feedback gains $k_{ij}$.

We shall illustrate these points with an example problem.

**Example 3.2.7** Let us consider the reactor problem of Example 3.2.5 except now $c_{Af}$ is fixed and only $c_{Bf}$ may be adjusted to control $c_A$ and $c_B$. In this case let us define

$$x_1 = \frac{c_A}{c_{Af}} - \frac{1}{1 + \mathrm{Da}_1} \qquad x_2 = \frac{c_B}{c_{Af}} \qquad t = \frac{t'F}{V} \qquad \mathrm{Da}_1 = \frac{k_1 V}{F}$$

$$\mathrm{Da}_3 = \frac{k_3 V}{F} \qquad u_1 = \frac{c_{Bf}}{c_{Af}} + \frac{\mathrm{Da}_1}{1 + \mathrm{Da}_1}$$

so that the modeling equations take the form

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 \qquad x_1(t_0) = x_{10} \tag{3.2.73}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + u_1 \qquad x_2(t_0) = x_{20} \tag{3.2.74}$$

In this instance, the **A** matrix is given as before and

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{L}_c = \left[\begin{array}{c|c} 0 & 0 \\ 1 & -(1 + \mathrm{Da}_3) \end{array}\right]$$

Here the controllability matrix $\mathbf{L}_c$ is singular and the system is *not* completely controllable. Only the state $x_2$ can be controlled by $u_1$, and $x_1$ is completely free. However, from the solution of Eq. (3.2.73), one sees that

$$x_1(t) = x_{10} \exp[-(1 + \mathrm{Da}_1)(t - t_0)]$$

so that the eigenvalue associated with $x_1$ is always negative and $x_1$ is stable. Thus the system of Eqs. (3.2.73) and (3.2.74) is stabilizable. In practical terms, this means that while both concentrations $A$ and $B$ *cannot* be controlled by adjusting the feed rate of $B$ (only $B$ can be controlled in this manner), the concentration of $A$ is stable and will approach the steady state unaffected by whatever control action is taken. Thus one could successfully design a control system for Eqs. (3.2.73) and (3.2.74) even though the system is not controllable.

Let us review the implications of *controllability* and *stabilizability* in the light of these three examples. First, a control system can always be designed for a *completely controllable system* and is *sometimes* impossible to design for one

which is not *completely controllable*. If the system is *stabilizable* (but not completely controllable) and the uncontrollable eigenvalues of the system are sufficiently large and negative, then an acceptable control system design is possible. However, if the system is *not stabilizable*, then control is generally impossible.

For Examples 3.2.5 to 3.2.7, the use of control variables $c_{Af}$, $c_{Bf}$ together or $c_{Af}$ alone will allow the CSTR to be completely controlled, but $c_{Bf}$ alone cannot completely control the reactor. Physically this is due to the fact that there is kinetic coupling from $c_A$ to $c_B$ but not in reverse, i.e., the rate of formation of $B$ depends on $c_A$, and thus the control of $c_A$ by $c_{Af}$ is sufficient to control $c_B$ as well. In contrast, the rate of formation of $A$ is independent of $c_B$, and thus controlling $c_B$ exercises no control over $c_A$. Had the first reaction been reversible in Example 3.2.7, then the reverse coupling would exist and the system would have been controllable with $c_{Bf}$ alone.

### Normality

A stronger form of controllability is called *normality* [2]. A system is said to be *normal* if each element of the control vector $\mathbf{u}$ *alone* will achieve controllability. This will be true if and only if the *normality matrix* $\mathbf{L}_{c_i}$,

$$\mathbf{L}_{c_i} \equiv \left[\mathbf{b}_i \vdots \mathbf{A}\mathbf{b}_i \vdots \mathbf{A}^2\mathbf{b}_i \vdots \cdots \vdots \mathbf{A}^{n-1}\mathbf{b}_i\right] \tag{3.2.75}$$

has rank $n$ *for all* $i$, where $\mathbf{b}_i$, $i = 1, 2, \ldots, m$, are the columns of the matrix $\mathbf{B}$. Thus the CSTR control system discussed in Example 3.2.5 is *completely controllable* but not *normal* because $u_2$ alone will not cause the system to be controllable. For scalar control variables, normality and controllability are identical properties.

### The Interaction Problem

The fundamental problem in designing multivariable feedback controllers lies in the steady-state and dynamic interactions which occur between the various input and output variables. If the system had no coupling between variables and the number of control variables equaled the number of outputs to be controlled, then the linear system of Eq. (3.2.1) in the transform domain would have a diagonal open-loop transfer function

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} g_{11}(s) & & \mathbf{0} \\ & g_{22}(s) & \\ \mathbf{0} & & \ddots \quad g_{ll}(s) \end{bmatrix} \tag{3.2.76}$$

and for $\mathbf{G}_c$ in the diagonal form, Eq. (3.2.50), the closed-loop transfer function, Eq. (3.2.58), relating $\mathbf{y}$ to $\mathbf{y}_d$ would then be diagonal and each control loop could be tuned separately by classical methods.

Unfortunately, most multivariable systems have significant coupling between outputs and controls, and these pose great difficulties in control system design. To illustrate these problems, consider the following example.

**Example 3.2.8** You are working for XYZ Chemical Co. in their engineering department and the plant supervisor comes to you with a control problem. In the plant is a large binary distillation column, sketched schematically in Fig. 3.4. There are four product streams drawn from this tower, with concentrations of the heavier component in the top three sidestreams denoted by $y_1$, $y_2$, and $y_3$. The bottoms must satisfy an overall material balance. The compositions of these products at each drawoff point have specifications $y_{1d}$, $y_{2d}$, and $y_{3d}$ and are controlled by adjusting the drawoff rates, $u_1$, $u_2$, and $u_3$. Currently the tower is run by manual control because the present single-loop PID controllers do not work well. They produce considerable offset and occasionally cause the sidestream compositions to continuously oscillate. Under manual operation there are no oscillations, but there are still significant deviations from the set point which require hours of time and many trial-and-error manual adjustments to eliminate. As an indication of the problem, the plant supervisor asks you to consider Fig. 3.5, where the operator successfully adjusts $y_2$ to a new set point by increasing flow rate $u_2$ and decreasing flow rate $u_1$ by an equal amount. This causes the concentration of the less volatile component at sidestream 2, $y_2$,

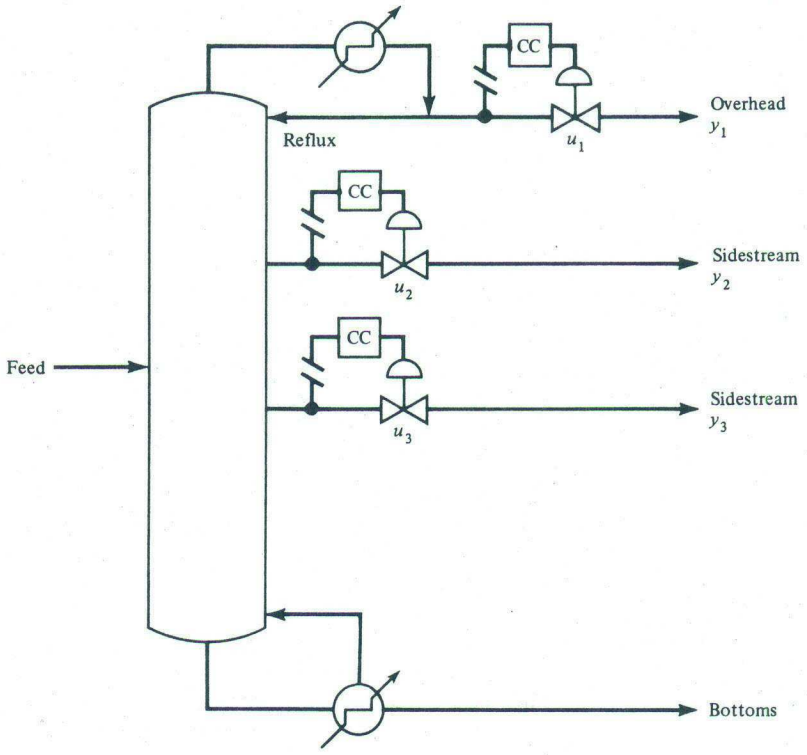


**Figure 3.4** Multi-sidestream distillation column.

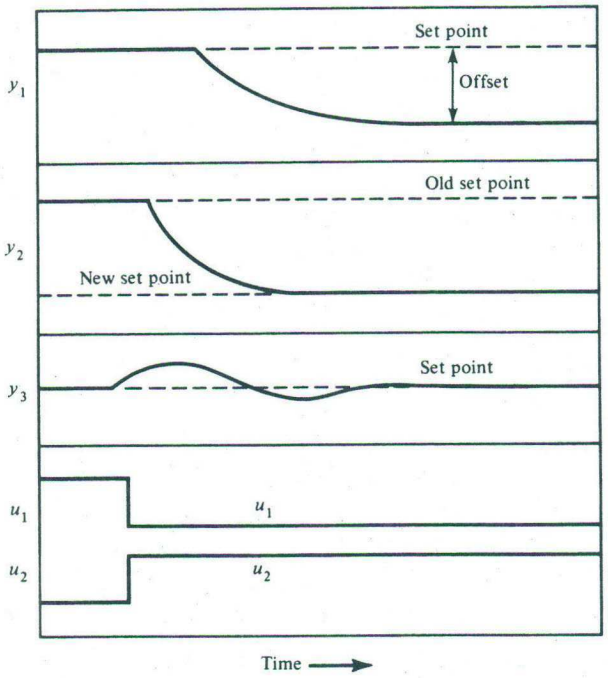


**Figure 3.5** Open-loop response of the distillation column to manual adjustments of $u_1$ and $u_2$.

to decrease to the new set-point value.* However as can be seen, the concentration drawn overhead does not remain constant but decreases also. The concentration $y_3$ at sidestream 2 first increases and finally decreases to approximately the old operating point. Thus, by successfully meeting the new specification on $y_2$, the operator has thrown $y_1$ off specification and must now work on correcting this by adjusting $u_1$. This begins a whole new round of detrimental interactions, and meanwhile the column is making many tons per hour of off-specification product.

Note that because of dynamic interactions which can cause temporary deviations in the wrong direction, after each adjustment, the operator must wait for the transients to die out to determine if this action is a success. The plant engineer asks for your help to develop a better controller for this tower.†

As a good control engineer, you first request that some modeling work be carried out on the column. Thus step input test data are fit to a rather simple transfer function relating the flow rates $\mathbf{u}$ to the sidestream composition $\bar{\mathbf{y}}(s)$, i.e.,

$$\bar{\mathbf{y}}(s) = \mathbf{G}\bar{\mathbf{u}}(s)$$

* The operator uses this approach because he has learned that this usually doesn't cause $y_3$ to be thrown too far off specification.

† Chapter 6 contains a further treatment of this example problem.

where $\mathbf{G}(s)$ has been found as

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{0.7}{1+9s} & 0 & 0 \\ \dfrac{2.0}{1+8s} & \dfrac{0.4}{1+6s} & 0 \\ \dfrac{2.3}{1+10s} & \dfrac{2.3}{1+8s} & \dfrac{2.1}{1+7s} \end{bmatrix} \tag{3.2.77}$$

Some interpretation of this model shows that all the responses between flow rates $u_j$ and sidestream compositions $y_i$ can be considered first-order. However, one should note that in actuality the system is of much higher order (order $\simeq$ number of trays between drawoffs). In practice, time delay terms $e^{-\alpha_{ij}s}$ would often appear, but for smaller columns, these delays may be negligible and have been neglected here. The model shows that there are significant interactions between variables, and in fact, the diagonal terms are not even dominant. Note that the interactions are one way, i.e., the adjustment of a drawoff rate affects all stream concentrations below it, but none above. This is the easiest type of process interaction to deal with.

Let us try to understand the difficulties encountered by the operators. The present column feedback control scheme, which was taken out of service, is of the form shown in Fig. 3.3, where

$$\mathbf{G}_c = \begin{bmatrix} g_{11_c}(s) & 0 & 0 \\ 0 & g_{22_c}(s) & 0 \\ 0 & 0 & g_{33_c}(s) \end{bmatrix}$$

represents three single-loop PI controllers. From Eq. (3.2.56), the closed-loop transfer function for the column is

$$\bar{\mathbf{y}} = \mathbf{T}\bar{\mathbf{y}}_d \tag{3.2.78}$$

where

$$\mathbf{T} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c)^{-1}\mathbf{G}\mathbf{G}_c = \mathbf{I} - (\mathbf{I} + \mathbf{G}\mathbf{G}_c)^{-1} \tag{3.2.58}$$

and

$$\mathbf{G}\mathbf{G}_c = \begin{bmatrix} \dfrac{0.7g_{11_c}}{1+9s} & 0 & 0 \\ \dfrac{2.0g_{11_c}}{1+8s} & \dfrac{0.4g_{22_c}}{1+6s} & 0 \\ \dfrac{2.3g_{11_c}}{1+10s} & \dfrac{2.3g_{22_c}}{1+8s} & \dfrac{2.1g_{33_c}}{1+7s} \end{bmatrix} \tag{3.2.79}$$

Thus

$$\mathbf{T} = \begin{bmatrix} \dfrac{0.7g_{11_c}}{1 + 9s + 0.7g_{11_c}} & 0 & 0 \\ \dfrac{2.0g_{11_c}(1 + 9s)(1 + 6s)}{(1 + 8s)(1 + 9s + 0.7g_{11_c})(1 + 6s + 0.4g_{22_c})} & \dfrac{0.4g_{22_c}}{1 + 6s + 0.4g_{22_c}} & 0 \\ \dfrac{\left[\dfrac{2.3g_{11_c}}{1 + 10s}\left(\dfrac{0.4g_{22_c}}{1 + 6s} + 1\right) - \dfrac{4.6g_{11_c}g_{22_c}}{(1 + 8s)^2}\right](1 + 9s)(1 + 6s)(1 + 7s)}{(1 + 9s + 0.7g_{11_c})(1 + 6s + 0.4g_{22_c})(1 + 7s + 2.1g_{33_c})} & \dfrac{2.3g_{22_c}(1 + 6s)(1 + 7s)}{(1 + 8s)(1 + 6s + 0.4g_{22_c})(1 + 7s + 2.1g_{33_c})} & \dfrac{2.1g_{33_c}}{1 + 7s + 2.1g_{33_c}} \end{bmatrix} \tag{3.280}$$

Observe that there are significant off-diagonal terms in **T** corresponding to strong dynamic and steady-state interactions in the system. This easily explains how operating difficulties with the present control scheme could arise if the controllers were not carefully tuned. Note that because the interactions are only one way, the controllers could, in principle, be tuned one at a time, starting at the top of the column. Nevertheless, the control system performance could be greatly improved by a better control system design which would minimize the effects of interaction.

Let us now consider another example problem where the transfer function is full and interactions occur in both directions.

**Example 3.2.9** Recall the simple mixing tank control problem shown in Fig. 3.2 and discussed in Example 3.2.1. If we assume that both the liquid level and tank temperature are measured, then the state variables and output variables are identical. Using Eq. (3.2.28) to convert to the Laplace domain,

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$\mathbf{G}_d(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\Gamma}$$

or

$$\mathbf{G}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + \frac{1}{2}\frac{F(h_s)}{A_c h_s} & 0 \\ 0 & s + \frac{F(h_s)}{A_c h_s} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{A_c} & \frac{1}{A_c} \\ \frac{T_H - T_s}{A_c h_s} & \frac{T_C - T_s}{A_c h_s} \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{bmatrix} \frac{1}{A_c\left(s + \frac{1}{2}\frac{F(h_s)}{A_c h_s}\right)} & \frac{1}{A_c\left(s + \frac{1}{2}\frac{F(h_s)}{A_c h_s}\right)} \\ \frac{T_H - T_s}{A_c h_s\left(s + \frac{F(h_s)}{A_c h_s}\right)} & \frac{T_C - T_s}{A_c h_s\left(s + \frac{F(h_s)}{A_c h_s}\right)} \end{bmatrix}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + \frac{1}{2}\frac{F(h_s)}{A_c h_s} & 0 \\ 0 & s + \frac{F(h_s)}{A_c h_s} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{A_c} & 0 \\ \frac{T_{ds} - T_s}{A_c h_s} & \frac{F_{ds}}{A_c h_s} \end{bmatrix}$$

$$\mathbf{G}_d(s) = \begin{bmatrix} \frac{1}{A_c\left(s + \frac{1}{2}\frac{F(h_s)}{A_c h_s}\right)} & 0 \\ \frac{T_{ds} - T_s}{A_c h_s\left(s + \frac{F(h_s)}{A_c h_s}\right)} & \frac{F_{ds}}{A_c h_s\left(s + \frac{F(h_s)}{A_c h_s}\right)} \end{bmatrix}$$

Thus changing either tank inlet flow rate, $u_1$ or $u_2$, influences both the level and the temperature; hence coupling is in both directions.

Efforts have been made to establish measures of interaction in multivariable processes. Perhaps the most widely used measure is the Bristol array [8], which measures the degree of steady-state interaction. To illustrate the application of the Bristol array, consider the 2 × 2 multivariable system shown in Fig. 3.6. The conventional method of tuning the control system would be to first open loop 2 and tune loop 1 so that $y_1$ has a good response, then open loop 1 and tune loop 2 until $y_2$ has a good response. If there were *no interaction* ($g_{12} = g_{21} = 0$), one could then close both loops and expect the control system to work well. However, in the presence of interactions, the overall control system performance could be quite poor when both loops are closed. Bristol [8] developed a general measure of this interaction in the steady state in a form of the so-called *Bristol*

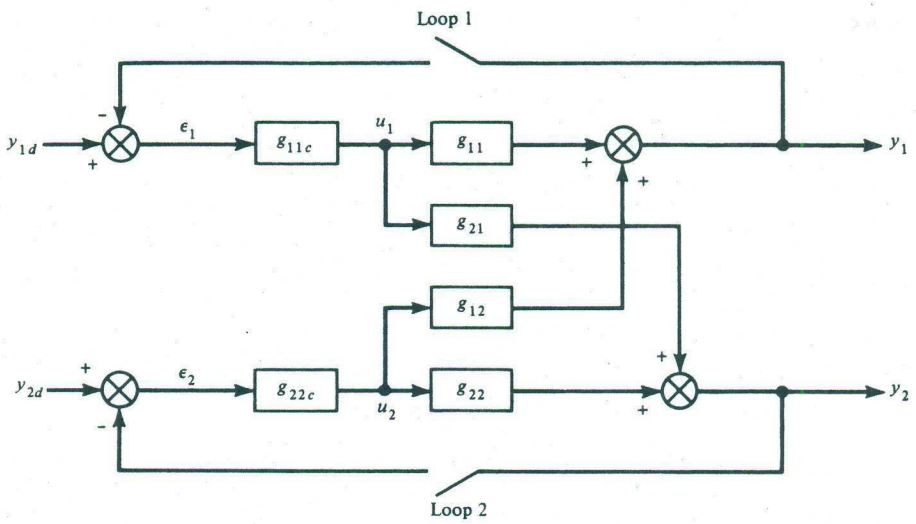


**Figure 3.6** $2 \times 2$ multivariable system undergoing sequential tuning of conventional controllers.

*array*:

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{m1} & \cdots & \cdots & \lambda_{mm} \end{bmatrix}$$

whose elements are defined as the steady-state ratio:

$$\lambda_{ij} \equiv \frac{\left\{ \dfrac{\partial y_i}{\partial u_j} \right\}_{\text{all loops open}}}{\left\{ \dfrac{\partial y_i}{\partial u_j} \right\}_{\substack{\text{all loops closed} \\ \text{except for } u_j}}} \qquad i, j = 1, 2, \ldots, m$$

The elements of the Bristol array are then the ratio of the steady-state open-loop response and the steady-state closed-loop response when a particular manipulated variable is adjusted. It is straightforward to determine the elements of the Bristol array, for the numerator is simply the $i, j$th element of the open-loop steady-state transfer function, i.e., $\lim_{s \to 0} g_{ij}(s)$. The denominator is calculated assuming that all the closed loops work perfectly, so that $y_k$ is constant for $k \neq j$. Thus

$$\left\{ \frac{\partial y_i}{\partial u_j} \right\}_{\substack{\text{all loops closed} \\ \text{except for } u_j}} \equiv \left\{ \frac{\partial y_i}{\partial u_j} \right\}_{y_k \text{ constant}, i \neq k} = \left\{ \frac{\partial u_j}{\partial y_k} \right\}^{-1}_{y_k \text{ constant}, i \neq k}$$

However, $\{\partial u_j / \partial y_i]$ is just the $j, i$th element of the inverse of the steady-state

process transfer function, i.e.,

$$\frac{\partial u_j}{\partial y_i} = \lim_{s \to 0} \left[ \mathbf{G}^{-1}(s) \right]_{ji}$$

Thus

$$\lambda_{ij} \equiv \left[ \mathbf{G}(0) \right]_{ij} \left[ \mathbf{G}(0)^{T-1} \right]_{ij}$$

where $[\ ]_{ij}$ denotes the $i,j$th element of the matrix in question.

To illustrate the application of the Bristol array, consider the $2 \times 2$ system shown in Fig. 3.6 with

$$\lim_{s \to 0} \mathbf{G}(s) = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Then

$$\left[ \mathbf{G}(0)^T \right]^{-1} = \frac{\begin{bmatrix} k_{22} & -k_{21} \\ -k_{12} & k_{11} \end{bmatrix}}{k_{11}k_{22} - k_{12}k_{22}}$$

and

$$\mathbf{\Lambda} = \frac{\begin{bmatrix} k_{11}k_{22} & -k_{21}k_{12} \\ -k_{21}k_{12} & k_{11}k_{22} \end{bmatrix}}{k_{11}k_{22} - k_{12}k_{21}}$$

Several things can be noted from this example which are general properties of the Bristol array:

1. The sum of any row or any column in the Bristol array is unity.
2. When the transfer function matrix is diagonal or is triangular, the Bristol array is the identity matrix.

From the definition of the interaction measure, it is clear that the best possible situation is to have all the off-diagonal terms near zero and the diagonal terms close to unity, i.e.,

$$\mathbf{\Lambda}_{\text{ideal}} = \begin{bmatrix} 1 & & & \mathbf{0} \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ \mathbf{0} & & & 1 \end{bmatrix}$$

This means that there is little interaction and that the closed-loop behavior is similar to the open-loop behavior. As the off-diagonal terms increase in absolute

magnitude and the diagonal terms depart from 1.0, more and more interaction is indicated.

*Positive interaction* arises when all elements of $\Lambda$ are positive: this means that there is some interaction, and one must choose the loop pairings $(u_i, y_i)$ to make the diagonal terms as close to unity as possible.

*Negative interaction* occurs when some of the elements of $\Lambda$ are negative. This means that changing $u_j$ in the closed-loop situation has just the opposite effect from changing $u_j$ in the open-loop case—a potentially dangerous situation.

The Bristol array can be used as a guide in choosing the pairing of control and output variables. Some general rules for this pairing are:

1. From examining $\mathbf{G}(0)$, pair $\mathbf{u}$, $\mathbf{y}$ such that diagonal elements are largest relative to off-diagonal elements.
2. From examining $\Lambda$, pair $\mathbf{u}$, $\mathbf{y}$ such that diagonal terms are dominant and close to unity in absolute value. If some diagonal terms are negative, then all must be negative for good performance.

**Example 3.2.10** Let us consider the pairing of control loops for the distillation column in Example 3.2.8, where

$$\mathbf{G}(0) = \begin{bmatrix} 0.7 & 0 & 0 \\ 2.0 & 0.4 & 0 \\ 2.3 & 2.3 & 2.1 \end{bmatrix} \qquad \mathbf{G}(0)^{-1} = \begin{bmatrix} 1.43 & 0 & 0 \\ -7.14 & 2.5 & 0 \\ 6.26 & -2.74 & 0.48 \end{bmatrix}$$

Thus

$$\Lambda = \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$$

and the Bristol array predicts no serious steady-state intereactions because the coupling is only in one direction. Tuning can be carried out one loop at a time, i.e., loop 1 ($u_1$ versus $y_1$) can be tuned, then loop 2 ($u_2$ versus $y_2$), and then loop 3 ($u_3$ versus $y_3$). This procedure should, in principle, allow reasonable control system performance when all loops are closed. However, as shown in Chap. 6, there are still interaction problems with these columns, and more sophisticated multivariable controller designs can be helpful.

**Example 3.2.11** Let us now consider the stirred tank of Example 3.2.1 where we pair the control and output variables as shown in Fig. 3.2; i.e., the level is controlled by adjusting the hot stream, and the tank temperature by manipulating the cold stream. Recall from Example 3.2.9 that

$$\bar{\mathbf{y}} = \mathbf{G}(s)\bar{\mathbf{u}}$$

where

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{1}{A_c\left[s + \dfrac{1}{2}\dfrac{F(h_s)}{A_c h_s}\right]} & \dfrac{1}{A_c\left[s + \dfrac{1}{2}\dfrac{F(h_s)}{A_c h_s}\right]} \\ \dfrac{T_H - T_s}{A_c h_s\left[s + \dfrac{F(h_s)}{A_c h_s}\right]} & \dfrac{T_C - T_s}{A_c h_s\left[s + \dfrac{F(h_s)}{A_c h_s}\right]} \end{bmatrix}$$

and

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $y_1$ represents the tank level and $y_2$ the tank temperature. Here $u_1$ is the flow rate of the hot stream and $u_2$ the flow rate of the cold stream. If we pair $u_1$ with $y_1$ and $u_2$ with $y_2$, as shown, it is interesting to calculate the Bristol array to check the interaction. Note that

$$\mathbf{G}(0) = \frac{1}{F(h_s)} \begin{bmatrix} 2h_s & 2h_s \\ T_H - T_s & T_C - T_s \end{bmatrix}$$

$$\mathbf{G}^T(0)^{-1} = \frac{F(h_s)}{2h_s(T_C - T_H)} \begin{bmatrix} T_C - T_s & -(T_H - T_s) \\ -2h_s & 2h_s \end{bmatrix}$$

so that the Bristol array becomes

$$\Lambda = \begin{bmatrix} \dfrac{T_C - T_s}{T_C - T_H} & \dfrac{-(T_H - T_s)}{T_C - T_h} \\ \dfrac{-(T_H - T_s)}{T_C - T_H} & \dfrac{T_C - T_s}{T_C - T_H} \end{bmatrix}$$

where we recall that $T_C$, $T_H$, and $T_s$ are the temperatures of the cold inlet stream, the hot inlet stream, and the steady-state exit temperature, respectively. Note that since $T_C \leq T_s \leq T_H$, all the elements of $\Lambda$ are positive, so that only *positive interactions* are possible. Furthermore, if the steady-state operating temperature $T_s$ is close to the hot-stream temperature $T_H$, the Bristol array predicts diagonal elements close to unity, and the present loop pairing will be good.

If, on the other hand, the steady-state temperature $T_s$ is close to the cold-stream temperature $T_C$, then the diagonal elements in the Bristol array are almost zero and the loop pairing should be switched for good performance. In the event that $T_s$ is midway between $T_C$ and $T_H$, then

$$\Lambda = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

and neither loop pairing will be very good. Very difficult interactions will be present. This example illustrates the fact that the amount of interaction can depend on the steady-state operating condition in some problems.

It should be emphasized that the Bristol array is only a measure of *steady-state interactions*. Although a great deal of work has been devoted to the study of *dynamic interactions*, there is, as yet, no generally accepted measure of dynamic interaction comparable to the Bristol array. Such a measure would be useful because it is sometimes possible to have significant dynamic interactions in the absence of important steady-state interactions.

## Noninteracting Control

The performance of a multivariable control system can often be significantly improved by some type of compensation which accounts for interactions. Such improvement is possible even for systems which appear to have only weak interactions according to the Bristol array. There are numerous techniques for designing multivariable feedback controllers with compensation for interactions. One of the classical approaches to the problem is to design a *noninteracting controller* [9, 10]. To illustrate noninteracting control, let us assume that it is desired to control the process outputs $\mathbf{y}(t)$ by adjusting the controls $\mathbf{u}(t)$. The feedback control structure, making use of single-loop controllers and a *noninteraction compensator* $\mathbf{G}_I$, may be seen in Fig. 3.7. These single-loop controllers give $\mathbf{G}_c$ the diagonal form

$$\mathbf{G}_c = \begin{bmatrix} g_{11_c} & & & \mathbf{0} \\ & g_{22_c} & & \\ & & \ddots & \\ \mathbf{0} & & & g_{ll_c} \end{bmatrix} \tag{3.2.50}$$

The closed-loop transfer function for the noninteracting control scheme shown in Fig. 3.7 is

$$\bar{\mathbf{y}} = (\mathbf{I} + \mathbf{G}\mathbf{G}_I\mathbf{G}_c)^{-1}(\mathbf{G}\mathbf{G}_I\mathbf{G}_c\bar{\mathbf{y}}_d + \mathbf{G}_d\bar{\mathbf{d}}) \tag{3.2.81}$$

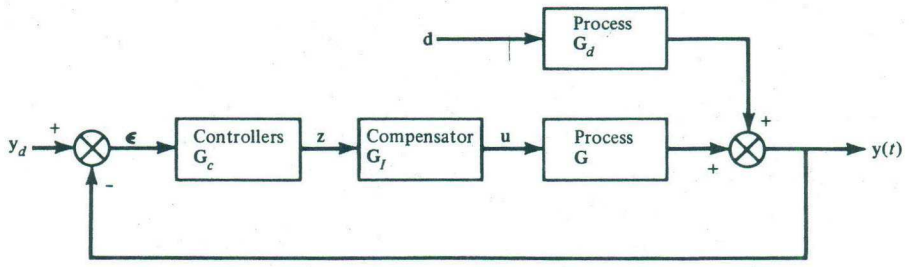


**Figure 3.7** Multivariable controller structure which includes interaction compensator.

or

$$\bar{\mathbf{y}} = \mathbf{T}\bar{\mathbf{y}}_d + \mathbf{T}_d\bar{\mathbf{d}} \tag{3.2.57}$$

Here $\mathbf{G}_I$ is a compensator which must be designed to eliminate as much interaction as possible. Ideally $\mathbf{G}_I(s)$ should be chosen to make

$$\mathbf{T} = (\mathbf{I} + \mathbf{G}\mathbf{G}_I\mathbf{G}_c)^{-1}\mathbf{G}\mathbf{G}_I\mathbf{G}_c \tag{3.2.82}$$

diagonal and drive $\mathbf{T} \to \mathbf{I}$ for $s = 0$ ($t \to \infty$) for some choice of the controller tuning parameters (such as controller gain). Obviously noninteracting control only makes sense when $\mathbf{G}$, $\mathbf{G}_I$, $\mathbf{G}_c$ are square matrices (i.e., when the number of controls and outputs are the same).*

Recalling that $\mathbf{G}_c$ is diagonal, a sufficient condition for $\mathbf{T}$ to be diagonal and have $\mathbf{T}(0) \to \mathbf{I}$ as the controller gains increase is to require

$$\mathbf{G}\mathbf{G}_I = \text{diag } \mathbf{G}(s)$$

or

$$\mathbf{G}_I = \mathbf{G}^{-1} \text{ diag } \mathbf{G}(s) \tag{3.2.83}$$

where diag $\mathbf{G}(s)$ is a diagonal matrix having the diagonal elements of $\mathbf{G}(s)$ along the main diagonal, i.e.,

$$\text{diag } \mathbf{G}(s) = \begin{bmatrix} g_{11}(s) & & & \mathbf{0} \\ & g_{22}(s) & & \\ & & \ddots & \\ \mathbf{0} & & & g_{ll}(s) \end{bmatrix}$$

If this compensation were done perfectly, the closed-loop transfer function would take the form

$$\bar{\mathbf{y}}(s) = \left(\mathbf{I} + \mathbf{G}\mathbf{G}^{-1} \text{ diag } \mathbf{G}\mathbf{G}_c\right)^{-1}\left(\mathbf{G}\mathbf{G}^{-1} \text{ diag } \mathbf{G}\mathbf{G}_c\bar{\mathbf{y}}_d + \mathbf{G}_d\bar{\mathbf{d}}\right)$$

or

$$\bar{y}_i(s) = \frac{g_{ii_c}(s)g_{ii}(s)}{1 + g_{ii_c}(s)g_{ii}(s)}y_{id}(s) + \frac{1}{1 + g_{ii_c}(s)g_{ii}(s)}\sum_{j=1}^{k} g_{ij_d}(s)\bar{d}_j(s) \qquad i = 1, 2, \ldots, l \tag{3.2.84}$$

Notice that there is total decoupling for set-point changes, and that even though each disturbance may influence all the outputs, the effect of the disturbances on any output $\bar{y}_i$ is damped by a single controller $g_{ii_c}$.

* If there are more controls than outputs, $m > l$, then a subset of controls may be chosen to accomplish decoupling, while if there are more outputs than controls, $l > m$, then only partial decoupling is possible.

Sometimes it is difficult or impossible to accomplish perfect dynamic compensation of the form of Eq. (3.2.83); however, one may always carry out *steady-state decoupling*, which eliminates the steady-state interactions. This requires $\lim_{s\to 0} \mathbf{T}(s)$ in Eq. (3.2.82) to be diagonal. The *steady-state compensator* is given by

$$\mathbf{G}_{I_{ss}} = \lim_{s\to 0} \mathbf{G}_I(s) = \lim_{s\to 0} \left[\mathbf{G}(s)^{-1} \operatorname{diag} \mathbf{G}(s)\right] = \mathbf{G}_{ss}^{-1} \operatorname{diag} \mathbf{G}_{ss} \quad (3.2.85)$$

In this case the closed-loop transfer functions takes the form

$$\bar{\mathbf{y}} = \left(\mathbf{I} + \mathbf{G}\mathbf{G}_{ss}^{-1} \operatorname{diag} \mathbf{G}_{ss}\mathbf{G}_c\right)^{-1}\left(\mathbf{G}\mathbf{G}_{ss}^{-1} \operatorname{diag} \mathbf{G}_{ss}\mathbf{G}_c\bar{\mathbf{y}}_d + \mathbf{G}_d\bar{\mathbf{d}}\right) \quad (3.2.86)$$

With such *steady-state decoupling*, the steady-state interactions are eliminated, so that by increasing the controller gains, steady-state offset may more easily be decreased. Even so, there is still a period of dynamic interaction which could cause single-loop controllers to fight each other if they were too tightly tuned. For the same reasons, significant integral action in the controllers is usually not desired, for this often leads to controllers fighting with each other in an attempt to eliminate offset.

In our discussion, we have presented noninteracting control via *output feedback*. It is also possible to use *state feedback control* to accomplish decoupling if all the state variables can be measured or inferred [11].

We shall illustrate noninteracting controller design with an example problem.

**Example 3.2.12** Let us consider the distillation column control problem discussed in Example 3.2.8 and design both static and dynamic noninteracting controllers for the column. For *steady-state decoupling*, $\mathbf{G}_{I_{ss}} = \mathbf{G}_{ss}^{-1} \operatorname{diag} \mathbf{G}_{ss}$, where we recall

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{0.7}{1+9s} & 0 & 0 \\ \dfrac{2.0}{1+8s} & \dfrac{0.4}{1+6s} & 0 \\ \dfrac{2.3}{1+10s} & \dfrac{2.3}{1+8s} & \dfrac{2.1}{1+7s} \end{bmatrix} \quad (3.2.87)$$

$$\mathbf{G}_{ss} = \begin{bmatrix} 0.7 & 0 & 0 \\ 2.0 & 0.4 & 0 \\ 2.3 & 2.3 & 2.1 \end{bmatrix} \qquad \operatorname{diag} \mathbf{G}_{ss} = \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 2.1 \end{bmatrix} \quad (3.2.88)$$

and thus

$$\mathbf{G}_{I_{ss}} = \begin{bmatrix} 1.43 & 0 & 0 \\ -7.14 & 2.50 & 0 \\ 6.26 & -2.74 & 0.48 \end{bmatrix} \begin{bmatrix} 0.7 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 2.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -7.14 & 1 & 0 \\ 6.26 & -2.74 & 1 \end{bmatrix}$$

If we neglect disturbances for the moment, the noninteracting control scheme for the distillation column is shown in Fig. 3.8, where for static

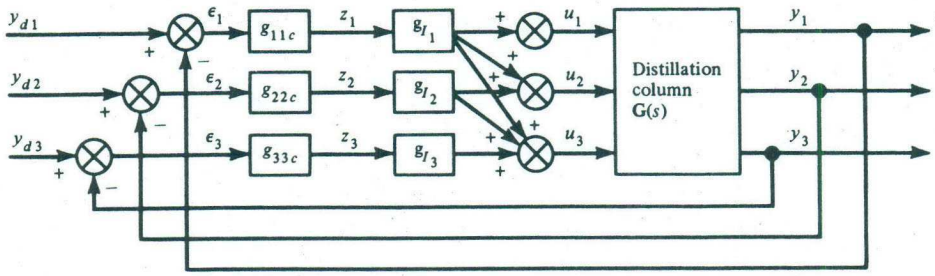


**Figure 3.8** Noninteracting control of a distillation column.

decoupling,

$$\mathbf{g}_{I_1} = \begin{bmatrix} 1.00 \\ -7.14 \\ 6.26 \end{bmatrix} \quad \mathbf{g}_{I_2} = \begin{bmatrix} 0 \\ 1.0 \\ -2.74 \end{bmatrix} \quad \mathbf{g}_{I_3} = \begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix} \tag{3.2.89}$$

Thus $u_1$ responds only to errors in $y_1$, while $u_2$ responds to errors in both $y_1$ and $y_2$, and $u_3$ responds to errors in all three output variables.

The *dynamic compensator design* for the column is of the form

$$\mathbf{G}_I = \mathbf{G}(s)^{-1} \operatorname{diag} \mathbf{G}(s) \tag{3.2.83}$$

where

$$\mathbf{G}^{-1}(s) = \begin{bmatrix} 1.43(1+9s) & 0 & 0 \\ -7.14\dfrac{(1+9s)(1+6s)}{1+8s} & 2.50(1+6s) & 0 \\ \left[\dfrac{7.82(1+9s)(1+6s)(1+7s)}{(1+8s)^2}\right. & \dfrac{-2.74(1+6s)(1+7s)}{(1+8s)} & 0.48(1+7s) \\ \left.-1.56\dfrac{(1+9s)(1+7s)}{(1+10s)}\right] & & \end{bmatrix} \tag{3.2.90}$$

and

$$\operatorname{diag} \mathbf{G}(s) = \begin{bmatrix} \dfrac{0.7}{1+9s} & 0 & 0 \\ 0 & \dfrac{0.4}{1+6s} & 0 \\ 0 & 0 & \dfrac{2.1}{1+7s} \end{bmatrix}$$

Hence

$$\mathbf{G}_I(s) = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{-7.14(1+9s)(1+6s)}{1+8s} & 1 & 0 \\ \dfrac{7.82(1+9s)(1+6s)(1+7s)}{(1+8s)^2} & \dfrac{-2.74(1+6s)(1+7s)}{1+8s} & 1 \\ -\dfrac{-1.56(1+9s)(1+7s)}{1+10s} & & \end{bmatrix}$$

and the compensator blocks in Fig. 3.8 take the form

$$\mathbf{g}_{I_1} = \begin{bmatrix} 1.0 \\ -7.14\dfrac{(1+9s)(1+6s)}{1+8s} \\ \dfrac{7.82(1+9s)(1+6s)(1+7s)}{(1+8s)^2} - 1.56\dfrac{(1+9s)(1+7s)}{1+10s} \end{bmatrix} \tag{3.2.91}$$

$$\mathbf{g}_{I_2} = \begin{bmatrix} 0 \\ 1.0 \\ \dfrac{-2.74(1+6s)(1+7s)}{(1+8s)} \end{bmatrix} \qquad \mathbf{g}_{I_3} = \begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix}$$

Note that for this example problem dynamic compensation requires differentiation of the signal coming from the single-loop controllers. The actual performance of steady-state and dynamic noninteracting control designs for this example problem will be described in Chap. 6.

Noninteracting controller design does not always work out as well as it did for the previous example. There can be conditions where implementation is difficult or impossible. For example, if the transfer function $\mathbf{G}(s)$ contains *time delays*, then the dynamic compensator will often contain *time leads*, requiring a knowledge of the outputs at some future time. Clearly this is impossible to implement exactly. However, in some cases, such a control scheme involving a predictor can yield reasonable results.

A problem encountered even more frequently results from the fact that perfect compensation requires a perfect transfer function model. Because the process model is often only approximate, the actual control scheme implemented will have the closed-loop transfer function

$$\bar{\mathbf{y}} = (\mathbf{I} + \mathbf{G}^*\mathbf{G}^{-1} \operatorname{diag} \mathbf{G}\mathbf{G}_c)^{-1}\mathbf{G}^*\mathbf{G}^{-1} \operatorname{diag} \mathbf{G}\mathbf{G}_c\bar{\mathbf{y}}_d \tag{3.2.92}$$

where $\mathbf{G}^*$ is the actual process and $\mathbf{G}^{-1}$ is the inverse of the process model. If the differences between the model and the process are too large, the control scheme behaves badly and can even become unstable.

These stability problems are most serious if there are right-half-plane zeros in the transfer function. These zeros become unstable poles in $\mathbf{G}_I$, and imperfect cancellation of these elements due to imperfections in the model can result in unstable poles in the transfer function.

Another potential disadvantage of noninteracting control is that a great deal of the control flexibility is used to achieve noninteraction, sometimes at the expense of overall dynamic response. In cases where some of the interactions greatly aid dynamic response, another type of controller design which takes advantage of these beneficial interactions (and only eliminates the most troublesome couplings) might be a better choice. We shall describe some alternative methods in what follows.

## Set-Point Compensation

As an alternative means of eliminating *steady-state interactions* due to set-point changes, one may use *set-point compensation*. This scheme, which can be applied either directly to analog controllers by the operator or through supervisory computer control, compensates for steady-state offset due to set-point adjustments. Recall that for the control scheme shown in Fig. 3.3,

$$\bar{\mathbf{y}} = \mathbf{T}\bar{\mathbf{y}}_d + \mathbf{T}_d\bar{\mathbf{d}} \tag{3.2.57}$$

where $\mathbf{G}_c$ is a diagonal controller matrix and

$$\mathbf{T} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c)^{-1}\mathbf{G}\mathbf{G}_c \tag{3.2.58}$$

is, in general, not diagonal. However, suppose Fig. 3.3 is altered to Fig. 3.9 by

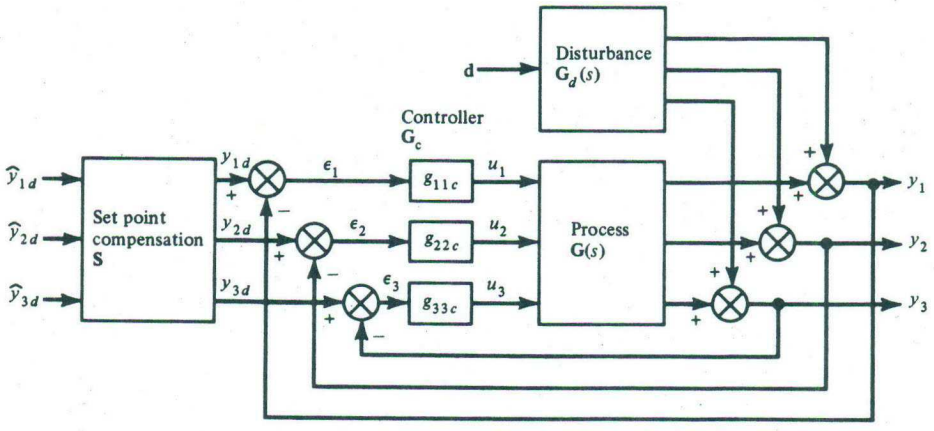


**Figure 3.9** Multivariable control with set-point compensation.

the addition of the set-point compensator

$$\mathbf{y}_d = \mathbf{S}\hat{\mathbf{y}}_d \tag{3.2.93}$$

where $\hat{\mathbf{y}}_d$ is the actual set point desired and $\mathbf{S}$ is the set-point compensation matrix. For example, for the three-input, three-output system in Fig. 3.9,

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Substitution of Eq. (3.2.93) into Eq. (3.2.57) yields

$$\bar{\mathbf{y}} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c)^{-1}\left(\mathbf{G}\mathbf{G}_c\mathbf{S}\hat{\mathbf{y}}_d + \mathbf{G}_d\bar{\mathbf{d}}\right) \tag{3.2.94}$$

and $\mathbf{S}$ must be chosen so that

$$\hat{\mathbf{T}} = (\mathbf{I} + \mathbf{G}\mathbf{G}_c)^{-1}\mathbf{G}\mathbf{G}_c\mathbf{S} \tag{3.2.95}$$

is diagonal and approaches the identity matrix at steady state. Thus by the ultimate value theorem of Laplace transforms,

$$\mathbf{S} = \lim_{s\to 0}\left[(\mathbf{I} + \mathbf{G}\mathbf{G}_c)^{-1}\mathbf{G}\mathbf{G}_c\right]^{-1} \tag{3.2.96}$$

The application of the set-point compensator shall be illustrated by the following example.

**Example 3.2.13** Let us consider the distillation column control problem of Example 3.2.12 and apply the set-point compensation scheme shown in Figure 3.9. If the controller matrix $\mathbf{G}_c$ consists of three proportional controllers

$$\mathbf{G}_c = \begin{bmatrix} k_{c_1} & 0 & 0 \\ 0 & k_{c_2} & 0 \\ 0 & 0 & k_{c_3} \end{bmatrix} \tag{3.2.97}$$

and $\mathbf{G}_{ss}$ is given by Eq. (3.2.88), then

$$\mathbf{S} = (\mathbf{G}_{ss}\mathbf{G}_c)^{-1}(\mathbf{I} + \mathbf{G}_{ss}\mathbf{G}_c) = (\mathbf{G}_{ss}\mathbf{G}_c)^{-1} + \mathbf{I} \tag{3.2.98}$$

Now

$$(\mathbf{G}_{ss}\mathbf{G}_c)^{-1} \equiv \mathbf{G}_c^{-1}\mathbf{G}_{ss}^{-1}$$

where

$$\mathbf{G}_c^{-1} = \begin{bmatrix} 1/k_{c_1} & 0 & 0 \\ 0 & 1/k_{c_2} & 0 \\ 0 & 0 & 1/k_{c_3} \end{bmatrix} \tag{3.2.99}$$

and

$$\mathbf{G}_{ss}^{-1} = \begin{bmatrix} 1.43 & 0 & 0 \\ -7.14 & 2.5 & 0 \\ 6.26 & -2.74 & 0.48 \end{bmatrix} \tag{3.2.100}$$

Thus the set-point compensator is

$$\mathbf{S} = \begin{bmatrix} \dfrac{1.43}{k_{c_1}} + 1 & 0 & 0 \\ \dfrac{-7.14}{k_{c_2}} & \dfrac{2.5}{k_{c_2}} + 1 & 0 \\ \dfrac{6.26}{k_{c_3}} & \dfrac{-2.74}{k_{c_3}} & \dfrac{0.48}{k_{c_3}} + 1 \end{bmatrix} \tag{3.2.101}$$

Note that for sufficiently high controller gains (i.e., $k_{c_i} \to \infty$), $\mathbf{S} \to \mathbf{I}$ and no steady-state compensation is necessary.

As an exercise, show that the set-point compensator $\mathbf{S}$ is the identity matrix $\mathbf{I}$ if PI control is used; i.e., if

$$\mathbf{G}_c = \begin{bmatrix} k_{c_1}\left(1 + \dfrac{1}{\tau_{I_1} s}\right) & & \mathbf{0} \\ & k_{c_2}\left(1 + \dfrac{1}{\tau_{I_2} s}\right) & \\ \mathbf{0} & & k_{c_3}\left(1 + \dfrac{1}{\tau_{I_3} s}\right) \end{bmatrix} \tag{3.2.102}$$

Explain why this must be true.

One obvious drawback of the set-point compensator is that it will not improve the response to disturbances because it does not appear in the feedback control loop. However, in making new set-point changes to compensate for sustained disturbances, a set-point compensator will minimize the effects of steady-state interactions. This is especially valuable if one is using supervisory control with local analog controllers in order to minimize the effects of interaction. In this case set-point compensation is rather easy to implement.

## Modal Feedback Control

Another approach to multivariable controller design is to use *modal feedback control*. This technique makes use of the linear nature of the system model to design a control scheme which allows one to specify the closed-loop eigenvalues of the system. To illustrate this technique, let us consider the system described in

state space by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \Gamma\mathbf{d} \tag{3.2.1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{3.2.2}$$

where for this discussion let us assume the number of controls, and the number of outputs, are the same as the number of states. Thus **A**, **B**, **C** are constant $n \times n$ matrices, and we shall assume **A** has real, distinct eigenvalues. These limitations are not crucial to the method (see [12]), but make the explanations to follow easier.

Let us further assume a proportional controller on the output

$$\mathbf{u}(t) = -\mathbf{G}_c\mathbf{y} = -\mathbf{G}_c\mathbf{C}\mathbf{x} \tag{3.2.103}$$

Now let us review the concepts of eigenvectors and eigenvalues. If $\Lambda$ is a diagonal matrix of eigenvalues of the $n \times n$ matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix} \tag{3.2.104}$$

then

$$\mathbf{R}\Lambda = \mathbf{A}\mathbf{R} \tag{3.2.105}$$

$$\Lambda\mathbf{L} = \mathbf{L}\mathbf{A} \tag{3.2.106}$$

where **R** and **L** are the normalized right and left eigenvectors for the matrix. This means that **R**, **L** are the matrices of the $n$ solutions of the equations

$$\mathbf{A}\mathbf{r}_i = \lambda_i\mathbf{r}_i \qquad i = 1, 2, \ldots, n \tag{3.2.107}$$

$$\mathbf{l}_i^T\mathbf{A} = \lambda_i\mathbf{l}_i \qquad i = 1, 2, \ldots, n \tag{3.2.108}$$

where the eigenvalues $\lambda_i$, $i = 1, \ldots, n$, are solutions of the equations

$$|\mathbf{A} - \lambda_i\mathbf{I}| = 0 \qquad i = 1, 2, \ldots, n \tag{3.2.109}$$

The $n$ vectors $\mathbf{l}_i$, $\mathbf{r}_i$ are each divided by a constant to make them orthonormal, i.e.,

$$\mathbf{l}_i^T\mathbf{r}_j = \delta_{ij} \qquad \mathbf{r}_i^T\mathbf{l}_j = \delta_{ij} \tag{3.2.110}$$

or

$$\mathbf{L}\mathbf{R} = \mathbf{R}\mathbf{L} = \mathbf{I} \tag{3.2.111}$$

where

$$\mathbf{R} = [\mathbf{r}_1 \vdots \mathbf{r}_2 \vdots \cdots \vdots \mathbf{r}_n]$$

$$\mathbf{L} = [\mathbf{l}_1 \vdots \mathbf{l}_2 \vdots \cdots \vdots \mathbf{l}_n]^T \tag{3.2.112}$$

Thus the properties in Eqs. (3.2.105) and (3.2.106) follow directly from Eqs. (3.2.107) and (3.2.111).

Making use of Eqs. (3.2.105) and (3.2.106), one can show that

$$\mathbf{\Lambda} = \mathbf{LAR} \tag{3.2.113}$$

and

$$\mathbf{R\Lambda L} = \mathbf{A} \tag{3.2.114}$$

Thus substituting Eqs. (3.2.103) and (3.2.114) into Eqs. (3.2.1), one obtains

$$\dot{\mathbf{x}} = (\mathbf{R\Lambda L} - \mathbf{BG}_c\mathbf{C})\mathbf{x} \tag{3.2.115}$$

Now if we choose the control matrix $\mathbf{G}_c$ to be

$$\mathbf{G}_c = \mathbf{B}^{-1}\mathbf{RK} \tag{3.2.116}$$

where $\mathbf{K}$ is a diagonal proportional controller matrix

$$\mathbf{K} = \begin{bmatrix} k_1 & & & \mathbf{0} \\ & k_2 & & \\ & & \ddots & \\ \mathbf{0} & & & k_n \end{bmatrix} \tag{3.2.117}$$

and choose the output matrix $\mathbf{C} = \mathbf{L}$, then Eq. (3.2.115) becomes

$$\dot{\mathbf{x}} = \mathbf{R}(\mathbf{\Lambda} - \mathbf{K})\mathbf{L}\mathbf{x} \tag{3.2.118}$$

By noting that

$$\mathbf{y} = \mathbf{L}\mathbf{x} \tag{3.2.119}$$

Eq. (3.2.118) takes the form

$$\dot{\mathbf{y}} = (\mathbf{\Lambda} - \mathbf{K})\mathbf{y} \tag{3.2.120}$$

and clearly $(\mathbf{\Lambda} - \mathbf{K})$ is a diagonal matrix, so the outputs have no interaction and

$$y_i = \alpha_i e^{(\lambda_i - k_i)t} \qquad i = 1, 2, \ldots, l \tag{3.2.121}$$

In addition the closed-loop solution to the state equations takes the form

$$\mathbf{x} = \sum_{i+1}^{n} \alpha_i \mathbf{r}_i e^{(\lambda_i - k_i)t} \tag{3.2.122}$$

where $\alpha_i$ is a constant determined from the initial conditions. Note that by adjusting $k_i$ we can make the closed-loop eigenvalues as large and negative as we wish, and there is no interaction between the $y_i$. Adjusting the $i$th controller constant $k_i$ affects only the $i$th mode. This means we can control the modes of the process, and even though there is state interaction, the output $\mathbf{y}$ has no interaction. The disadvantages of this control scheme are that only proportional control is possible, freedom is needed to choose $\mathbf{C} = \mathbf{L}$, and experience seems to indicate that tuning is a problem [10]. The block diagram for the control scheme is shown in Fig. 3.10.

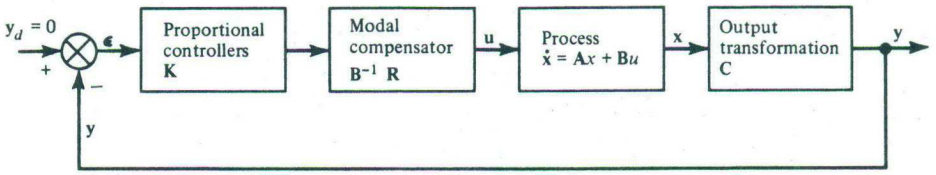


**Figure 3.10** Block diagram for modal feedback control.

In the case where there are more states than controls, it is possible to apply modal control to the first $m$ eigenvectors (where $m$ is the number of control variables) with zero output interaction [12].

**Example 3.2.14** Let us illustrate the concepts of modal control through consideration of the simple CSTR of Example 3.2.5, in which the reaction $A \rightarrow B \rightarrow C$ is taking place. It is desired to control the outlet concentration of $A$ and $B$ through manipulation of feed concentrations $c_{Af}$, $c_{Bf}$. In dimensionless form (see Example 3.2.5) the modeling equations become

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 + u_1 \tag{3.2.123}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + u_2 \tag{3.2.124}$$

where we recall that $x_1$ and $x_2$ are the dimensionless forms of $c_A$, $c_B$ and $u_1$, $u_2$ are dimensionless feed concentrations. The $\mathrm{Da}_i$ are the Damköhler numbers for each of the reaction steps. Let us suppose that both $x_1$ and $x_2$ are available as outputs, so any output of the form

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{3.2.2}$$

is possible.

Now if we apply simple single-loop proportional feedback control on the states (where we assume the problem has been scaled to let $\mathbf{x}_d = \mathbf{0}$ be the set point)

$$\begin{aligned} u_1 &= -k_{11}x_1 \\ u_2 &= -k_{22}x_2 \end{aligned} \tag{3.2.125}$$

then the system equations are

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1 + k_{11})x_1 \tag{3.2.126}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3 + k_{22})x_2 \tag{3.2.127}$$

Even though $k_{11}$ may be used to control $x_1$, there is a strong influence of $x_1$ on the state $x_2$.

Let us now apply modal control to the problem. We begin by determining the eigenvalues and eigenvectors of the state matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} -(1 + \mathrm{Da}_1) & 0 \\ \mathrm{Da}_1 & -(1 + \mathrm{Da}_3) \end{bmatrix} \tag{3.2.128}$$

Thus

$$|\mathbf{A} - \lambda\mathbf{I}| = \left| \begin{bmatrix} -(1 + \mathrm{Da}_1 + \lambda) & 0 \\ \mathrm{Da}_1 & -(1 + \mathrm{Da}_3 + \lambda) \end{bmatrix} \right| = 0 \tag{3.2.129}$$

yields

$$\lambda^2 + (2 + \mathrm{Da}_1 + \mathrm{Da}_3)\lambda + (1 + \mathrm{Da}_1)(1 + \mathrm{Da}_3) = 0 \tag{3.2.130}$$

or

$$\lambda_1, \lambda_2 = -(1 + \mathrm{Da}_1), -(1 + \mathrm{Da}_3) \tag{3.2.131}$$

Now from standard references on matrix algebra (e.g., [1]), one obtains the right and left eigenvectors as the nontrivial solutions to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{3.2.132}$$

which from standard solutions to homogeneous algebraic equations becomes the nontrivial columns of the adjoint of $(\mathbf{A} - \lambda\mathbf{I})$; i.e.,

$$\mathrm{adj}(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} (1 + \mathrm{Da}_3 + \lambda) & 0 \\ -\mathrm{Da}_1 & -(1 + \mathrm{Da}_1 + \lambda) \end{bmatrix} \tag{3.2.133}$$

so that for

$$\lambda_1 = -(1 + \mathrm{Da}_1) \qquad \mathbf{r}_1 = \begin{bmatrix} \mathrm{Da}_1 - \mathrm{Da}_3 \\ -\mathrm{Da}_1 \end{bmatrix} \tag{3.2.134}$$

$$\lambda_2 = -(1 + \mathrm{Da}_3) \qquad \mathbf{r}_2 = \begin{bmatrix} 0 \\ \mathrm{Da}_1 \end{bmatrix} \tag{3.2.135}$$

Similarly, the left-hand eigenvectors (or eigenrows) are the nontrivial rows of adj $(\mathbf{A} - \lambda\mathbf{I})$; i.e., for

$$\lambda_1 = -(1 + \mathrm{Da}_1) \qquad \mathbf{l}_1 = \begin{bmatrix} \mathrm{Da}_1 \\ 0 \end{bmatrix} \tag{3.2.136}$$

$$\lambda_2 = -(1 + \mathrm{Da}_3) \qquad \mathbf{l}_2 = \begin{bmatrix} \mathrm{Da}_1 \\ \mathrm{Da}_1 - \mathrm{Da}_3 \end{bmatrix} \tag{3.2.137}$$

Now because each eigenvector is uniquely determined only up to a multiplicative constant, one can divide $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathrm{Da}_1(\mathrm{Da}_1 - \mathrm{Da}_3)$ to make the matrix of right- and left-hand eigenvectors orthonormal [i.e., satisfy

Eq. (3.2.111)]. Thus

$$\mathbf{R} = \begin{bmatrix} \dfrac{1}{\mathrm{Da}_1} & 0 \\ \dfrac{-1}{\mathrm{Da}_1 - \mathrm{Da}_3} & \dfrac{1}{\mathrm{Da}_1 - \mathrm{Da}_3} \end{bmatrix} \tag{3.2.138}$$

$$\mathbf{L} = \begin{bmatrix} \mathrm{Da}_1 & 0 \\ \mathrm{Da}_1 & \mathrm{Da}_1 - \mathrm{Da}_3 \end{bmatrix} \tag{3.2.139}$$

and one may verify that $\mathbf{LR} = \mathbf{I}$. Now if we let $\mathbf{C} = \mathbf{L}$ and choose $\mathbf{G}_c$ from Eq. (3.2.102), then

$$\begin{aligned} y_1 &= \mathrm{Da}_1 x_1 \\ y_2 &= \mathrm{Da}_1 x_1 + (\mathrm{Da}_1 - \mathrm{Da}_3) x_2 \end{aligned} \tag{3.2.140}$$

and the feedback control law becomes

$$\mathbf{u} = -\mathbf{RKLx} \tag{3.2.141}$$

where

$$\mathbf{RKL} = \begin{bmatrix} k_{11} & 0 \\ \dfrac{\mathrm{Da}_1(k_{22} - k_{11})}{\mathrm{Da}_1 - \mathrm{Da}_3} & k_{22} \end{bmatrix} \tag{3.2.142}$$

or

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1 + k_{11})x_1 \tag{3.2.143}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1\left(1 + \frac{k_{11} - k_{22}}{\mathrm{Da}_1 - \mathrm{Da}_3}\right)x_1 - (1 + \mathrm{Da}_3 + k_{22})x_2 \tag{3.2.144}$$

where one may note that the outputs given by

$$\frac{dy_1}{dt} = -(1 + \mathrm{Da}_1 + k_{11})y_1 \tag{3.2.145}$$

$$\frac{dy_2}{dt} = -(1 + \mathrm{Da}_3 + k_{22})y_2 \tag{3.2.146}$$

show no interactions.

Let us now discuss the features of modal feedback control for this problem. First of all, the outputs [Eq. (3.2.140)] can be controlled independently without any interaction, and this is an advantage if a meaningful output set point could be devised. However, this is not a problem, because in general $\mathbf{y}_d = \mathbf{Lx}_d$ and $\mathbf{x}_d = \mathbf{Ry}_d$; thus one may change back and forth with no difficulties.

Another advantage of the controller matrix [Eq. (3.2.128)] is that if $k_{11}$ and $k_{22}$ are chosen so that

$$k_{22} - k_{11} = \mathrm{Da}_1 - \mathrm{Da}_3 \tag{3.2.147}$$

then Eq. (3.2.144) shows that the state interactions can be eliminated entirely. This is not generally a property of modal control, but is due to the particular structure of this example problem.

Because the matrix **A** must be known and all the states accessible, modal control design in the transform domain is a little artificial. However, one may obviously use it if one desires [10]. For a more complex example of modal control, see the paper by Davison and Chadha [13].

### Further Design Techniques

There are a whole host of multivariable controller design techniques available (see [10] and [14] for an overview). These methods, usually implemented in an interactive mode with a computer, allow the iterative design of the feedback gains until good multivariable controller response is obtained. In addition to optimal control (to be discussed in the next section) and the modal control and noninteractive control design procedures already discussed, other proposed methods include:

1. The *commutative-controller* technique [10]
2. The *inverse Nyquist array* technique [10, 14]
3. The *characteristic locus* technique [10]

The reader is urged to consult these references for the details of the design procedure. A very nice series of case studies showing the performance of some of these designs when applied to the control of a double-effect evaporator may be found in Ref. [15].

## 3.3 OPTIMAL CONTROL THEORY AND PRACTICE*

Another major class of lumped parameter controller design methods involves optimal control. We shall begin our discussion of optimal control theory with the consideration of optimal *open-loop* control policies. The general class of problems we wish to consider can be represented by the nonlinear modeling equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad 0 \le t \le t_f \tag{3.3.1}$$

* Part of the material in this section is adapted from Ref. [16]. Reprinted by permission of John Wiley & Sons, Inc.

where $\mathbf{x}(t)$ is an $n$-dimensional vector of the state variables and $\mathbf{u}(t)$ is an $m$-dimensional vector of control variables which we wish to choose optimally.

The initial and terminal conditions will depend on the physical nature of the problem. If we specify only the initial state, then

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{3.3.2}$$

An example of such an initial condition would be to specify the initial composition charged to a batch chemical reactor. Thus, if we were to fix the initial conditions only, the result would be a straightforward initial-value problem.

In other practical systems we may also desire to specify the final state

$$\mathbf{x}(t_f) = \mathbf{x}_f \tag{3.3.3}$$

(an example of which would be the requirement that the final product be of a given composition in the batch reactor); then we have a two-point boundary-value problem. Thus, we must find an optimal control $\mathbf{u}(t)$ which also causes $\mathbf{x}(t_f) = \mathbf{x}_f$.

Other possible conditions may require that some components of $\mathbf{x}$ be specified at the initial time and others at the final time. Alternatively, one may wish that some transversality conditions

$$\psi(\mathbf{x}(t_f)) = 0 \tag{3.3.4}$$

be satisfied at the final time. In physical terms, such a transversality condition may mean that rather than requiring a given final state of the process, we may wish to specify some relationship between the final states. This might correspond to the situation where there are tradeoffs possible in the final product specifications for a chemical reactor or other process.

In order to specify what is meant by optimal, we must select an objective functional* $I[\mathbf{u}(t)]$,

$$I[\mathbf{u}(t)] = G(\mathbf{x}(t_f)) + \int_0^{t_f} F(\mathbf{x}, \mathbf{u})\, dt \tag{3.3.5}$$

which we wish to maximize or minimize. We shall see that Eq. (3.3.5) is sufficiently general to allow the treatment of a wide class of practical problems.

Although the definitions of $G(\mathbf{x}(t_f))$ and $\int_0^{t_f} F(\mathbf{x}, \mathbf{u})\, dt$ have been given implicitly above, it may be helpful to illustrate the form that these functions can take for a given application.

If we were to consider the control of a batch chemical reactor, then the first component of the objective functional might take the following form:

$$G(\mathbf{x}(t_f)) = [\mathbf{x}(t_f) - \mathbf{x}_s]^T [\mathbf{x}(t_f) - \mathbf{x}_s] \tag{3.3.6}$$

where $\mathbf{x}_s$ is a vector describing the desired end composition; thus in this instance

* It is perhaps helpful to make clear that our objective is a functional (the transformation of a function into a value for $I$) rather than a function (the transformation of a parameter into a value for $I$).

the function $G(\mathbf{x}(t_f))$ is just the square deviation from the desired end composition. The second term in the objective $\int_0^{t_f} F(\mathbf{x}, \mathbf{u})\, dt$, might be used to describe the sum of the loss of reactant or product due to side reactions, the cost of control action (e.g., steam input), etc.; thus we may write

$$\int_0^{t_f} F(\mathbf{x}, \mathbf{u})\, dt = C_1 \int_0^{t_f} \begin{pmatrix} \text{rate of reactant} \\ \text{loss} \end{pmatrix} dt + C_2 \int_0^{t_f} \begin{pmatrix} \text{rate of product} \\ \text{loss} \end{pmatrix} dt + C_3 \int_0^{t_f} (\text{cost of control})\, dt \tag{3.3.7}$$

Here $C_1$, $C_2$, and $C_3$ are the appropriate cost factors. It is noted that the influence of the modeling Eq. (3.2.1) appears implicitly in the three integrals appearing on the right-hand side of Eq. (3.3.7).

For this example, the chemical compositions, together with the temperature, would constitute the state variables, whereas the heating rate, together with the rate of addition of catalyst or reactants, would constitute the control variables.

As an alternative form that

$$\int_0^{t_f} F(\mathbf{x}, \mathbf{u})\, dt$$

might take, let us leave $\mathbf{x}(t_f)$ in Eq. (3.3.3) unspecified and use the objective functional to force $\mathbf{x}$ to a given, desired final value. For example, the minimization of

$$\int_0^{t_f} (x - x_s)^2\, dt$$

will cause $x$ to approach $x_s$ in a very short time and will minimize the integral squared error.

In some optimal control problems, there also arise constraints of the form

$$\mathbf{g}(\mathbf{x}, \mathbf{u}) \leq \mathbf{0} \tag{3.3.8}$$

$$\mathbf{h}(\mathbf{x}, \mathbf{u}) = \mathbf{0} \tag{3.3.9}$$

and there are techniques for handling these.† However, because of the tremendous complexity that constraints of this form add, and because a great many practical problems only involve constraints of the form

$$\mathbf{u}_* \leq \mathbf{u} \leq \mathbf{u}^* \tag{3.3.10}$$

we shall only be concerned with upper and lower bounds on our control for the present.

In many practical problems, one may wish to choose $t_f$ (e.g., the batch time) optimally as well. As seen in the next section, this presents no theoretical difficulties.

† For example, see the text by Bryson and Ho [6].

## Necessary Conditions for Optimality

In this section we shall derive in a formal manner the necessary conditions for optimality for the lumped parameter system (3.3.1). This section is important not only for the useful final result, but also because the essence of variational methods is introduced in the derivation.

Let us consider the system with $n$ state variables $x_i(t)$, $m$ control variables $u_j(t)$, $j = 1, 2, \ldots, m$, and with dynamic behavior described by the ordinary differential equations*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.3.11}$$

We wish to find the control vector $\mathbf{u}(t)$, $0 \leq t \leq t_f$, such that the objective functional given by Eq. (3.3.5) is maximized.† Suppose that we have a set of nominal values for the control variables

$$\bar{\mathbf{u}}(t) = \begin{bmatrix} \bar{u}_1(t) \\ \bar{u}_2(t) \\ \vdots \\ \bar{u}_m(t) \end{bmatrix}$$

which we think may be optimal. Let us express any other control as a perturbation about $\bar{\mathbf{u}}(t)$,

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) + \delta\mathbf{u}(t) \tag{3.3.12}$$

and represent the state $\mathbf{x}(t)$ resulting from $\mathbf{u}(t)$ as a perturbation about the state $\bar{\mathbf{x}}(t)$ caused by the control $\bar{\mathbf{u}}(t)$; that is,

$$\mathbf{x}(t) = \bar{\mathbf{x}}(t) + \delta\mathbf{x}(t) \tag{3.3.13}$$

By checking the value of $I$ in Eq. (3.3.5) for all perturbations $\delta\mathbf{u}(t)$, we could determine whether $\bar{\mathbf{u}}(t)$ is optimal. However, there are variations $\delta\mathbf{x}(t)$ which are produced by the perturbations $\delta\mathbf{u}(t)$, so that one must consider whether Eq. (3.3.11) is satisfied. If the perturbations $\delta\mathbf{u}(t)$ are chosen small enough, that is, if

$$|\delta\mathbf{u}(t)| \leq \varepsilon_1$$

then a first-order expansion about $\bar{\mathbf{u}}(t)$ would be adequate to represent the system. Thus we linearize Eqs. (3.3.11) and (3.3.5) about the nominal controls $\bar{\mathbf{u}}$

* It is straightforward to show (e.g., [2]) that any set of higher-order ordinary differential equations can be reduced to a set of first-order equations such as Eqs. (3.3.11).

† It is perhaps helpful to note that if we wished to *minimize* the objective $I$ in Eq. (3.3.5), it is only necessary to maximize $-I$. This is due to the happy relationship

$$\min_{\mathbf{u}(t)} (I) \equiv -\max_{\mathbf{u}(t)} (-I)$$

to obtain Eqs. (3.3.14) and (3.3.15), respectively:

$$\frac{d(\delta \mathbf{x})}{dt} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\cdot} \delta \mathbf{x} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\cdot} \delta \mathbf{u} \qquad \delta \mathbf{x}(0) = \delta \mathbf{x}_0 \tag{3.3.14}$$

$$\begin{aligned} \delta I &= I[\bar{\mathbf{u}}(t) + \delta \mathbf{u}(t)] - I[\bar{\mathbf{u}}(t)] \\ &= \left(\frac{\partial G}{\partial \mathbf{x}}\right)_{\cdot} \delta \mathbf{x}(t_f) + \int_0^{t_f} \left[\left(\frac{\partial F}{\partial \mathbf{x}}\right)_{\cdot} \delta \mathbf{x} + \left(\frac{\partial F}{\partial \mathbf{u}}\right)_{\cdot} \delta \mathbf{u}\right] dt \\ &\quad + \left[F(t_f) + \frac{\partial G}{\partial \mathbf{x}} \mathbf{f}(t_f)\right]_{\cdot} \delta t_f \end{aligned} \tag{3.3.15}$$

where the notation $(\ )_{\cdot}$ reminds us that the partial derivatives are evaluated along the nominal trajectory $\bar{\mathbf{u}}(t)$, $\bar{\mathbf{x}}(t)$.

The last term in Eq. (3.3.15) arises because we may wish to choose $t_f$ optimally; thus variations $\delta t_f$ are allowed as well as variations $\delta \mathbf{u}(t)$.

Let us now adjoin to the objective functional the linearized constraint [Eqs. (3.3.14) and (3.3.15)] by using the $n$-dimensional adjoint variables (i.e., time-dependent Lagrange multipliers) $\boldsymbol{\lambda}(t)$. If we require that Eq. (3.3.14) be satisfied everywhere, then the subtraction of

$$\int_0^{t_f} \boldsymbol{\lambda}^T(t) \left[\frac{d(\delta \mathbf{x}(t))}{dt} - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\cdot} \delta \mathbf{x} - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\cdot} \delta \mathbf{u}\right] dt = 0 \tag{3.3.16}$$

from Eq. (3.3.15) yields

$$\begin{aligned} \delta I = &\left(\frac{\partial G}{\partial \mathbf{x}}\right)_{\cdot} \delta \mathbf{x}(t_f) + \left[F(t_f) + \frac{\partial G}{\partial \mathbf{x}} \mathbf{f}(t_f)\right]_{\cdot} \delta t_f \\ &+ \int_0^{t_f} \left[\left(\frac{\partial F}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_{\cdot} \delta \mathbf{x} \right. \\ &\left. + \left(\frac{\partial F}{\partial \mathbf{u}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)_{\cdot} \delta \mathbf{u}\right] dt - \int_0^{t_f} \boldsymbol{\lambda}^T(t) \frac{d(\delta \mathbf{x})}{dt}\, dt \end{aligned} \tag{3.3.17}$$

By integrating the last term by parts, we obtain

$$\begin{aligned} \delta I = &\left[F(t_f) + \frac{\partial G}{\partial \mathbf{x}} \mathbf{f}(t_f)\right]_{\cdot} \delta t_f + \boldsymbol{\lambda}^T(0) \delta \mathbf{x}_0 + \left[\left(\frac{\partial G}{\partial \mathbf{x}}\right)_{\cdot} - \boldsymbol{\lambda}^T(t_f)\right] \delta \mathbf{x}(t_f) \\ &+ \int_0^{t_f} \left\{\left[\left(\frac{\partial H}{\partial \mathbf{x}}\right)_{\cdot} + \frac{d\boldsymbol{\lambda}^T}{dt}\right] \delta \mathbf{x} + \left(\frac{\partial H}{\partial \mathbf{u}}\right)_{\cdot} \delta \mathbf{u}\right\} dt \end{aligned} \tag{3.3.18}$$

where $H$ (sometimes called the Hamiltonian) is defined by

$$H \equiv F(x, u) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{3.3.19}$$

Equation (3.3.18) represents the influence of variations $\delta \mathbf{u}(t)$ on $\delta I$, both directly and through $\delta \mathbf{x}(t)$. In order to express the direct influence of $\delta \mathbf{u}(t)$ alone, let us define the heretofore arbitrary functions $\boldsymbol{\lambda}(t)$ such that they satisfy

$$\frac{d\boldsymbol{\lambda}^T}{dt} = -\left(\frac{\partial H}{\partial \mathbf{x}}\right)_{\cdot} \tag{3.3.20}$$

In effect, this allows the influence of the system equations [Eq. (3.3.11)] to be transmitted by $\boldsymbol{\lambda}(t)$ and is felt in $(\partial H/\partial \mathbf{u})$, which carries $\boldsymbol{\lambda}(t)$.

The remaining terms outside the integral in Eq. (3.3.18) will depend on the boundary conditions of the physical system. Let us first consider the case where $t_f$ is specified (so that $\delta t_f = 0$), $x_0$ is fixed, and $\mathbf{x}(t_f)$ is unspecified. In this case the variation $\delta \mathbf{x}_0 = \mathbf{0}$, and $\delta \mathbf{x}(t_f)$ is completely arbitrary. However, the condition

$$\lambda_i(t_f) = \left(\frac{\partial G}{\partial x_i}\right)_{t_f} \tag{3.3.21}$$

will shift the influence of $\delta \mathbf{x}(t)$ to $\boldsymbol{\lambda}(t)$ and cause it to arise in $(\partial H/\partial \mathbf{u})$. Notice that Eq. (3.3.21) completes the definition of $\boldsymbol{\lambda}(t)$ when combined with Eq. (3.3.20). If only some of the components of $\mathbf{x}(t_f)$ are unspecified, then $\delta x_i(t_f) = 0$ for those specified, and Eq. (3.3.21) holds for those unspecified. Similarly, if some components of $\mathbf{x}_0$ were to be unspecified, then $\lambda_i(0) = 0$ would hold for those components.

If in addition we wish to choose $t_f$ optimally, then the first term in Eq. (3.3.18) remains. Now if all the $x_i(t_f)$ are unspecified, then Eq. (3.3.21) must hold, and in addition

$$H(t_f) = F(t_f) + \frac{\partial G}{\partial \mathbf{x}} \mathbf{f}(t_f) \tag{3.3.22}$$

must vanish when $t_f$ is chosen optimally.

In the case where some of the $x_i(t_f)$ are fixed at $x_{if}$, then by a Taylor series expansion

$$x_i(t_f + \delta t_f) = x_i(t_f) + \left(f_i(t_f)\right) \delta t_f = \bar{x}_i(t_f) = x_{if} \tag{3.3.23}$$

as shown in Fig. 3.11. Thus the difference between $x_i(t_f)$ and $\bar{x}_i(t_f)$ to a first-order approximation is given by

$$\delta x_i(t_f) = x_i(t_f) - \bar{x}_i(t_f) = -\left(f_i(t_f)\right) \delta t_f \tag{3.3.24}$$

and again we see that

$$H(t_f) = F(t_f) + \boldsymbol{\lambda}^T(t_f) \mathbf{f}(t_f) \tag{3.3.25}$$

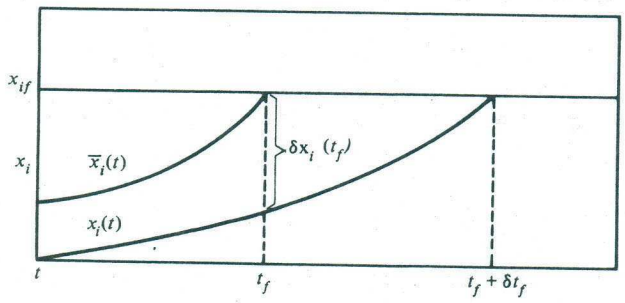


**Figure 3.11** The expansion of $x_i(t)$ about $t_f$.

must vanish for $t_f$ to be optimal. Notice that in this case the $\lambda_i(t_f)$ associated with fixed $x_i(t_f)$ are unspecified, while those associated with unspecified $x_i(t_f)$ are given by Eq. (3.3.21).

Having removed the boundary condition terms from Eq. (3.3.18), we obtain

$$\delta I = \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial \mathbf{u}} \right) \delta \mathbf{u}(t) \right] dt \tag{3.3.26}$$

and we now see the direct influence of variations $\delta \mathbf{u}(t)$ on $\delta I$. A necessary condition for optimality of $\bar{\mathbf{u}}(t)$ is that $\delta I \leq 0$ for all possible small variations $\delta \mathbf{u}(t)$. It is clear from Eq. (3.3.26) that the only way this can be true is that

$$\left( \frac{\partial H}{\partial \mathbf{u}} \right) = 0 \tag{3.3.27}$$

at every $t$.

Suppose that some components of $\bar{\mathbf{u}}(t)$ include segments along the constraints $u_i^*$, $u_i^*$. Obviously, variations $\delta u_i(t)$ can only be negative at the upper bound $u_i^*$ and only positive along lower bounds $u_i^*$. An examination of Eq. (3.3.26) shows that a necessary condition for optimality at upper and lower bounds is

$$\begin{aligned} &\text{for} && \bar{u}_i(t) = u_i^* && \left( \frac{\partial H}{\partial u_i} \right) \geq 0 \\ &\text{and for} && \bar{u}_i(t) = u_i^* && \left( \frac{\partial H}{\partial u_i} \right) \leq 0 \end{aligned} \tag{3.3.28}$$

Equations (3.3.28) can be reduced to the requirement that $H$ have a local maximum at the constraints.

The results derived here may now be summarized as follows:

**Theorem 3.1: Weak maximum principle** In order for a control $\bar{\mathbf{u}}(t)$, $\mathbf{u}_* \leq \bar{\mathbf{u}}(t) \leq \mathbf{u}^*$, to be optimal in the sense that it maximizes the objective $I$ in Eq. (3.3.5) while satisfying the system Eqs. (3.3.11), it is necessary that Eq. (3.3.27) be satisfied for unconstrained portions of the path and $H$ as defined by Eq. (3.3.19) be maximized along constrained portions of the control trajectory.

Thus, given

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.3.11}$$

and

$$I[\mathbf{u}(t)] = G(\mathbf{x}(t_f)) + \int_0^{t_f} F(\mathbf{x}, \mathbf{u})\, dt \tag{3.3.5}$$

the necessary condition for $\bar{\mathbf{u}}(t)$ to maximize

$$I[\mathbf{u}(t)]$$

is that

$$\frac{\partial H}{\partial \mathbf{u}} = 0 \tag{3.3.27}$$

on the unconstrained portion of the path and

$$H \equiv F(\mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{3.3.19}$$

be a maximum on the constrained portion of the path. Here $H$ is the Hamiltonian defined by Eq. (3.2.19), and $\boldsymbol{\lambda}$ is the time-dependent Lagrange multiplier, which is defined by

$$\frac{d\boldsymbol{\lambda}^T}{dt} = -\frac{\partial H}{\partial \mathbf{x}} \tag{3.3.20}$$

and

$$\lambda_i(t_f) = \frac{\partial G}{\partial x_i} \tag{3.3.21}$$

for those state variables unspecified at $t = t_f$.

In addition it is necessary that the Hamiltonian $H(t)$ remain constant along the optimal trajectory, and that $H(t)$ take the constant value of zero when the terminal time $t_f$ is unspecified [see Eq. (3.3.25)].

A much stronger version of these necessary conditions, whose derivation is available elsewhere [17, 2] is summarized in the following theorem.

**Theorem 3.2: Strong maximum principle** In order for a control $\mathbf{u}(t)$ (constrained to lie in some constraint set $\Omega$) to be optimal for the problem given by Eqs. (3.3.5) and (3.3.11), it is necessary that $H$ be maximized by $\mathbf{u}(t)$ almost everywhere.

This much stronger result can also be shown sufficient for optimality under certain convexity assumptions. For further details, see the work of Lee and Markus [2].

The results developed in this section are very similar to those arising from dynamic programming or the classical calculus of variations. While the relationship can be made quite explicit, we shall not pursue the discussion further here. The reader is referred to Dreyfus [18] and Leitman [19] for a treatment of these relationships.

**Example 3.3.1** Consider the radiant heating of a small billet or slab having a uniform temperature distribution so that the modeling equations are

$$\frac{dT}{dt'} = C_1(T_s^4 - T^4) \qquad T(0) = T_0 \tag{3.3.29}$$

where $T_s$ is the radiant source temperature bounded by $T_* \leq T_s \leq T^*$, and $T_0$ is the initial temperature. Let us determine the optimal source temperature $T_s(t')$ so as to bring the billet to temperature $T_1$ in minimum time while

minimizing the heat losses. This objective can be expressed as

$$\min_{T_s(t')} I[T_s(t')] = t_f' + C\int_0^{t_f'} T_s(t')^4\, dt' \tag{3.3.30}$$

where $t_f'$ is left free and $C$ denotes the relative value of heat losses to operating time.

SOLUTION Let us define the variables

$$x_0 = T_0 \qquad x = T \qquad u = T_s^4 \qquad t = C_1 t'$$
$$x_s = T_1 \qquad u_* = T_*^4 \qquad u^* = T^{*4}$$

so that our problem becomes

$$\min_{u(t)} \left\{ I[u] = \int_0^{t_f} [1 + Cu(t)]\, dt \right\} \tag{3.3.31}$$

subject to

$$\frac{dx}{dt} = u - x^4 \qquad x(0) = x_0 \qquad x(t_f) = x_s \tag{3.3.32}$$

and

$$u_* \le u \le u^* \tag{3.3.33}$$

We can now define the Hamiltonian

$$H = 1 + Cu(t) + \lambda(u - x^4) = 1 - \lambda x^4 + (\lambda + C)u \tag{3.3.34}$$

and adjoint variables $\lambda(t)$ by

$$\frac{d\lambda}{dt} = 4\lambda x^3 \tag{3.3.35}$$

From the fact that $H$ is linear in $u$, it is clear that

$$u(t) = \begin{cases} u^* & \text{if} \quad \lambda + C < 0 \\ u_* \le u \le u^* & \text{if} \quad \lambda + C = 0 \\ u_* & \text{if} \quad \lambda + C > 0 \end{cases} \tag{3.3.36}$$

because this is the only policy which minimizes $H$ and thus satisfies the maximum principle. We note at this juncture that had our objective been to maximize rather than minimize $H$, then Eq. (3.3.36) would have taken the following form:

$$u(t) = \begin{cases} u_* & \text{if} \quad \lambda + C < 0 \\ u_* \le u \le u^* & \text{if} \quad \lambda + C = 0 \\ u^* & \text{if} \quad \lambda + C > 0 \end{cases}$$

Let us now deduce the exact optimal policy. If $u_* < u < u^*$ somewhere on the optimal policy, say, the region $0 \le t \le t_1$ in Fig. 3.12*a* (which indicates a slow increase in source temperature with time until the final

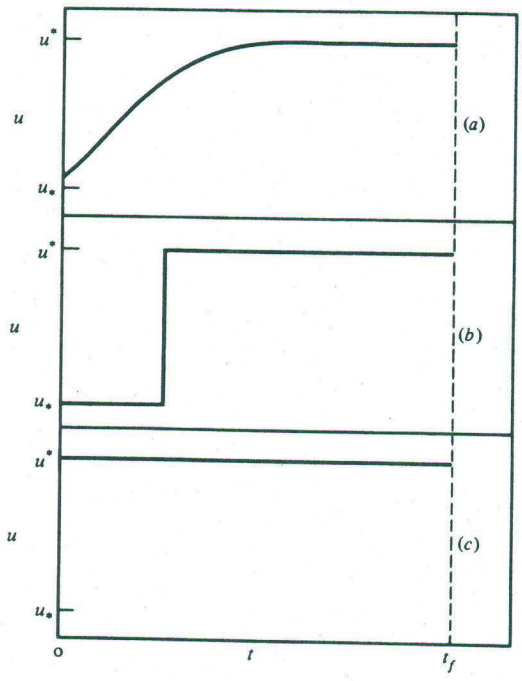


**Figure 3.12** Possible optimal control policies for the billet reheating problem: (*a*) gradual increase of $u$ from $u_*$ to $u^*$; (*b*) stepwise increase from the lower bound to the upper bound; and (*c*) optimum at the upper bound.

value is reached), then from Eq. (3.3.36), $\lambda + C = 0$ and $d\lambda/dt = 0$ on $0 \le t \le t_1$. However, Eq. (3.3.35) does not allow this, because $d\lambda/dt = 0$ implies $\lambda = 0$, a contradiction because $C \neq 0$. Thus the policy shown in Fig. 3.12*a* (as well as any other policy where $u_* < u < u^*$) is not optimal.

Examination of Eq. (3.3.35) shows that $\lambda + C$ can change sign only once and that is when $\lambda(0) + C > 0$ and $\lambda(0) < 0$. This would produce the "optimal" policy shown in Fig. 3.12*b*. However, the fact that $H$ must be identically zero along the optimal trajectory when $t_f$ is unspecified, leads to

$$u_{\text{opt}} = \frac{-1 + \lambda x^4}{C + \lambda} \tag{3.3.37}$$

Clearly if $\lambda < 0$ and $\lambda + C > 0$, then $u_{\text{opt}} < 0$ which is physically impossible. Therefore, the policy given in Fig. 3.12*b* cannot be optimal. The policy $u = u_*$ is not optimal because for the temperature to increase, $u_{\text{opt}} > x^4$ must be true; however when $u = u_*$, then both $\lambda > 0$ and $\lambda + C > 0$ are satisfied everywhere, and $u_{\text{opt}} = -1/(\lambda + C) + \lambda x^4/(\lambda + C)$ and $u_{\text{opt}} < x^4$.

The only remaining possibility for the optimal policy is shown in Fig. 3.12*c*, in which the radiant heat source is kept at its maximum value until the billet reaches the desired temperature. In this case both $\lambda < 0$ and $\lambda + C < 0$ for all times. All that remains is to evaluate Eq. (3.3.29) with $T_s = T^*$ in order to determine the actual minimum time.

Although this problem may be a simple one, it shows that in some cases the optimal control may be deduced without performing any calculations.

## Computational Techniques

There are a large variety of computational methods available for determining the optimal *open-loop* and *closed-loop* control policy. Some of the methods are based on numerically satisfying the necessary conditions of optimality derived in the last section, while others involve more direct search algorithms. In this section we shall discuss a few of the more commonly used algorithms.

*Control vector iteration procedures* for open-loop optimal control synthesis are very similar in philosophy to techniques used for parameter optimization [16]. Basically one makes use of Eq. (3.3.26).

$$\delta I = \int_0^{t_f} \left[ \sum_{i=1}^{m} \left( \frac{\partial H}{\partial u_i} \right) \delta u_i(t) \right] dt \tag{3.3.38}$$

Suppose that $\bar{u}_i(t)$ is not optimal, so that $\partial H/\partial u_i \neq 0$; how can we determine a correction $\delta u_i(t)$ so as to improve $I$ (i.e., cause $\delta I > 0$)? It can be shown [2] that choosing $\delta u_i(t)$ corrected in the gradient direction at each time $t$ produces the greatest local improvement in $I$. Thus on selecting

$$\delta u_i(t) = \varepsilon \left( \frac{\partial H}{\partial u_i} \right) \qquad \varepsilon > 0 \tag{3.3.39}$$

one obtains

$$\delta I = \varepsilon \int_0^{t_f} \sum_{i=1}^{m} \left( \frac{\partial H}{\partial u_i} \right)^2 dt > 0 \tag{3.3.40}$$

which guarantees $\delta I > 0$ for $\varepsilon$ small enough that the linear approximation is not violated.

These results can be incorporated in the following modified gradient method:

1. Guess $\bar{\mathbf{u}}(t)$, $0 \leq t \leq t_f$.
2. With this value of $\bar{\mathbf{u}}(t)$, integrate the state Eqs. (3.3.1) forward in time to produce $\bar{\mathbf{x}}(t)$, $0 \leq t \leq t_f$.
3. With these values of $\bar{\mathbf{x}}(t)$, $\bar{\mathbf{u}}(t)$, integrate the adjoint Eqs. (3.3.20) backward in time, $0 \leq t \leq t_f$.
4. Correct $\bar{\mathbf{u}}(t)$ by Eqs. (3.3.39) where $\varepsilon$ is chosen arbitrarily. Evaluate $I$ for this new control $\mathbf{u}(t)$.
5. If $I[\mathbf{u}(t)] > I[\bar{\mathbf{u}}(t)]$ double $\varepsilon$ and repeat step 4: If $I[\mathbf{u}(t)] < I[\bar{\mathbf{u}}(t)]$, halve $\varepsilon$ and repeat step 4. Do this until a concave function $I(\varepsilon)$ is formed (see Fig. 3.13).
6. Fit a quadratic $I(\varepsilon)$ to these results and predict the optimal value of $\varepsilon$, i.e., $\varepsilon_{\text{opt}}$.

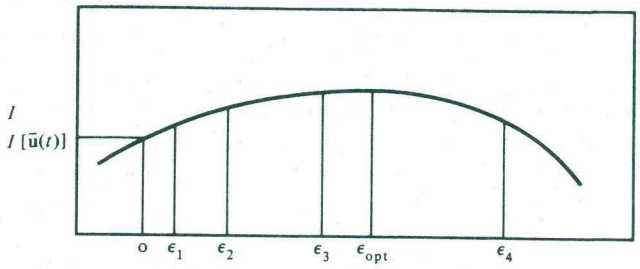


**Figure 3.13** Determination of the optimal $\varepsilon$ by a quadratic approximation.

7. Let

$$\bar{\mathbf{u}}^{\text{new}} = \bar{\mathbf{u}}^{\text{old}} + \varepsilon_{\text{opt}}\left(\frac{\partial H}{\partial \mathbf{u}}\right) \tag{3.3.41}$$

   and return to step 2.
8. Iterate until convergence is attained.

Experience has shown that these methods will lead to rapid progress in the first few iterations, but tend to become very slow as the optimum is approached. Even though convergence to the optimum can be proved theoretically, the rate of convergence can be so slow that the exact optimum is never found in a finite number of iterations. For this reason, several second-order methods have been proposed (e.g., [20–21]) which are similar to those for parameter optimization. In addition, conjugate gradient procedures have been developed [22, 23] which showed improved convergence properties over the standard gradient methods.

As an illustration of the application of control vector iteration techniques, let us consider an example of chemical reactor control.

**Example 3.3.2** Let us consider a batch chemical reactor in which we can control the reaction temperature exactly* and in which we wish to carry out the following reaction:

$$A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C \tag{3.3.42}$$

We consider the kinetics and the temperature dependence of the rate constants known, and our objective is to find the optimal temperature control policy, for a fixed batch time, which will maximize the production of the intermediate $B$. We note that this is one of the classical batch reactor control problems and that the general scheme given by Eq. (3.3.42) is of considerable practical importance in a number of chemical processing operations, e.g., the oxidation of hydrocarbons or the chlorination of aromatics. In all these cases we may wish to maximize the production of an intermediate and thus wish to prevent the reaction from going to completion.

* In many practical situations such close temperature control is, in fact, quite feasible.

In order to define the problem, let us assume that the reaction is of second order with respect to the first step and of first order with respect to the second step. Thus the material balance on the reacting species may be written as

$$\frac{dc_1}{dt} = -k_1(T)c_1^2 \qquad c_1(0) = 1.0 \tag{3.3.43}$$

$$\frac{dc_2}{dt} = k_1(T)c_1^2 - k_2(T)c_2 \qquad c_2(0) = 0 \tag{3.3.44}$$

where

$$c_1 = [\mathrm{A}] \qquad c_2 = [\mathrm{B}] \qquad k_i(T) = \mathrm{A}_{i0}e^{-E_i/RT} \qquad i = 1, 2$$

Let us consider that the temperature is bounded by

$$T_* \le T(t) \le T^* \tag{3.3.45}$$

The object is to find the open-loop temperature control $T(t)$ which maximizes the amount of species $B$ present after 1 h of reaction. Thus our objective becomes

$$\max_{T(t)} [I = c_2(1)] \tag{3.3.46}$$

The additional parameters of the system required to define the problem are given as follows:

$$A_{10} = 4000.0 \text{ L/ (mol)(s)}$$

$$A_{20} = 6.2 \times 10^5/\text{s}$$

$$E_1 = 5000 \text{ cal/ (g)(mol)}$$

$$E_2 = 10{,}000 \text{ cal/ (g)(mol)}$$

$$T_* = 298°\text{K}$$

$$T^* = 398°\text{K}$$

batch time: 1 h

SOLUTION For this problem, the Hamiltonian is

$$H = (\lambda_2 - \lambda_1)k_1(T)c_1^2 - \lambda_2 k_2(T)c_2 \tag{3.3.47}$$

where the adjoint variables $\lambda_1$, $\lambda_2$ are given by

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial c_1} = 2(\lambda_1 - \lambda_2)k_1(T)c_1 \qquad \lambda_1(1) = 0 \tag{3.3.48}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial c_2} = \lambda_2 k_2(T) \qquad \lambda_2(1) = 1 \tag{3.3.49}$$

and the gradient $\partial H/\partial T$ is

$$\frac{\partial H}{\partial T} = \frac{1}{RT^2}[(\lambda_2 - \lambda_1)E_1 k_1 c_1^2 - E_2\lambda_2 k_2(T)c_2] \tag{3.3.50}$$

The modified gradient algorithm was applied to the problem from two different initial guesses of $T(t)$. The resulting optimal temperature program is shown in Fig. 3.14 together with some of the intermediate iterations. Figure 3.15 shows the optimal yield $c_2(1)$ as a function of the number of iterations—for two initial guesses of the temperature program. As can be seen, the same optimal temperature program is found in both cases within three to four iterations. An inspection of the graph shows quite clearly that the optimal policy would produce very marked improvements in the yield of the desired intermediate species $B$. This improvement is found to be $\sim 30$ percent and 300 percent compared with operation of constant temperatures corresponding to the initial guesses of 398 °K and 298 °K, respectively. This example is thus an illustration of situations in which optimal control may produce significant improvements in performance.

Another approach to the problem of numerically solving the necessary conditions of optimality is to convert the control problem to a two-point boundary-value problem (TPBVP) through the elimination of the control variable. These are called *indirect* or *direct substitution* methods. The first step in such a procedure is to eliminate the control vector $\mathbf{u}(t)$ by solving Eq. (3.3.27)

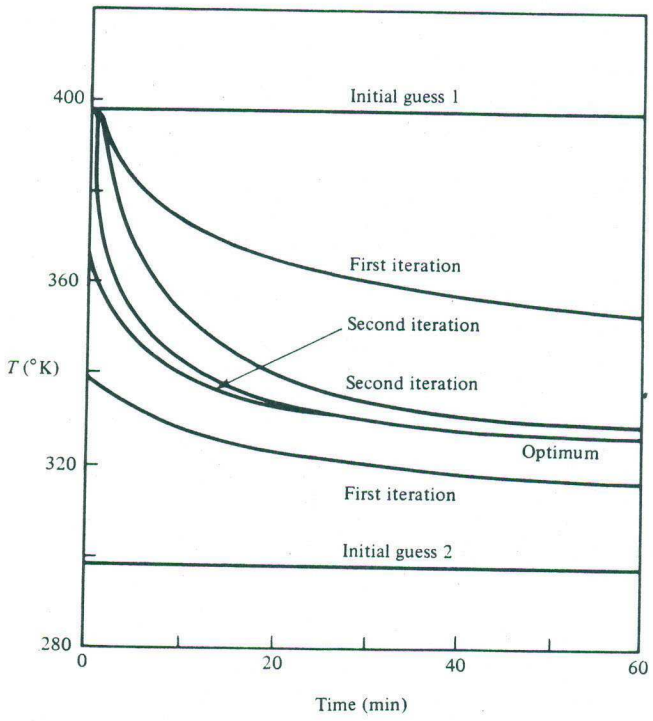


**Figure 3.14** The optimal temperature program in Example 3.3.1.

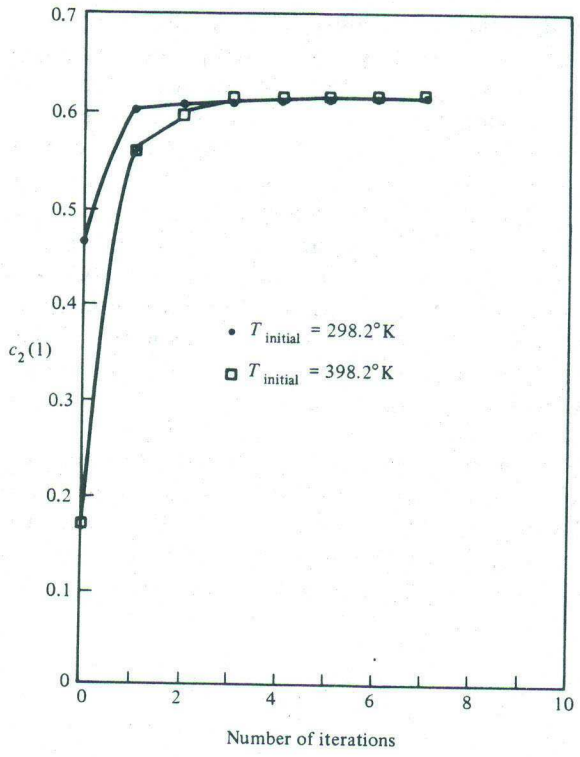


**Figure 3.15** Plot of $c_2(1)$ against the number of iterations.

for $\mathbf{u}(t)$ explicitly:

$$u_i(t) = g_i(\mathbf{x}, \boldsymbol{\lambda}) \qquad i = 1, 2, \ldots, m \tag{3.3.51}$$

It is clear that this may not always be possible; however, it can be done for simple problems. If Eq. (3.3.51) is then substituted into Eqs. (3.3.11) and (3.3.20), the result is a set of $2n$ equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}_1(\mathbf{x}, \boldsymbol{\lambda}) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.3.52}$$

$$\frac{d\boldsymbol{\lambda}}{dt} = \mathbf{f}_2(\mathbf{x}, \boldsymbol{\lambda}) \qquad \boldsymbol{\lambda}(t_f) = \left(\frac{\partial G}{\partial \mathbf{x}}\right)_{t_f} \tag{3.3.53}$$

with split boundary conditions.* This problem, having boundary conditions at two values of the independent variables [a two-point boundary-value problem (TPBVP)], has a solution which produces the optimal values of $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$. When these are substituted into Eq. (3.3.51), one obtains the optimal control $\mathbf{u}(t)$.

What has been effected by the elimination of $\mathbf{u}(t)$ is the trading of a trajectory optimization problem for a TPBVP. TPBVPs are notoriously difficult

* The boundary conditions for $\lambda(t_f)$ given here assume that all the $x(t_f)$ are unspecified. In some instances we may end up with two sets of boundary conditions for Eqs. (3.3.52) and none for Eqs. (3.3.53), as will be illustrated by the example given at the end of the end of the section. However, these problems, too, are two-point boundary-value problems.

to solve, even numerically, and thus most of the techniques associated with this approach are techniques for solving TPBVPs. Let us discuss several types of these techniques.

One technique for solving TPBVPs is the method of *boundary-condition iteration*. This approach tries to find, by some iterative procedure, the missing boundary conditions $\mathbf{x}(t_f)$ or $\boldsymbol{\lambda}(0)$ so that Eqs. (3.3.52) and (3.3.53) can be integrated together in the same direction of time. For simple scalar cases a mapping can be done of guessed values of $\boldsymbol{\lambda}(0)$ versus the resulting values of $\boldsymbol{\lambda}(t_f)_{\text{calc}} - (\partial G/\partial \mathbf{x})_{t_f}$ as sketched in Fig. 3.16. Obviously this graphical technique will not work well for multivariable problems. However, a number of techniques have been proposed for solving these problems [24] by perturbation methods or by minimizing the error in the boundary conditions by a direct search.

There is a basic difficulty with the boundary-condition-iteration approach which often arises in practical problems. The numerical integration of Eqs. (3.3.52) and (3.3.53) in the same direction is very often unstable. The reason for this behavior is that the state equations are usually stable when integrated forward, but unstable in the reverse direction. Similarly, the adjoint equations are usually unstable when integrated forward, but stable in the reverse direction. This can cause great numerical difficulties which are quite independent of the choice of the proper boundary conditions. Nevertheless, the method has been used successfully for some problems. The reader is urged to refer to standard references (e.g., [24 to 26]) for further methods of solving TPBVPs.

Let us illustrate the direct-substitution–boundary-condition-iteration approach by considering a slight variation on the problem posed in Example 3.3.1.

**Example 3.3.3** As before, let us consider the radiant heating of a small billet, as described in Eq. (3.3.29).

$$\frac{dT}{dt} = C_1(T_s^4 - T^4) \qquad T(0) = T_0 \tag{3.3.29}$$

where $T_s$ is the source temperature, $T_0$ is the initial temperature of the billet, and $T_1$ is the final desired temperature.

Our objective is to find the optimal source temperature $T_s(t)$ so as to bring the billet to the desired temperature in a minimum time, while minimizing the rate of wear of the refractory roof.

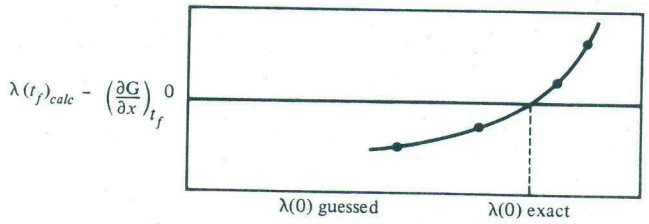


**Figure 3.16** Solution of a two-point boundary value problem by a mapping procedure.

The rate of wear may be expressed as

$$C_2 e^{T_s^4} \tag{3.3.54}$$

where $C_2$ is a constant.

Thus the optimality criterion may be written as

$$\min_{T_s(t')} \left\{ I[T_s(t')] = t_f' + C_2 \int_0^{t_f'} e^{T_s^4(t')} dt' \right\} \tag{3.3.55}$$

where, as before, $t_f'$ is left free and $C_2$ denotes the relative value of the roof erosion to operating time.

Let us define the variables:

$$x_0 = T_0 \qquad x = T \qquad u(t) \equiv u = T_s^4 \qquad t = C_1 t' \qquad C = \frac{C_2}{C_1}$$

$$x_1 = T_1 \qquad \text{and} \qquad u_* = T_*^4 \qquad u^* = T^{*4}$$

where, as before, $u^*$ and $u_*$ denote the upper and lower limits of the operating temperature, respectively.

The problem may now be written as

$$\min_{u(t)} \left[ I(u) = \int_0^{t_f} 1 + Ce^u \, dt \right] \tag{3.3.56}$$

subject to

$$\frac{dx}{dt} = u - x^4 \qquad x(0) = x_0 \qquad x(t_f) = x_1 \tag{3.3.57}$$

Let us follow the procedure set out in Eqs. (3.3.51) to (3.3.53).

The Hamiltonian is given as

$$H = \lambda(u - x^4) + Ce^u + 1 \tag{3.3.58}$$

Thus, from Eq. (3.3.27) we have that

$$\frac{\partial H}{\partial u} = 0 \tag{3.3.27}$$

that is,

$$\lambda + Ce^u = 0$$

$$u = \ln\left(-\frac{\lambda}{C}\right) \tag{3.3.59}$$

On recalling Eq. (3.3.20), i.e.,

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x}$$

by differentiation we obtain

$$\frac{d\lambda}{dt} = 4\lambda x^3 \tag{3.3.60}$$

Thus the optimal $u(t)$ is defined by

$$-\frac{dx}{dt} = x^4 - \ln\left(-\frac{\lambda}{C}\right) \tag{3.3.61}$$
$$x = x_0 \qquad t = 0$$
$$x = x_1 \qquad t = t_f$$

and Eq. (3.3.60). Equations (3.3.60) and (3.3.61) may be solved numerically, e.g., by the boundary-condition-iteration technique described above. The result would then be

$$x = x(t)$$
$$\lambda = \lambda(t)$$

from which $u = u(t)$ is readily obtained from Eq. (3.3.59). While in general one would have to iterate to determine the optimal $t_f$ from the condition $H(t) = 0$, in this simple problem Eq. (3.3.58) may be combined with Eq. (3.3.59) algebraically to yield

$$\lambda\left[\ln\left(-\frac{\lambda}{C}\right) - x^4 - 1\right] + 1 = 0$$

This expression may be solved together with Eq. (3.3.61) from $x = x_0$ to $x = x_1$ to yield the solution in one integration.

We note that the applicability of the direct-substitution technique depends critically on the types of functional relationships involved. Had the rate of wear, Eq. (3.3.54), been given by an alternative expression, such as

$$\text{Rate of wear} = C_2' e^{T_s} \tag{3.3.62}$$

then $H$ would have taken the following form:

$$H = \lambda(u - x^4) + C \exp(u^{1/4}) \tag{3.3.63}$$

thus

$$\frac{\partial H}{\partial u} = 0 = \lambda + \tfrac{1}{4} C u^{-3/4} \exp(u^{1/4}) \tag{3.3.64}$$

Clearly, Eq. (3.3.64) cannot be solved explicitly for $u$. While one could proceed with solving the problem for certain parametric relationships between $u$ and $\lambda$, this is likely to be cumbersome in the majority of cases.

An alternative approach which uses direct search methods is termed *control vector parameterization*. For open-loop control, one could represent $u_i(t)$ by a set of trial functions $\phi_{ij}(t)$, that is,

$$u_i(t) = \sum_{j=1}^{s} a_{ij}\phi_{ij}(t) \tag{3.3.65}$$

and use parameter optimization techniques to determine the optimal set of

coefficients $a_{ij}$. A second approach, which allows a type of optimal closed-loop control, involves generating $u_i$ in a feedback form, i.e., by expanding in a set of trial functions of the state variables

$$u_i(t) = \sum_{j=1}^{p} b_{ij}\phi_{ij}(x_1, x_2, \ldots, x_n) \tag{3.3.66}$$

and determining the optimal constants $b_{ij}$. Limited computational experience [16, 27] has shown that this feedback control scheme has much better convergence properties than the open-loop version. However, the resulting feedback controller parameters may not be generally optimal; they have been computed for only *one* initial condition. To be a good set for controller design, they must give good feedback controller performance for a wide range of initial conditions.

Both of these parameterization approaches have the advantages that no adjoint equations need to be solved, and standard parameter optimization techniques such as those discussed in [16] can be used to determine the coefficients. An even greater advantage, as demonstrated in Fig. 3.17, is the ability of the technique to optimize complex process models by allowing a parameter optimization scheme to select the experiments to be performed on this model. This avoids having to modify existing process models in order to perform optimal control calculations—a significant practical advantage.

The principal disadvantage of parameterization methods is that the functional form of the optimal control must be specified in advance. This requires much more physical insight than is needed for the previous methods discussed. In the absence of a physical feeling for the general shape of the optimal control, a very general functional form [Eq. (3.3.65)] must be used and the optimization performed with respect to a large number of coefficients. On the other hand, if

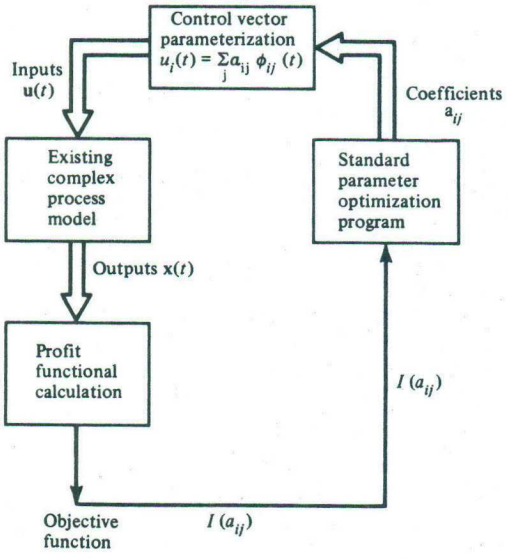


**Figure 3.17** The use of control vector parameterization with an existing complex process model.

one has good reason to suspect a particular form of the optimal policy (e.g., a falling temperature profile in Example 3.3.2), then a simple functional form with only a few coefficients should be adequate. A word of caution is in order, however: surprises do arise from optimal control studies (that is why we do them), and in practical problems one should use several types of functional forms to ensure that the functional form chosen is, in fact, general enough. Several other parameterization approaches have been reported and seem to have some merit. See Ref. [27] for a discussion and comparison of these methods.

Let us illustrate this method with an example problem.*

**Example 3.3.4** Consider the problem of determining the optimal start-up control scheme for a nonlinear CSTR in which the exothermic, first-order reaction $A \rightarrow B$ is taking place. The modeling equations take the form

$$\frac{dc}{dt} = \frac{1}{\theta}(c_f - c) - k_0 e^{(-E/RT)}c \qquad 0 < t < t_f$$

$$\frac{dT}{dt} = \frac{1}{\theta}(T_f - T) + \frac{(-\Delta H)}{\rho C_p} k_0 e^{(-E/RT)}c - \alpha u(T - T_c) \qquad 0 < t < t_f$$

where $c$ is the reactant composition, $T$ the reactor temperature, and $u$ the reactor jacket heat-transfer coefficient, which is influenced by adjusting the coolant flow rate. In this problem, the control variable $u$ is bounded by $u_* \leq u \leq u^*$ and can be parameterized in time by letting

$$u = u_* + (u^* - u_*) \sum_{j=1}^{6} (-1)^{j+1} H(t - b_j) + \sum_{i=1}^{3} a_i (t - b_6)^i$$

where $H(t)$ is the Heaviside step function

$$H(t) = \begin{Bmatrix} 1 & t > 0 \\ 0 & t < 0 \end{Bmatrix}$$

This expression for $u(t)$ allows as many as six switches between the upper and lower bounds on $u$, followed by a smooth cubic trajectory whose shape is determined by the $a_i$. This produces an open-loop start-up program.

Alternatively, one can parameterize in a feedback controller form

$$u(t) = u_d + K_1(c - c_d) + K_2(T - T_d)$$

and search for the two parameters $K_1$, $K_2$.

The objective to be minimized is

$$I = 10^6 \int_0^{t_f} \left[ (c - c_d)^2 + 2 \times 10^{-5}(T - T_d)^2 + 10^{-9}(u - u_s)^2 \right] dt$$

where $c_d = 0.408$, $T_d = 330°\text{K}$, and $u_s = 370$ represent the set-point values for composition, temperature, and control.

For both schemes, the control variable parameterization procedure then searches for the parameters ($a_i$, $b_j$ for time parameterization or $K_1$, $K_2$ for

* This example is taken from Ref. [27] and adapted with permission of the Canadian Journal of Chemical Engineering.

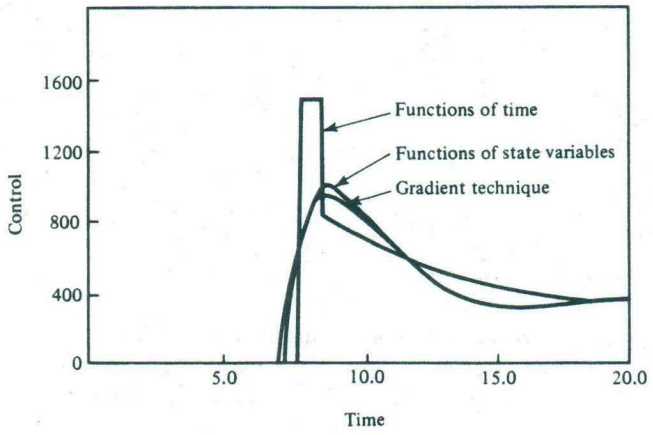


**Figure 3.18** Optimal stirred tank reactor start-up [27]; (a) Hamiltonian gradient method ($I_{min}$ = 987,502); (b) open-loop time parameterization ($I_{min}$ = 987,955); (c) closed-loop state feedback parameterization ($I_{min}$ = 987,436).

state feedback) which minimize the objective $I$. The resulting control trajectories are shown in Fig. 3.18 and compared with the optimal control found from Hamiltonian gradient techniques based on the maximum principle. Note that the value of the objective is almost identical for the three methods even though the control policy varies slightly. The resulting concentration and temperature responses, seen in Fig. 3.19, indicate that the start-up procedure is quite good from a practical point of view.

Let us now discuss briefly the advantages and disadvantages of these various computational approaches. The control vector iteration procedure has the advantage that it can be applied with little algebraic manipulation to even the most complex problems. In addition, because the state equations are solved

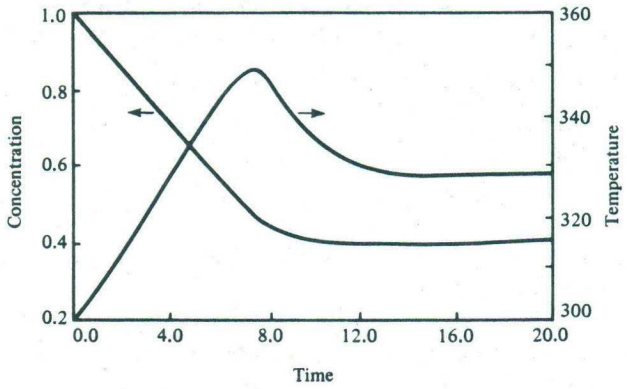


**Figure 3.19** CSTR concentration and temperature responses to optimal start-up control [27].

exactly at each stage, each iteration produces a feasible solution. This has the attraction that one may stop at any iteration with a suboptimal, but reasonably good, usable solution.

The direct-substitution procedures have the disadvantage that a fair amount of algebraic manipulation is required to produce the TPBVP, and then sophisticated procedures are needed for solution. The complexity of these methods makes them difficult for the novice to apply. One advantage of the boundary-condition-iteration approach is that every iteration produces an optimum solution—to the wrong problem. If $\mathbf{x}(t_f)$ is the boundary condition to be adjusted so that Eqs. (3.3.52) and (3.3.53) are integrated backward together, then a calculated value of $\mathbf{x}(0)$ is produced at each iteration. Thus each iteration produces the optimal solution for that calculated initial condition $\mathbf{x}(0)$. This property would be useful if one wished to obtain the optimal policies for a variety of initial conditions.

The control vector parameterization procedure seems to be the most attractive for the novice. Very little sophistication is required, and standard techniques for parameter optimization may be applied. The one major disadvantage seems to be that there is no guarantee that the parameterized optimal control will be very close to the *exact* optimum unless the trial functions are chosen in a sufficiently general way. The number of trial functions needs to be as small as possible to minimize the number of coefficients to be optimized, and yet the functional form must be capable of representing the exact optimum. Thus care must be exercised in the choice of trial functions.

A final word on the practical problems of convergence is in order. As in parameter optimization problems, trajectory optimization algorithms always stop progressing before the exact optimum is reached. However, efficient algorithms will usually stop very close to the true optimum. Thus, to ensure that the optimum has indeed been found, one must be able to produce the same "optimal policy" from several initial guesses. This would seem to ensure that, at least, a local optimum has been found. One should be aware that multiple optima are possible, and in rare cases these have been found in real problems. Thus, even though several starting points must always be used to ensure that the algorithm has converged, this is not an absolute guarantee that the global optimum has been found.

## Problems Linear in the Control

There are a number of classes of optimal control problems which allow special techniques to be used. One such case is systems which have Hamiltonians linear in the control; i.e.,

$$H = h_0(\mathbf{x}, \boldsymbol{\lambda}) + \sum_{i=1}^{m} h_i(\mathbf{x}, \boldsymbol{\lambda}) u_i(t) \tag{3.3.67}$$

then the structure of the optimal control policy is clear without further computation. For example, if $u(t)$ is constrained by $\mathbf{u}_* \leq \mathbf{u} \leq \mathbf{u}^*$, then from the *strong*

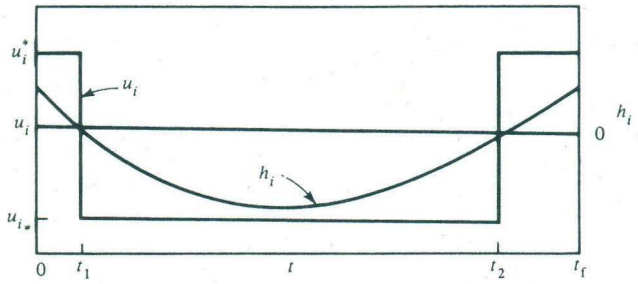


**Figure 3.20** Bang-bang control policy.

*maximum principle* we see that the optimal control has the form

$$u_i(t) = \begin{cases} u_i^* & \text{if } h_i > 0 \\ u_{i*} & \text{if } h_i < 0 \end{cases} \tag{3.3.68}$$

This behavior, plotted in Fig. 3.20, is called a *bang-bang control policy*. The points $t_1$, $t_2$ where $h_i(t)$ changes sign are called switching times.

There is another special situation which occurs when $h_i = 0$ over some interval of time. An examination of Fig. 3.21 shows us that $h_i = 0$ over $t_1 \leq t \leq t_2$. Since this causes the control to vanish from $H$ in that interval, it is not clear how we can determine the optimal value of $u_i(t)$ over $t_1 \leq t \leq t_2$. The control over this interval is called a *singular arc*, and these especially difficult problems are called *singular control problems*. Whenever one encounters a problem such that the Hessian matrix

$$\mathbf{H} = \frac{\partial^2 H}{\partial \mathbf{u}^2} \tag{3.3.69}$$

is singular over some interval $t_1 < t < t_2$, then one has encountered a singular optimal control problem and must exercise special caution. The reader is referred to [6] and [28] for a deeper discussion of these problems and the techniques available for their solution.

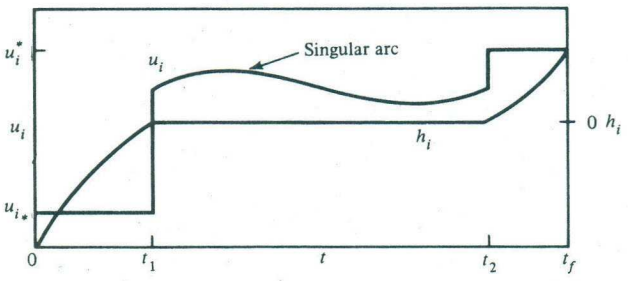


**Figure 3.21** Control policy with a singular arc.

## Optimal State Feedback Control of Linear Systems—Linear Quadratic Problem

Let us now consider a special classical problem in optimal control theory—the *linear-quadratic problem*—which leads to an optimal state feedback control law. There have been numerous papers written on this problem (see [6] and [29]). The problem may be posed as follows: We assume that the state of the system $\mathbf{x}$ can be represented by the linear differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.3.70}$$

and that it is desired to control the system at the set point* $\mathbf{x}_d = \mathbf{0}$ without excessive control action. The quadratic objective functional is

$$I = \tfrac{1}{2}\mathbf{x}^T\mathbf{S}_f\mathbf{x}|_{t_f} + \tfrac{1}{2}\int_{t_0}^{t_f}(\mathbf{x}^T\mathbf{F}\mathbf{x} + \mathbf{u}^T\mathbf{E}\mathbf{u})\,dt \tag{3.3.71}$$

where $\mathbf{A}(t)$, $\mathbf{B}(t)$ are system matrices, $\mathbf{S}_f$, $\mathbf{F}(t)$ are symmetric, positive semidefinite weighting matrices, and $\mathbf{E}(t)$ is a symmetric positive definite matrix.

Now if optimal control theory is applied to this problem, one obtains as the Hamiltonian

$$H = \tfrac{1}{2}(\mathbf{x}^T\mathbf{F}\mathbf{x} + \mathbf{u}^T\mathbf{E}\mathbf{u}) + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u})$$

with conditions

$$\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \mathbf{x}} \qquad \boldsymbol{\lambda}(t_f) = \mathbf{S}_f\mathbf{x}(t_f) \tag{3.3.72}$$

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} \tag{3.3.73}$$

Equation (3.3.73) becomes

$$\mathbf{E}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{0}$$

or

$$\mathbf{u} = -\mathbf{E}^{-1}\mathbf{B}^T\boldsymbol{\lambda} \tag{3.3.74}$$

Thus Eqs. (3.3.70) and (3.3.72) become

$$\left.\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{E}^{-1}\mathbf{B}^T\boldsymbol{\lambda} & \mathbf{x}(t_0) &= \mathbf{x}_0 \\ \dot{\boldsymbol{\lambda}} &= -\mathbf{F}\mathbf{x} - \mathbf{A}^T\boldsymbol{\lambda} & \boldsymbol{\lambda}(t_f) &= \mathbf{S}_f\mathbf{x}(t_f) \end{aligned}\right\} \tag{3.3.75}$$

Now this linear TPBVP can be solved in several ways. However, the solution can be conveniently represented by the form

$$\boldsymbol{\lambda}(t) = \mathbf{S}(t)\mathbf{x}(t) \tag{3.3.76}$$

which has been termed the *Riccati transformation* [29]. Here $\mathbf{S}(t)$ is a symmetric positive definite $n \times n$ matrix. Substitution of Eq. (3.3.76) into Eqs. (3.3.75)

* If we wish to achieve any nonzero state set point $\mathbf{x} = \mathbf{x}_d$, then defining $\check{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d$ will convert the problem to this form. See Example 3.3.5 below.

yields

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{E}^{-1}\mathbf{B}^T\mathbf{S}\mathbf{x} \tag{3.3.77}$$

$$(\dot{\mathbf{S}}\mathbf{x} + \mathbf{S}\dot{\mathbf{x}}) = -\mathbf{F}\mathbf{x} - \mathbf{A}^T\mathbf{S}\mathbf{x} \tag{3.3.78}$$

The elimination of $\dot{\mathbf{x}}$ by substituting Eq. (3.3.77) into Eq. (3.3.78) yields

$$(\dot{\mathbf{S}} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - \mathbf{S}\mathbf{B}\mathbf{E}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{F})\mathbf{x}(t) = \mathbf{0} \tag{3.3.79}$$

Now for Eq. (3.3.79) to hold for all nonzero $\mathbf{x}(t)$, the coefficient matrix of $\mathbf{x}(t)$ must vanish, yielding the *Riccati equation*

$$\frac{d\mathbf{S}}{dt} = -\mathbf{S}\mathbf{A} - \mathbf{A}^T\mathbf{S} + \mathbf{S}\mathbf{B}\mathbf{E}^{-1}\mathbf{B}^T\mathbf{S} - \mathbf{F} \qquad \mathbf{S}(t_f) = \mathbf{S}_f \tag{3.3.80}$$

for $\mathbf{S}(t)$. Note that the boundary conditions on $\mathbf{S}$ follow directly from a comparison of Eqs. (3.3.76) and (3.3.72). By making use of Eq. (3.3.74), one obtains the *state feedback control law*

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t) \tag{3.3.81}$$

where

$$\mathbf{K}(t) = \mathbf{E}^{-1}\mathbf{B}^T\mathbf{S}(t) \tag{3.3.82}$$

To summarize, a proportional state feedback controller with time-varying gain has been derived which will control the system [Eq. (3.3.70)] so that the objective [Eq. (3.3.71)] is minimized. Some points to note are:

1. The time-varying gain $\mathbf{K}(t)$ can be determined offline [by solving for $\mathbf{S}(t)$ beginning at $t = t_f$] because $\mathbf{K}(t)$ does not depend on $\mathbf{x}(t)$ or $\mathbf{u}(t)$.
2. If we let $t_f \to \infty$, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{E}$ are constant, then $\mathbf{S}(t)$ becomes a constant and is the solution of

$$\mathbf{S}\mathbf{B}\mathbf{E}^{-1}\mathbf{B}^T\mathbf{S} - \mathbf{S}\mathbf{A} - \mathbf{A}^T\mathbf{S} - \mathbf{F} = \mathbf{0} \tag{3.3.83}$$

   Thus $\mathbf{K}(t)$ is also a constant. In this case the controller is a constant-gain proportional controller.
3. The precise physical meaning of the objective [Eq. (3.3.71)] is somewhat vague. Clearly the quadratic weighting of the state leads to desirable controller performance when $I$ is minimized; however, the quadratic weighting of the control has less justification, particularly if controller power is not costly. Sometimes the quadratic weighting can be used in place of explicit control constraints in order to yield a feedback controller, but it is difficult to say in what sense this is optimal. Additionally, weighting $\mathbf{E}$ too large causes the state performance to decline, and weighting $\mathbf{E}$ too small causes the control $\mathbf{u}(t)$ to take on extremely large values. The crucial limitations of this controller are that $\mathbf{E}$ must stay positive definite and $\mathbf{x}$, $\mathbf{u}$ must be unconstrained.
4. If one makes the objective functional [Eq. (3.3.71)] more general to include cross terms between the state and the control, one can write Eq. (3.3.71) as

$$I = \tfrac{1}{2}\mathbf{x}^T\mathbf{S}_f\mathbf{x}\big|_{t_f} + \frac{1}{2}\int_{t_0}^{t_f}(\mathbf{x}^T, \mathbf{u}^T)\begin{bmatrix}\mathbf{F}(t) & \mathbf{N}(t)\\ \mathbf{N}^T(t) & \mathbf{E}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} dt \tag{3.3.84}$$

In this case it is straightforward to show that the feedback control law takes the form of Eq. (3.3.81), where

$$\mathbf{K}(t) = \mathbf{E}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{S}) \tag{3.3.85}$$

and

$$\begin{aligned} \frac{d\mathbf{S}}{dt} &= -\mathbf{SA} - \mathbf{A}^T\mathbf{S} + (\mathbf{SB} + \mathbf{N})\mathbf{E}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{S}) - \mathbf{F} \\ \mathbf{S}(t_f) &= \mathbf{S}_f \end{aligned} \tag{3.3.86}$$

This more general form will be useful in the derivation of nonlinear optimal feedback controllers.

**Example 3.3.5** Determine the optimal feedback control law for a process described by Eq. (3.3.70) with the objective to minimize

$$I = \tfrac{1}{2}(\mathbf{x} - \mathbf{x}_d)^T\mathbf{S}_f(\mathbf{x} - \mathbf{x}_d)|_{t_f} + \frac{1}{2}\int_{t_0}^{t_f}\big[(\mathbf{x} - \mathbf{x}_d)^T\mathbf{F}(\mathbf{x} - \mathbf{x}_d) + (\mathbf{u} - \mathbf{u}_d)^T\mathbf{E}(\mathbf{u} - \mathbf{u}_d)\big]\,dt \tag{3.3.87}$$

where $\mathbf{x}_d$, $\mathbf{u}_d$ correspond to a desired *steady-state* position.

Let us define

$$\check{\mathbf{x}} = \mathbf{x} - \mathbf{x}_d \qquad \check{\mathbf{u}} = \mathbf{u} - \mathbf{u}_d \tag{3.3.88}$$

then Eq. (3.3.87) takes the form of Eq. (3.3.71) in the variables $\check{\mathbf{x}}$, $\check{\mathbf{u}}$ and Eq. (3.3.70) becomes

$$\frac{d\check{\mathbf{x}}}{dt} = \mathbf{A}\check{\mathbf{x}} + \mathbf{B}\check{\mathbf{u}} + \cancelto{0}{\mathbf{A}\mathbf{x}_d + \mathbf{B}\mathbf{u}_d} \tag{3.3.89}$$

However, the last two terms in Eq. (3.3.88) vanish because they represent a steady state. Therefore we have the control law

$$\check{\mathbf{u}} = -\mathbf{K}(t)\check{\mathbf{x}} \tag{3.3.90}$$

where $\mathbf{K}(t)$ is given by Eq. (3.3.82). In terms of the original variables, this is

$$\mathbf{u}(t) = \mathbf{u}_d - \mathbf{K}(t)(\mathbf{x} - \mathbf{x}_d) \tag{3.3.91}$$

**Example 3.3.6** Let us consider the CSTR in which an isothermal multicomponent chemical reaction is being carried out. The chemical reaction system is

$$A \underset{k_2}{\overset{k_1}{\rightleftarrows}} B \xrightarrow{k_3} C$$

with the rates of reaction given by

$$r_1 = k_1 c_A - k_2 c_B$$
$$r_2 = k_3 c_B$$

where $k_1$, $k_2$, $k_3$ are constants.

The modeling equations for this system take the form

$$V\frac{dc_A}{dt'} = F(c_{Af} - c_A) - V(k_1c_A - k_2c_B) \qquad c_A(0) = c_{A0}$$

$$V\frac{dc_B}{dt'} = F(c_{Bf} - c_B) + V[k_1c_A - (k_2 + k_3)c_B] \qquad c_B(0) = c_{B0}$$

It is desired to control $c_A$, $c_B$ as close as possible to a desired set point $c_{Ad}$, $c_{Bd}$ by adjusting $c_{Bf}$. At the same time, there is a target steady-state value of $c_{Bf}$, i.e., $c_{Bfd}$, which we would like to achieve as well. The feed concentration of $c_{Af}$ is considered fixed.

By defining the dimensionless variables

$$t = \frac{t'F}{V} \qquad u = \frac{c_{Bf}}{c_{Af}} \qquad x_1 = \frac{c_A}{c_{Af}} \qquad x_2 = \frac{c_B}{c_{Af}}$$

$$\mathrm{Da}_1 = \frac{k_1V}{F} \qquad \mathrm{Da}_2 = \frac{k_2V}{F} \qquad \mathrm{Da}_3 = \frac{k_3V}{F}$$

one obtains the model

$$\frac{dx_1}{dt} = 1 - (1 + \mathrm{Da}_1)x_1 + \mathrm{Da}_2x_2$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1x_1 - (1 + \mathrm{Da}_2 + \mathrm{Da}_3)x_2 + u$$

Now let us define the quadratic objective

$$I = \tfrac{1}{2}\int_0^{t_f}\left[(x_1 - x_{1d})^2 + \alpha(x_2 - x_{2d})^2 + \beta(u - u_d)^2\right] dt$$

where $x_{1d}$, $x_{2d}$, $u_d$ satisfy the steady-state equations

$$0 = 1 - (1 + \mathrm{Da}_1)x_{1d} + \mathrm{Da}_2x_{2d}$$

$$0 = \mathrm{Da}_1x_{1d} - (1 + \mathrm{Da}_2 + \mathrm{Da}_3)x_{2d} + u_d$$

By defining $\check{x}_1 = x_1 - x_{1d}$, $\check{x}_2 = x_2 - x_{2d}$, $\check{u} = u - u_d$, one obtains the linear quadratic problem having a model

$$\frac{d\check{x}_1}{dt} = -(1 + \mathrm{Da}_1)\check{x}_1 + \mathrm{Da}_2\check{x}_2$$

$$\frac{d\check{x}_2}{dt} = \mathrm{Da}_1\check{x}_1 - (1 + \mathrm{Da}_2 + \mathrm{Da}_3)\check{x}_2 + \check{u}$$

and an objective to be minimized

$$I = \tfrac{1}{2}\int_0^{t_f}\left(\check{x}_1^2 + \alpha\check{x}_2^2 + \beta\check{u}^2\right) dt$$

The solution then can be found from Eqs. (3.3.91), (3.3.82), and (3.3.86),

where

$$\mathbf{A} = \begin{bmatrix} -(1 + \mathrm{Da}_1) & \mathrm{Da}_2 \\ \mathrm{Da}_1 & -(1 + \mathrm{Da}_2 + \mathrm{Da}_3) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \qquad E = \beta$$

Thus

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

is given by the solution of

$$\dot{S}_{11} = 2S_{11}(1 + \mathrm{Da}_1) - 2S_{12}\,\mathrm{Da}_1 + \frac{1}{\beta}S_{12}^2 - 1 \qquad S_{11}(t_f) = 0$$

$$\dot{S}_{12} = -S_{11}\,\mathrm{Da}_2 + S_{12}(2 + \mathrm{Da}_1 + \mathrm{Da}_2 + \mathrm{Da}_3) - S_{22}\,\mathrm{Da}_1 + \frac{1}{\beta}S_{12}S_{22} \qquad S_{12}(t_f) = 0$$

$$\dot{S}_{22} = 2S_{22}(1 + \mathrm{Da}_2 + \mathrm{Da}_3) - 2S_{12}\,\mathrm{Da}_2 + \frac{1}{\beta}S_{22}^2 - \alpha \qquad S_{22}(t_f) = 0$$

and $S_{21} = S_{12}$. Also, $\mathbf{K}(t)$ is

$$\mathbf{K}(t) = \frac{1}{\beta}[S_{12}S_{22}]$$

Thus the final controller is of the form

$$u(t) = u_d - \{K_1(t)[x_1(t) - x_{1d}] + K_2(t)[x_2(t) - x_{2d}]\}$$

Note that the Riccati equations for $S_{11}$, $S_{12}$, and $S_{22}$ may be solved offline and $K_1(t)$, $K_2(t)$ stored in the computer for real-time use.

To illustrate the performance of the linear quadratic controller for this problem, computational results shall be presented for the parameter values:

$$\mathrm{Da}_1 = 3.0 \quad \mathrm{Da}_2 = 0.5 \quad \mathrm{Da}_3 = 1.0 \quad x_{1d} = 0.3 \quad x_{2d} = 0.4 \quad u_{1d} = 1.0$$
$$x_1(0) = 1.0 \quad x_2(0) = 0.0 \quad t_f = 2.0 \quad \alpha = 1.0 \quad \beta = 0.25$$

Solving the Riccati equations for $\mathbf{S}(t)$ by solving backward from $\mathbf{S}_f = \mathbf{0}$ gives the time-dependent controller gains $K_1(t)$, $K_2(t)$ shown in Fig. 3.22. Notice that $K_1$ and $K_2$ are constant over much of the time period and only change significantly as $t$ approaches $t_f$. The linear-quadratic optimal feedback controller response is shown in Fig. 3.23, where it is seen that the controller does cause the system to approach the desired set points $x_{1d} = 0.3$, $x_{2d} = 0.4$ rather closely within the time period $0 < t < 2$. Also note that $\beta$ has been chosen large enough so that $u(t)$ does not violate the physical

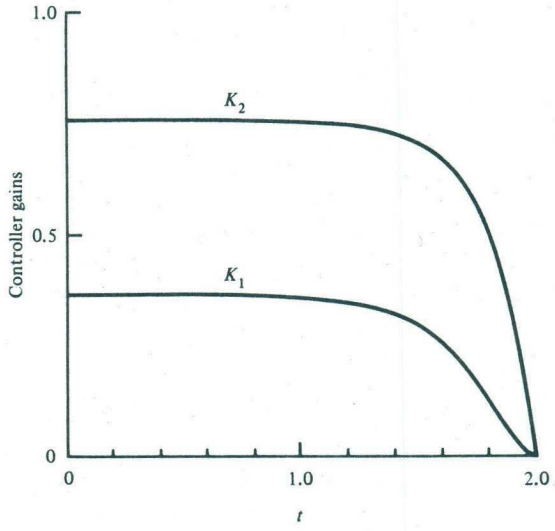


**Figure 3.22** Optimal time-dependent controller gains for Example 3.3.6.

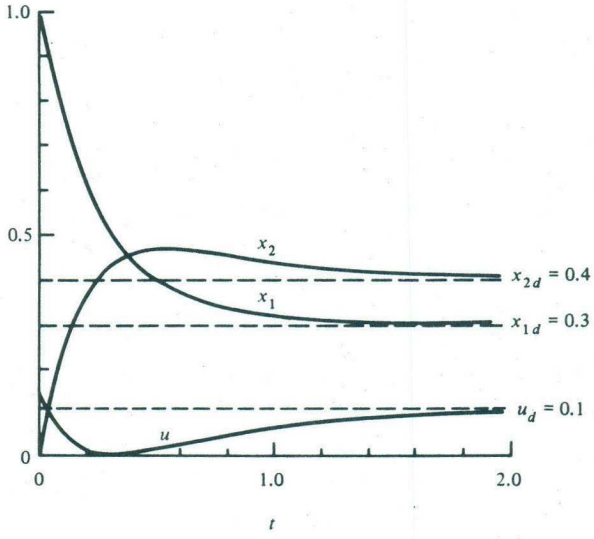


**Figure 3.2.3** Linear-quadratic optimal feedback controller performance for Example 3.3.6.

constraint $u(t) > 0$. For values of $\beta$ too small, $u(t)$ can become negative, violating this constraint, while for $\beta$ too large, little dynamic control action is possible because $K_1(t)$, $K_2(t)$ remain close to zero. Thus for good optimal linear-quadratic controller design, one must tune the weighting parameters (such as $\alpha$ and $\beta$ here) in order to obtain the desired controller response.

From the previous example one may note that the linear-quadratic formulation produces a proportional state feedback controller. From classical control

theory one recognizes that proportional controllers lead to offset when there are set-point changes or load changes in the process; thus, it would be desirable to formulate the optimal feedback control problem so as to allow *integral control action* which would eliminate these offsets. There are several possible ways of doing this.

One method of including integral action is to include $d\mathbf{u}/dt$ terms in the objective functional so that Eq. (3.3.71) becomes

$$I = \tfrac{1}{2}\mathbf{x}^T\mathbf{S}_f\mathbf{x}|_{t_f} + \frac{1}{2}\int_{t_0}^{t_f}(\mathbf{x}^T\mathbf{F}\mathbf{x} + \dot{\mathbf{u}}^T\mathbf{E}\dot{\mathbf{u}})\,dt \tag{3.3.92}$$

where the state equation (3.3.70) must be differentiated to yield (for constant matrices **A**, **B**)

$$\ddot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{u}}$$

Then the problem may be reformulated by letting

$$\mathbf{v}(t) = \dot{\mathbf{u}} \qquad \mathbf{w}_1 = \mathbf{x} \qquad \mathbf{w}_2 = \dot{\mathbf{x}} \qquad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \end{bmatrix} \tag{3.3.93}$$

so that

$$\dot{\mathbf{w}} = \left[\begin{array}{c|c} 0 & \mathbf{I} \\ \hline 0 & \mathbf{A} \end{array}\right]\mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{B} \end{bmatrix}\mathbf{v} \tag{3.3.94}$$

and the objective becomes

$$I = \frac{1}{2}\left\{\mathbf{w}^T\left[\begin{array}{c|c} \mathbf{S}_f & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]\mathbf{w}\right\}_{t=t_f} + \frac{1}{2}\int_{t_0}^{t_f}\left\{\mathbf{w}^T\left[\begin{array}{c|c} \mathbf{F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]\mathbf{w} + \mathbf{v}^T\mathbf{E}\mathbf{v}\right\}\,dt \tag{3.3.95}$$

By applying the feedback control law [Eq. (3.3.82)] to this reformulated problem, we obtain

$$\mathbf{v} = -\mathbf{K}(t)\mathbf{w} = -\left[\mathbf{K}_1 \,\vdots\, \mathbf{K}_2\right]\begin{bmatrix} \mathbf{w}_1 \\ \hline \mathbf{w}_2 \end{bmatrix} \tag{3.3.96}$$

which, upon integration, takes the form

$$\mathbf{u}(t) = -\mathbf{K}_2\mathbf{x}(t) - \int_{t_0}^{t_f}(\mathbf{K}_1 - \dot{\mathbf{K}}_2)\mathbf{x}(t)\,dt \tag{3.3.97}$$

which has somewhat complicated integral action. If one allows $t_f \to \infty$, then $\mathbf{K}_1$ and $\mathbf{K}_2$ are constants and the feedback control law is

$$\mathbf{u}(t) = -\mathbf{K}_2\mathbf{x}(t) - \mathbf{K}_1\int_{t_0}^{t_f}\mathbf{x}(t)\,dt \tag{3.3.98}$$

an "optimal" proportional-integral controller.*

One may justifiably question the physical meaning of minimizing the time derivative of the control action in Eq. (3.3.92) and how this relates to reducing offset in the state variables. At the moment this relation is unclear.

* Through a slightly different transformation of states, derivative action may be obtained as well (see [30]).

A second means of incorporating integral action into the controller [31, 32] is to augment the state variables to include $p$ new variables $\mathbf{z}(t)$ where

$$\dot{\mathbf{z}} = \mathbf{C}^*\mathbf{x} \tag{3.3.99}$$

are those state variables for which integral action is desired. Thus the new state $\hat{\mathbf{x}}$ of dimension $n + p$ is

$$\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \hline \mathbf{z} \end{bmatrix} \tag{3.3.100}$$

When the objective functional is also modified to accommodate the new state variable, i.e.,

$$I = \tfrac{1}{2}\hat{\mathbf{x}}^T\hat{\mathbf{S}}_f\mathbf{x} + \frac{1}{2}\int_0^{t_f}(\hat{\mathbf{x}}\hat{\mathbf{F}}\hat{\mathbf{x}} + \mathbf{u}^T\mathbf{E}\mathbf{u})\,dt \tag{3.3.101}$$

the linear-quadratic optimal control law takes the form

$$\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} = -\mathbf{K}_1\mathbf{x} - \mathbf{K}_2\mathbf{z} = -\mathbf{K}_1\mathbf{x} - \mathbf{K}_2\mathbf{C}^*\int \mathbf{x}\,dt \tag{3.3.102}$$

which naturally includes integral action. Note that necessarily $p < m$, that is, the number of state variables for which integral action is desired cannot be larger than the number of control variables.

Another derivation of proportional and integral "optimal" feedback control is given in Sec. 5.4, where stochastic control is discussed. There integral action arises naturally as a means of dealing with random process disturbances.

### Optimal Linear-Quadratic Feedback Control of Nonlinear Systems

It is possible to extend the results of the linear-quadratic problem to nonlinear systems so as to produce an optimal feedback control law. Let us consider the nonlinear optimal control problem given by the modeling equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.3.103}$$

and the control objective functional

$$I[\mathbf{u}(t)] = \mathbf{G}(\mathbf{x}(t_f)) + \int_{t_0}^{t_f} F(\mathbf{x}, \mathbf{u})\,dt \tag{3.3.104}$$

Now if one expands the objective functional to second-order about a nominal control and state trajectory $\bar{\mathbf{u}}(t)$, $\bar{\mathbf{x}}(t)$ while adjoining the state constraints, the variation in $I$ is

$$\begin{aligned}\delta I = {} & \left(\frac{\partial G}{\partial \mathbf{x}}\right)_{t_f}\delta\mathbf{x}(t_f) + \frac{1}{2}\delta\mathbf{x}^T(t_f)\left(\frac{\partial^2 G}{\partial \mathbf{x}^2}\right)_{t_f}\delta\mathbf{x}(t_f) - \boldsymbol{\lambda}^T(t_f)\,\delta\mathbf{x}(t_f) \\ & + \boldsymbol{\lambda}^T(t_0)\,\delta\mathbf{x}(t_0) + \int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial \mathbf{u}}\right)\delta\mathbf{u} + \frac{1}{2}\delta\mathbf{u}^T\left(\frac{\partial^2 H}{\partial \mathbf{u}^2}\right)\delta\mathbf{u} + \left(\frac{\partial H}{\partial \mathbf{x}}\right)\delta\mathbf{x}\right. \\ & \left. + \frac{1}{2}\delta\mathbf{x}^T\left(\frac{\partial^2 H}{\partial \mathbf{x}^2}\right)\delta\mathbf{x} + \delta\mathbf{x}^T\left(\frac{\partial^2 H}{\partial \mathbf{x}\,\partial \mathbf{u}}\right)\delta\mathbf{u} + \frac{d\boldsymbol{\lambda}^T}{dt}\,\delta\mathbf{x}\right]dt\end{aligned} \tag{3.3.105}$$

where it is useful to recall that

$$\delta \mathbf{x} = \mathbf{x} - \bar{\mathbf{x}} \qquad \delta \mathbf{u} = \mathbf{u} - \bar{\mathbf{u}} \qquad \text{and} \qquad H = F + \boldsymbol{\lambda}^T \mathbf{f}$$

Now let us suppose our nominal trajectory $\bar{\mathbf{u}}(t)$, $\bar{\mathbf{x}}(t)$ satisfies the first-order necessary conditions for open-loop optimality, i.e.,

$$\frac{\partial H}{\partial \bar{\mathbf{u}}} = 0 \qquad \frac{d\boldsymbol{\lambda}^T}{dt} = -\left(\frac{\partial H}{\partial \mathbf{x}}\right)_{\cdot} \qquad \boldsymbol{\lambda}^T(t_f) = \left(\frac{\partial G}{\partial \mathbf{x}(t_f)}\right)_{\cdot} \tag{3.3.106}$$

where we assume $\mathbf{x}(t_0)$ is specified ($\mathbf{x}(t_0) = \bar{\mathbf{x}}_0$) and $\mathbf{x}(t_f)$ is unspecified. Thus $\bar{\mathbf{x}}(t)$, $\bar{\mathbf{u}}(t)$ are the open-loop optimal controls for a particular initial condition $\mathbf{x}(t_0) = \bar{\mathbf{x}}_0$. In this case Eq. (3.3.105) has only the remaining terms:

$$\delta I = \int_{t_0}^{t_f} \left[ \frac{1}{2} \delta \mathbf{u}^T \left( \frac{\partial^2 H}{\partial \mathbf{u}^2} \right)_{\cdot} \delta \mathbf{u} + \delta \mathbf{x}^T \left( \frac{\partial^2 H}{\partial \mathbf{x}\, \partial \mathbf{u}} \right)_{\cdot} \delta \mathbf{u} + \frac{1}{2} \delta \mathbf{x}^T \left( \frac{\partial^2 H}{\partial \mathbf{x}^2} \right)_{\cdot} \delta \mathbf{x} \right] dt$$

$$+ \frac{1}{2} \delta \mathbf{x}^T(t_f) \left( \frac{\partial^2 G}{\partial \mathbf{x}^2} \right)_{\cdot_{t_f}} \delta \mathbf{x}(t_f) \tag{3.3.107}$$

Now let us note that if we change the initial condition $\mathbf{x}(t_0)$, the entire nonlinear open-loop optimal control policy must be recalculated, because unlike linear problems, in which superposition may be used to quickly adjust for changes in initial conditions, nonlinear optimal control problems depend nonlinearly on the initial conditions. However, it is possible to develop a perturbation approach to solve this problem. If we consider the system equations linearized about the open-loop optimal policy $\bar{\mathbf{x}}(t)$, $\bar{\mathbf{u}}(t)$ for a fixed initial condition $\mathbf{x}(t_0) = \bar{\mathbf{x}}_0$, we obtain the *perturbation equations*

$$\frac{d(\delta \mathbf{x}(t))}{dt} = \left( \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_{\cdot} \delta \mathbf{x} + \left( \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right)_{\cdot} \delta \mathbf{u} \qquad \delta \mathbf{x}(t_0) = \delta \mathbf{x}_0 \tag{3.3.108}$$

which represent the system behavior for initial conditions in some domain close to the nominal value $\bar{\mathbf{x}}_0$, i.e., for $|\delta \mathbf{x}(t_0)| < \varepsilon$. The degradation in the system performance due to the deviations is given by Eq. (3.3.107). Thus Eqs. (3.3.107) and (3.3.108) represent a *linear-quadratic* optimal control problem whose solution $\delta \mathbf{u}^*(t)$, $\delta \mathbf{x}^*(t)$ represents the optimal feedback corrections to the nominal open-loop control. This is illustrated graphically for a single control–single state process in Fig. 3.24. There the nominal open-loop optimal control trajectory $\bar{\mathbf{x}}(t)$, $\bar{\mathbf{u}}(t)$ originating from $\bar{\mathbf{x}}_0$, and the perturbation feedback corrections $\delta \mathbf{u}(t)$, $\delta \mathbf{x}(t)$ resulting from another initial disturbance $\mathbf{x}_0$, are shown.

The solution to the perturbation feedback control law follows directly from the results of the last section if we define

$$\mathbf{A}(t) = \left( \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_{\cdot} \qquad \mathbf{B}(t) = \left( \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right)_{\cdot} \qquad \mathbf{F}(t) = \left( \frac{\partial^2 H}{\partial \mathbf{x}^2} \right)_{\cdot}$$

$$\mathbf{E}(t) = \left( \frac{\partial^2 H}{\partial \mathbf{u}^2} \right)_{\cdot} \qquad \mathbf{N}(t) = \left( \frac{\partial^2 H}{\partial \mathbf{x}\, \partial \mathbf{u}} \right)_{\cdot} \qquad \mathbf{S}_f = \left( \frac{\partial^2 G}{\partial \mathbf{x}^2} \right)_{\cdot_{t_f}} \tag{3.3.109}$$

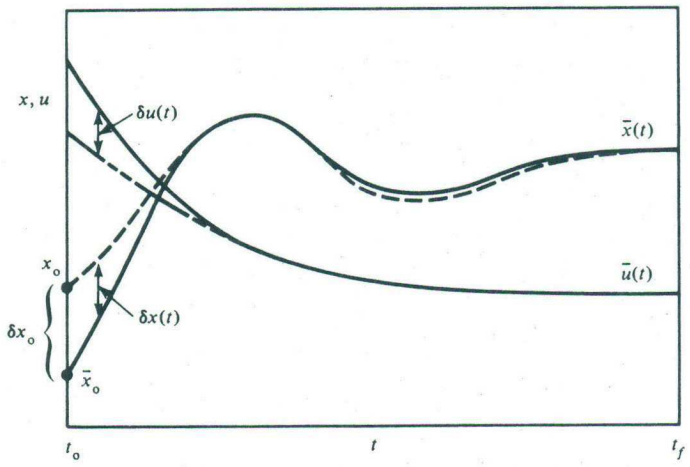


**Figure 3.24** Optimal perturbation feedback control of a nonlinear system.

and use the results of Eqs. (3.3.84) to (3.3.86). Thus

$$\delta\mathbf{u}(t) = -\mathbf{K}(t)\,\delta\mathbf{x}(t) \tag{3.3.110}$$

where

$$\mathbf{K}(t) = \mathbf{E}^{-1}(\mathbf{N}^T + \mathbf{B}^T\mathbf{S}) \tag{3.3.85}$$

or

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) + \mathbf{K}(t)\big[\mathbf{x}(t) - \bar{\mathbf{x}}(t)\big] \tag{3.3.111}$$

The implementation of such a feedback control scheme could be carried out as follows:

1. Calculate and store a set of optimal open-loop control policies $\bar{\mathbf{u}}(t)$, $\bar{\mathbf{x}}(t)$ corresponding to a very coarse grid of nominal initial conditions $\bar{\mathbf{x}}_0$.
2. Also precalculate and store the controller gains $\mathbf{K}(t)$ for each nominal initial condition.
3. The on-line control can then be carried out by identifying the closest nominal initial condition in the stored grid and using that corresponding $\mathbf{K}(t)$ in the control law {Eq. (3.3.111)].

Let us illustrate this technique with the following example.

**Example 3.3.7** Consider the simple problem of the control, by cooling-rate manipulation, of the temperature of a continuous-stirred tank reactor (CSTR) in which a zero-order reaction is taking place. The mathematical

model for the reactor temperature is

$$\rho C_p V \frac{dT}{dt'} = \rho C_p F(T_f - T) + (-\Delta H) V k_0 e^{-E/RT} - Q' \quad (3.3.112)$$

$$T(0) = T_0$$

Let

$$x = \frac{T}{T_f} \qquad t = \frac{t'F}{V} \qquad a = \frac{(-\Delta H)k_0 V}{\rho C_p T_f F} \qquad \gamma = \frac{E}{RT_f} \qquad u = \frac{Q'}{\rho C_p F T_f}$$

$$x_0 = \frac{T_0}{T_f} \qquad x_d = \frac{T_d}{T_f}$$

so that the dimensionless model equation is

$$\frac{dx}{dt} = 1 - x + ae^{-\gamma/x} - u \qquad x(0) = x_0 \quad (3.3.113)$$

Our objective is to minimize the objective functional

$$I = \frac{1}{2}\int_0^{t_f} \left[(x - x_d)^2 + \alpha u^2\right] dt \quad (3.3.114)$$

by designing an optimal feedback controller.

Let us assume that the open-loop optimal control policy $\bar{u}(t)$, $\bar{x}(t)$ has been determined for a given initial condition, $\bar{x}_0$ by a control vector iteration procedure:

1. Guess $u(t)$.
2. Integrate Eq. (3.3.102) from $x_0$ to $x_f$ to yield $x(t)$.
3. Integrate

$$\frac{d\lambda}{dt} = -(x - x_d) + \lambda\left(1 - \frac{a\gamma e^{-\gamma/x}}{x^2}\right) \quad (3.3.115)$$

   backwards from $\lambda(t_f) = 0$.
4. Correct $u$ by

$$u^{\text{new}} = u^{\text{old}} + \varepsilon \frac{\partial H}{\partial u} \qquad \varepsilon < 0$$

   where

$$H = \tfrac{1}{2}\left[(x - x_d)^2 + \alpha u^2\right] + \lambda(1 - x + ae^{-\gamma/x} - u)$$

   and

$$\frac{\partial H}{\partial u} = \alpha u - \lambda \quad (3.3.116)$$

   and go back to step 2.

Let the result of this iterative calculation be denoted $\bar{x}(t)$, $\bar{u}(t)$. Repeat this open-loop optimal control calculation for a grid of $x_0$ values over the expected range of disturbances.

Now we can compute the linearized "matrices"

$$A = \left(\frac{\partial f}{\partial x}\right)_{\cdot} = \left(-1 + \frac{a\gamma}{\bar{x}^2} e^{-\gamma/\bar{x}}\right)$$

$$B = \left(\frac{\partial f}{\partial u}\right)_{\cdot} = -1$$

$$F = \left(\frac{\partial^2 H}{\partial x^2}\right)_{\cdot} = 1 + \bar{\lambda}\left(\frac{a\gamma^2}{\bar{x}^4} - \frac{2a\gamma}{\bar{x}^3}\right)e^{-\gamma/\bar{x}}$$

$$E = \left(\frac{\partial^2 H}{\partial u^2}\right)_{\cdot} = \alpha$$

$$N = 0$$

Thus $S(t)$ satisfies

$$\frac{dS}{dt} = -2S\left(-1 + \frac{a\gamma}{\bar{x}^2} e^{-\gamma/\bar{x}}\right) - \left[1 + \bar{\lambda}\left(\frac{a\gamma^2}{\bar{x}^4} - \frac{2a\gamma}{\bar{x}^3}\right)e^{-\gamma/\bar{x}}\right] + \frac{S^2}{\alpha}$$

$$S(t_f) = 0$$

or recognizing from Eq. (3.3.105) that $\bar{\lambda} = \alpha\bar{u}$, one obtains

$$\frac{dS}{dt} = -2\left[\frac{a\gamma}{\bar{x}^2} e^{-\gamma/\bar{x}} - 1\right]S + \frac{S^2}{\alpha} - 1 - \alpha\bar{u}\left(\frac{a\gamma^2}{\bar{x}^4} - \frac{2a\gamma}{\bar{x}^3}\right)e^{-\gamma/\bar{x}}$$

$$S(t_f) = 0 \tag{3.3.117}$$

and

$$K(t) = -\frac{S(t)}{\alpha} \tag{3.3.118}$$

Thus the optimal feedback controller is

$$u(t) = \bar{u}(t) - K(t)[x(t) - \bar{x}(t)] \tag{3.3.119}$$

## 3.4 NONLINEAR MULTIVARIABLE CONTROL

Although most of existing control theory only applies to linear systems with constant coefficients, the great majority of practical process control problems involve nonlinear systems. Thus there is a need to explore the useful theory for nonlinear multivariable control problems. For systems described by nonlinear differential equations, the general form of the modeling equations is

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{d}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.4.1}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}) \tag{3.4.2}$$

where, as in the previous section, $\mathbf{x}(t)$ is an $n$ vector of states, $\mathbf{u}(t)$ is an $m$ vector

of controls, $\mathbf{d}(t)$ is a $k$ vector of disturbances, and $\mathbf{y}(t)$ is an $l$ vector of measured outputs. Clearly when

$$\mathbf{f} = \mathbf{Ax} + \mathbf{Bu} + \Gamma\mathbf{d} \tag{3.4.3}$$

$$\mathbf{h} = \mathbf{C} \tag{3.4.4}$$

the nonlinear system [Eqs. (3.4.1) and (3.4.2)] reduces to the linear problem treated in the last section.

General strategies of nonlinear multivariable controller design seem to fall into two major categories:

1. Linearization of the nonlinear equations so that the linear design procedures (such as those described in Secs. 3.2 and 3.3) may be applied.
2. Special-purpose methods which may be applied directly to the nonlinear system.

Both approaches to control system design have their merits and shall be discussed in more detail in what follows.

## Linearization

The easiest and most common approach to the design of control schemes for nonlinear multivariable systems is to linearize the modeling equations and apply standard linear design procedures. Although this approach is straightforward, it is useful to outline the essential features.

Rigorous conditions for *controllability*, *stabilizability*, etc., for nonlinear systems have been derived only for rather limited special cases (e.g., [2]). However, in most cases a good practical answer to such questions may be found through linearization of the nonlinear equations (3.4.1) and (3.4.2) and application of the linear theory to these linearized equations. The details of this approach are best illustrated by an example.

**Example 3.4.1** Let us consider the isothermal CSTR described in Example 3.2.6 *except* that here the reaction $A \rightarrow B$ is second-order and the reaction $B \rightarrow C$ is $\frac{1}{2}$-order. In this case the modeling equations are

$$V\frac{dc_A}{dt'} = F(c_{A_f} - c_A) - Vk_1c_A^2 \tag{3.4.5}$$

$$V\frac{dc_B}{dt'} = F(c_{B_f} - c_B) + V(k_1c_A^2 - k_3c_B^{1/2}) \tag{3.4.6}$$

and may be put in the form of Eqs. (3.4.1) and (3.4.2) by defining

$$x_1 = \frac{c_A}{c_{A_{\text{ref}}}} \qquad x_2 = \frac{c_B}{c_{A_{\text{ref}}}} \qquad u_1 = \frac{c_{A_f}}{c_{A_{\text{ref}}}} \qquad u_2 = \frac{c_{B_f}}{c_{A_{\text{ref}}}}$$

$$\text{Da}_1 = \frac{k_1c_{A_{\text{ref}}}V}{F} \qquad \text{Da}_3 = \frac{k_3V}{F(c_{A_{\text{ref}}})^{1/2}} \qquad t = \frac{t'F}{V} \tag{3.4.7}$$

Thus the nonlinear model becomes

$$\frac{dx_1}{dt} = -x_1 - \mathrm{Da}_1 x_1^2 + u_1 \tag{3.4.8}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1^2 - x_2 - \mathrm{Da}_3 (x_2)^{1/2} + u_2 \tag{3.4.9}$$

Let us linearize these equations around the steady state defined by $u_{1s} = 1$, $u_{2s} = 0$ and make use of deviation variables $\hat{x}_i = x_i - x_{is}$, $\hat{u}_i = u_i - u_{is}$, $i = 1, 2$. Expanding Eqs. (3.4.8) and (3.4.9) in a first-order Taylor series, one obtains the linearized model

$$\frac{d\hat{x}_1}{dt} = -(1 + 2\,\mathrm{Da}_1 x_{1s})\hat{x}_1 + \hat{u}_1 \tag{3.4.10}$$

$$\frac{d\hat{x}_2}{dt} = (2\,\mathrm{Da}_1 x_{1s})\hat{x}_1 - \left(1 + \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right)\hat{x}_2 + \hat{u}_2 \tag{3.4.11}$$

where $x_{1s}$, $x_{2s}$ are solutions to the steady-state equations

$$\begin{aligned} 0 &= -x_{1s} - \mathrm{Da}_1 x_{1s}^2 + 1 \\ 0 &= \mathrm{Da}_1 x_{1s}^2 - x_{2s} - \mathrm{Da}_3 (x_{2s})^{1/2} \end{aligned} \tag{3.4.12}$$

The linearized equations (3.4.10) and (3.4.11) may now be written in the form of Eq. (3.2.1), where

$$\mathbf{A} = \begin{bmatrix} -(1 + 2\,\mathrm{Da}_1 x_{1s}) & 0 \\ (2\,\mathrm{Da}_1 x_{1s}) & -\left(1 + \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right) \end{bmatrix} \tag{3.4.13}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.4.14}$$

and the test for controllability [Eq. (3.2.60)] applied. This requires that the controllability matrix

$$\mathbf{L}_c = [\mathbf{B} \vdots \mathbf{AB}] = \left[\begin{array}{cc:cc} 1 & 0 & -(1 + 2\,\mathrm{Da}_1 x_{1s}) & 0 \\ 0 & 1 & 2\,\mathrm{Da}_1 x_{1s} & -\left(1 + \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right) \end{array}\right]$$

have rank 2. Clearly this is satisfied for all steady states. Thus we may say that this nonlinear system [Eqs. (3.4.8) and (3.4.9)] is controllable.

**Example 3.4.2** Let us now determine the controllability of the problem in the previous example when we define $u_1 = c_{B_f}/c_{A_{\mathrm{ref}}}$ and $d_1 = c_{A_f}/c_{A_{\mathrm{ref}}}$, i.e., we have only one control and one disturbance. In this case the nonlinear

equations are

$$\frac{dx_1}{dt} = -x_1 - \mathrm{Da}_1 x_1^2 + d_1 \tag{3.4.15}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1^2 - x_2 - \mathrm{Da}_3 (x_2)^{1/2} + u_1 \tag{3.4.16}$$

When one linearizes around the steady state defined by $d_{1s} = 1$, $u_{1s} = 0$, the linearized equations take the form of Eq. (3.3.1), with **A** given by Eq. (3.4.13) and

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.4.17}$$

The controllability matrix [Eq. (3.2.60)] becomes

$$\mathbf{L}_c = [\mathbf{B} \vdots \mathbf{AB}] = \begin{bmatrix} 0 & \vdots & 0 \\ 1 & \vdots & -\left(1 + \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right) \end{bmatrix}$$

which clearly has only rank one. Therefore the nonlinear system [Eqs. (3.4.15) and (3.4.16)] is *not* controllable.

One should note that these two examples are completely similar in structure to Examples 3.2.5 and 3.2.7, and the conclusions regarding controllability are the same for both the linear and nonlinear systems. This emphasizes the fact that the dominant factor determining controllability is the system structure, not whether the system is linear or nonlinear. Thus, in practical problems, one may usually use "linearized" controllability tests with confidence.

As a final comment, one should note that we linearized about a *steady state* to produce linear *constant coefficient* linearized equations. Had we linearized about some nominal time-varying path $\mathbf{x}_s(t)$ (such as might be necessary in batch processes or other processes not having a steady state), then the linearized equations would have had time-varying coefficients and a nonautonomous test of controllability [Eq. (3.2.68)] would have been necessary.

Having linearized the nonlinear system and tested for controllability as illustrated above, one may now directly apply one of the multivariable controller design procedures discussed in Secs. 3.2 and 3.3. To illustrate, we shall apply modal control to a nonlinear example.

**Example 3.4.3** Let us design a modal feedback controller for the nonlinear CSTR described in Example 3.4.1. Notice that this is a nonlinear analog to the linear system discussed in Example 3.2.14. The nonlinear model is given by Eqs. (3.4.8) and (3.4.9), while the linearized state equations are given by Eqs. (3.4.10) and (3.4.11) and the output equation is

$$\mathbf{y} = \mathbf{C}\hat{\mathbf{x}} \tag{3.4.18}$$

where **C** must be chosen to provide modal decoupling. Analyzing the matrix

**A**, Eq. (3.4.13), one sees that the eigenvalues of the linearized system are

$$\lambda_1 = -(1 + 2\,\mathrm{Da}_1 x_{1s})$$
$$\lambda_2 = -\left[1 + \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right] \tag{3.4.19}$$

and by noting the similarity to Example 3.2.14, one may immediately write down an orthonormal set of right and left eigenvectors of **A**:

$$\mathbf{R} = \frac{1}{2\,\mathrm{Da}_1 x_{1s} - \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}}\begin{bmatrix} 2\,\mathrm{Da}_1 x_{1s} - (1/2)\left[\mathrm{Da}_3/(x_{2s})^{1/2}\right] & 0 \\ 2\,\mathrm{Da}_1 x_{1s} & \\ -1 & 1 \end{bmatrix} \tag{3.4.20}$$

$$\mathbf{L} = \begin{bmatrix} 2\,\mathrm{Da}_1 x_{1s} & 0 \\ 2\,\mathrm{Da}_1 x_{1s} & 2\,\mathrm{Da}_1 x_{1s} - \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}} \end{bmatrix} \tag{3.4.21}$$

Recalling the modal design procedure of Sec. 3.2, the feedback controller must take the form

$$\hat{\mathbf{u}} = -\mathbf{B}^{-1}\mathbf{R}\mathbf{K}\mathbf{y} \tag{3.4.22}$$

where **B** is given by Eq. (3.4.14), **K** is a proportional controller given by

$$\mathbf{K} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix}$$

and the outputs are given by

$$\mathbf{y} = \mathbf{L}\hat{\mathbf{x}} \tag{3.4.23}$$

In component form the controller design is

$$\hat{u}_1 = -\frac{k_{11}}{2\,\mathrm{Da}_1 x_{1s}} y_1 = -k_{11}\hat{x}_1$$
$$\hat{u}_2 = \frac{1}{2\,\mathrm{Da}_1 x_{1s} - (1/2)\left[\mathrm{Da}_3/(x_{2s})^{1/2}\right]}\left[k_{11}y_1 - k_{22}y_2\right]$$
$$= \frac{2(k_{11} - k_{22})\mathrm{Da}_1 x_{1s}}{2\,\mathrm{Da}_1 x_{1s} - (1/2)\left[\mathrm{Da}_3/(x_{2s})^{1/2}\right]}\hat{x}_1 - k_{22}\hat{x}_2 \tag{3.4.24}$$

where the outputs must be chosen as

$$y_1 = 2\,\mathrm{Da}_1 x_{1s}\hat{x}_1$$
$$y_2 = 2\,\mathrm{Da}_1 x_{1s}\hat{x}_1 + \left[2\,\mathrm{Da}_1 x_{1s} - \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right]\hat{x}_2 \tag{3.4.25}$$

Although this modal feedback control does decouple the outputs of the *linearized* system, the actual system is *nonlinear*, with closed-loop dynamics given by

$$\frac{dx_1}{dt} = -x_1 - \mathrm{Da}_1 x_1^2 + 1 - k_{11}(x_1 - x_{1s}) \tag{3.4.26}$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1^2 - x_2 - \mathrm{Da}_3 (x_2)^{1/2} + \frac{2(k_{11} - k_{22})\mathrm{Da}_1 x_{1s}}{2\mathrm{Da}_1 x_{1s} - (1/2)\left[\mathrm{Da}_3 / (x_{2s})^{1/2}\right]}(x_1 - x_{1s}) - k_{22}(x_2 - x_{2s}) \tag{3.4.27}$$

$$y_1 = 2\,\mathrm{Da}_1 x_{1s}(x_1 - x_{1s})$$

$$y_2 = 2\,\mathrm{Da}_1 x_{1s}(x_1 - x_{1s}) + \left[2\,\mathrm{Da}_1 x_{1s} - \frac{1}{2}\frac{\mathrm{Da}_3}{(x_{2s})^{1/2}}\right](x_2 - x_{2s}) \tag{3.4.28}$$

The controller performance can be illustrated by simulation of these nonlinear equations. Some results may be seen in Figs. 3.25 and 3.26 for the parameters $\mathrm{Da}_1 = 1.0$, $\mathrm{Da}_3 = 2.0$, $u_{1s} = 1.0$, and $u_{2s} = 0$, which from the solution of Eq. (3.4.12) yield $x_{1s} = 0.618$, $x_{2s} = 0.0308$. The initial conditions are $x_1 = 1.0$, $x_2 = 0$. Two cases were run: Case 1 with $k_{11} = 0.5$, $k_{22} = 1.5$, and Case 2 with $k_{11} = 5.0$, $k_{22} = 10.0$. The state variables are shown in Fig. 3.25, while the output variables are plotted in Fig. 3.26. Note

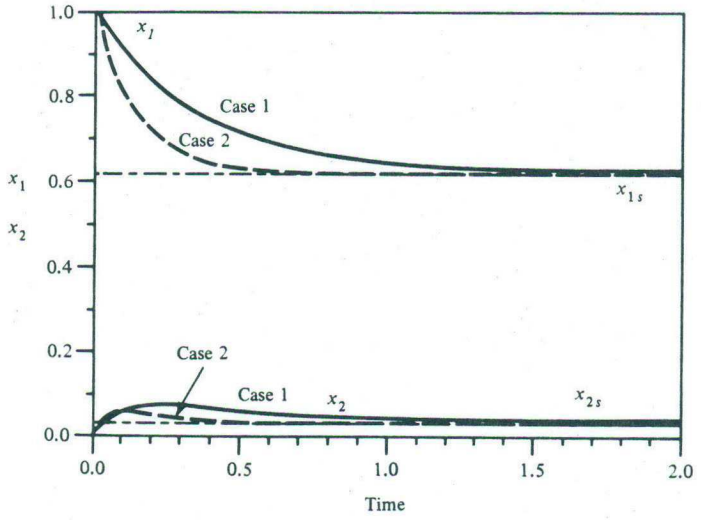


**Figure 3.25** Modal control of a nonlinear chemical reactor. Case 1: $k_{11} = 0.5$. $k_{22} = 1.5$; Case 2: $k_{11} = 5.0$, $k_{22} = 10.0$.

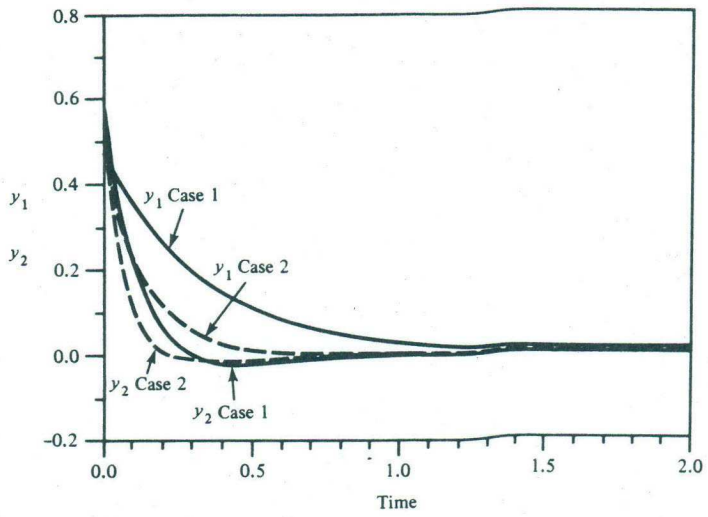


**Figure 3.26** Modal control of a nonlinear chemical reactor. Case 1: $k_{11} = 0.5$; $k_{22} = 1.5$. Case 2: $k_{11} = 5.0$; $k_{22} = 10.0$.

that the controller performs well in both cases even though the design is based on the linearized equations. As might be expected, the larger controller gains provide faster system response.

## Feedback Controller Parameterization

Although there are many special-purpose nonlinear system controller design procedures, most of these have very limited applicability and usually result from accumulated experience with a special type of nonlinear system. There is, however, one approach, termed *feedback controller parameterization*, which can be applied directly to any nonlinear system and has been found to perform well [27, 33].* The basic approach has been described previously in Sec. 3.3, and Example 3.3.4 illustrates things for a single control variable.

By defining a feedback control law

$$\mathbf{u} = \mathbf{g}(\mathbf{y}, \mathbf{b}) \tag{3.4.29}$$

with a specific structure and undetermined parameters $\mathbf{b}$, one can search for the optimal set of parameters which minimizes some objective functional such as

$$I = \mathbf{G}(\mathbf{y}(t_f)) + \int_0^{t_f} F(\mathbf{y}(t), \mathbf{u}(t))\, dt \tag{3.4.30}$$

Even though in principle there will be a different optimal set of parameters for each system initial condition, experience has shown that this dependence is often weak, and in practice one can determine a global set of optimal controller parameters. It is perhaps best to illustrate this type of control scheme by an example.

* In the control literature this approach is sometimes referred to as "specific optimal control."

**Example 3.4.4** Let us apply feedback control parameterization to the design of a multivariable proportional controller for the isothermal CSTR system treated in Example 3.4.1. We shall assume that set points are on both $x_1$ and $x_2$, and we wish a state feedback controller of the form

$$\hat{\mathbf{u}} = \mathbf{K}(\mathbf{x} - \mathbf{x}_s) \tag{3.4.31}$$

where

$$\mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{3.4.32}$$

is the multivariable proportional controller having four parameters $k_{ij}$ to be chosen optimally. The objective to be minimized is

$$I = \int_0^{t_f} \left[ (x_1 - x_{1s})^2 + \alpha(x_2 - x_{2s})^2 + \beta(\hat{u}_2)^2 \right] dt \tag{3.4.33}$$

where the last term arises because it is expensive to feed species $B$ to the reactor to achieve good control. The controls $u_1$, $u_2$ are bounded by $0 \le u_1 \le 2.0$, $0 \le u_2 \le 1.0$.

The control system synthesis involves the following steps:

1. Guess the elements of **K**.
2. Solve Eqs. (3.4.8), (3.4.9), (3.4.31), and (3.4.33).
3. Send the resultant value of $I$ to a parameter search routine and receive a new set of elements of **K**.
4. Return to step 2 and iterate.

For the parameters $\alpha = 2$, $\beta = 5$, $\mathrm{Da}_1 = 1.0$, $\mathrm{Da}_2 = 2.0$, $u_{1s} = 1.0$, $u_{2s} = 0$, $x_{1s} = 0.618$, $x_{2s} = 0.0308$, the optimal values of **K** found from the initial state $x_1 = 1.0$, $x_2 = 0$ were

$$\mathbf{K} = \begin{bmatrix} 3.0 & 0.35 \\ 0.2 & 2.0 \end{bmatrix}$$

while from the initial state $x_1 = 0.0$, $x_2 = 0.0$, the optimal proportional controller gains were

$$\mathbf{K} = \begin{bmatrix} 3.5 & 0.4 \\ 0 & 2.0 \end{bmatrix}$$

Note that there is little variation in **K** even though the initial start-up conditions are vastly different. The control system performance may be seen in Fig. 3.27. Clearly this design procedure is effective for this problem.

One disadvantage of this design procedure is that one must perform a complete dynamic simulation in order to compute $I$ at each iteration of the parameter search. Because 50 to 100 iterations are not atypical, large-scale problems could require long computing times. As yet there does not appear to be sufficient practical experience with this design procedure to determine if this is a serious limitation.

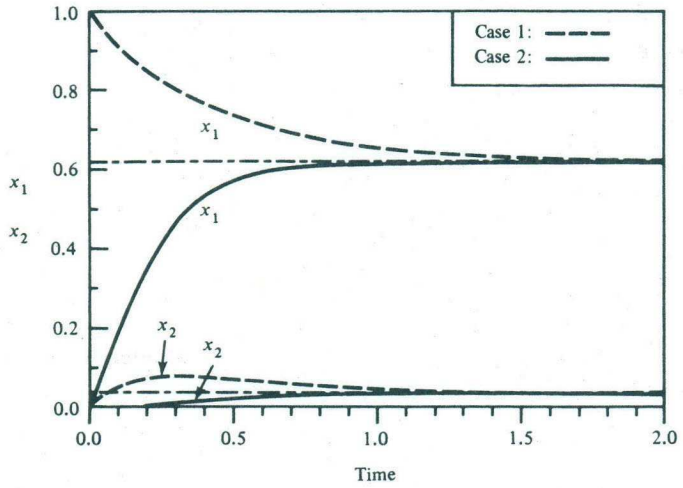


**Figure 3.27** Parameterized feedback controller performance. Case 1: $x_1(0) = 1.0$; $x_2(0) = 0$. Case 2: $x_1(0) = 0$; $x_2(0) = 0$.

## 3.5 DISCRETE TIME SYSTEMS

Although the natural process model for most dynamic systems takes the form of differential equations, there are instances when discrete time models are convenient to use. In this case the model is a difference equation, which for linear systems may be written

$$\mathbf{x}[(k+1)\,\Delta t] = \mathbf{\Phi}(k\,\Delta t)\mathbf{x}(k\,\Delta t) + \boldsymbol{\beta}\mathbf{u}(k\,\Delta t) \tag{3.5.1}$$

$$\mathbf{y}(k\,\Delta t) = \mathbf{C}\mathbf{x}(k\,\Delta t) \tag{3.5.2}$$

Discrete time models are particularly useful when implementing direct digital control (DDC) because the measurements $\mathbf{y}(k\,\Delta t)$ are taken at discrete times $t_k = k\,\Delta t$, $k = 1, 2, \ldots,$ and the controls are held piecewise constant over the interval $\Delta t$. In this case the difference equations (3.5.1) and (3.5.2) may be derived from the original differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.5.3}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{3.5.4}$$

where **A**, **B**, **C**, are constant matrices. By recalling the solution to Eq. (3.5.3),

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \tag{3.5.5}$$

we may solve for $\mathbf{x}(k\,\Delta t)$ by repeated application of Eq. (3.5.5) for piecewise constant controls to yield

$$\begin{aligned}
\mathbf{x}(\Delta t) &= e^{\mathbf{A}\,\Delta t}\mathbf{x}_0 + \int_0^{\Delta t} e^{\mathbf{A}(\Delta t-\tau)}\,d\tau\;\mathbf{B}\mathbf{u}(0) \\
\mathbf{x}(2\,\Delta t) &= e^{\mathbf{A}\,\Delta t}\mathbf{x}(\Delta t) + \int_{\Delta t}^{2\,\Delta t} e^{\mathbf{A}(2\,\Delta t-\tau)}\,d\tau\;\mathbf{B}\mathbf{u}(\Delta t) \\
\mathbf{x}[(k+1)\,\Delta t] &= e^{\mathbf{A}\,\Delta t}\mathbf{x}(k\,\Delta t) + \int_{k\,\Delta t}^{(k+1)\,\Delta t} e^{\mathbf{A}[(k+1)\,\Delta t-\tau]}\,d\tau\;\mathbf{B}\mathbf{u}(k\,\Delta t)
\end{aligned} \tag{3.5.6}$$

Thus comparing Eqs. (3.5.6) and (3.5.1), one sees that defining

$$\mathbf{\Phi} = e^{\mathbf{A}\,\Delta t} \tag{3.5.7}$$

$$\boldsymbol{\beta} = \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - \tau)}\, d\tau\, \mathbf{B} \tag{3.5.8}$$

gives the equivalent difference equation for implementation of DDC. By rearrangement of Eqs. (3.5.6), the discrete analog to Eq. (3.5.5) is

$$\mathbf{x}(k\,\Delta t) = \mathbf{\Phi}^k \mathbf{x}_0 + \boldsymbol{\beta} \sum_{i=0}^{k-1} \mathbf{\Phi}^{k-1-i} \mathbf{u}(i\,\Delta t) \tag{3.5.9}$$

The discrete equations (3.5.1) can also arise by taking the finite difference form of Eq. (3.5.3). This leads to

$$\frac{\mathbf{x}[(k+1)\,\Delta t] - \mathbf{x}(k\,\Delta t)}{\Delta t} = \mathbf{A}\mathbf{x}(k\,\Delta t) + \mathbf{B}\mathbf{u}(k\,\Delta t) \tag{3.5.10}$$

which by defining

$$\mathbf{\Phi} = (\mathbf{I} + \mathbf{A}\,\Delta t) \tag{3.5.11}$$

$$\boldsymbol{\beta} = \mathbf{B}\,\Delta t \tag{3.5.12}$$

reduces to Eq. (3.5.1). For the finite difference formulation to be valid, $\Delta t$ must be small compared with the smallest time constant of the system. By contrast, the DDC formulation [Eqs. (3.5.7) and (3.5.8)] is valid for any size $\Delta t$.

Although there is a full theory of discrete systems, involving difference equations in the time domain [34] and $z$ transforms in the transform space [35], we shall not go deeper in this direction here. The results parallel those already discussed for continuous systems, and the details are readily available to the interested reader (e.g., [6, 34, 35]).

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## PROBLEMS

**3.1** A high-priced specialty chemical is made in a batch reactor. The reactions of interest are

$$R \xrightarrow{k_1} P \tag{1}$$

$$R \xrightarrow{k_2} W \tag{2}$$

where $R$ is an expensive raw material, $P$ is the product, and $W$ is a waste byproduct. Both reactions (1) and (2) are irreversible and first-order in species $R$. The velocity constants $k_1$ and $k_2$ are given by

$$k_i = A_i e^{-E_i/RT} \qquad i = 1, 2$$

Material balances on species $R$ and $P$ in the batch reactor are

$$\frac{dR}{dt} = -(k_1 + k_2)R \qquad R(0) = R_0$$

$$\frac{dP}{dt} = k_1 R \qquad P(0) = 0$$

$$W = R_0 - R - P$$

As a control engineer you have been asked to design a temperature control program for the batch reactor to be carried out over a 1-h batch time and which maximizes the amount of $P$ produced at the end of the run. The mathematics can be simplified by letting

$$x_1 = \frac{R}{R_0} \qquad x_2 = \frac{P}{R_0} \qquad u = k_1 \qquad p = \frac{E_2}{E_1}$$

$$\alpha = \frac{A_2}{(A_1)^p}$$

so that $k_2 = \alpha u^p$. With these substitutions, the optimal control problem is to choose $u(t)$, $0 \le t \le 1$,

such that $x_2|_{t=1}$ is maximized and

$$\frac{dx_1}{dt} = -(u + \alpha u^p)x_1 \qquad x_1(0) = 1$$

$$\frac{dx_2}{dt} = ux_1 \qquad x_2(0) = 0$$

Note that it is permissible to use $u$ as the control variable, because

$$u = k_1 = A_1 e^{-E_1/RT}$$

yields a monotonic relationship between $u(t)$ and $T(t)$.

For the parameters $p = 2.0$, $\alpha = 0.5$, $A_1 = 10^6\ \mathrm{s}^{-1}$, $E_1 = 10{,}000$ cal/(g)(mol), and for temperature constraints of the form $0 \le u(t) \le 5$, determine the optimal open-loop temperature program $u(t) \Leftrightarrow T(t)$ in three different ways:

(*a*) Apply the maximum principle in order to calculate the optimal control using a gradient procedure. Determine the optimal program from two initial guesses of $u(t)$.

(*b*) Apply the control vector parameterization procedure to the problem by finding the optimal set of coefficients $a_0$, $a_1$, $a_2$ in the expression

$$u(t) = a_0 + a_1 t + a_2 t^2$$

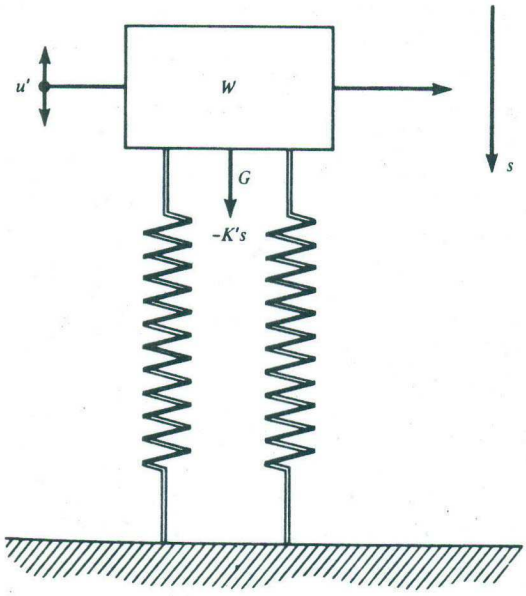
Use a multivariable search routine (see Ref. [16]) to determine the optimal values of $a_0$, $a_1$, and $a_2$. Compare your results with those found by the gradient method.

(*c*) Determine an optimal feedback control law of the form

$$u(x_1) = b_0 + b_1 x_1 + b_2 x_1^2$$

by using a multivariable search routine to determine the optimal controller parameters $b_0$, $b_1$, $b_2$. Compare with the results of parts (*a*) and (*b*).

**3.2** A simple suspension system is to be designed for a high-speed rapid transit vehicle. A simple laboratory model of the system is sketched in the figure below, where springs (spring constant $K'$) are used to cushion the vertical motions in the absence of feedback control. However, it is desired to design a feedback controller to improve the stability and performance of the suspension system. To accomplish this, a force $u'(t)$ may be applied to the bouncing system.



The modeling equations are

$$W\frac{ds^2}{dt} = -K's + G + u'$$

where $G$ is the force of gravity and $W$ is the mass of the system. By noting that at steady state ($u' = 0$)

$$0 = -K's_s + G$$

we may define normalized deviation variables and parameters

$$x_1 = s - s_s \qquad x_2 = \dot{s} \qquad u = \frac{u'}{W} \qquad K = \frac{K'}{W}$$

to yield the model

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -Kx_1 + u$$

Based on this information:

(*a*)Make use of the results in Sec. 3.3 to find the optimal linear quadratic feedback control law which minimizes

$$I = \int_0^{t_f}(x_1^2 + \alpha x_2^2 + \beta u^2)\,dt$$

Carry out the computations for $\alpha = 1$, $\beta = 0.25$, $K = 3$, $t_f = 2.0$, $x_1(0) = 1.0$, $x_2(0) = 1.0$.

(*b*) The traditional feedback controller for suspension systems is the shock absorber, which has the form $u = -C\dot{x}_1$ (pure derivative action). Compare your "optimal controller" with this traditional one.

**3.3** Carry out the computations to determine the near-optimal linear quadratic feedback controller for the problem discussed in Example 3.3.7. Use the following parameters: $a = 1000$, $\gamma = 10$, $x_d = 1.3$, $\bar{x}(0) = 1.5$, $t_f = 2.0$, and $\alpha = 0.25$. After the optimal open-loop policy has been found for $x(0) = 1.5$, determine the near-optimal feedback controller performance for $x(0) = 1.4$ and $x(0) = 1.6$. How does the controller perform for this highly nonlinear system?

**3.4** Consider the tank network shown in the figure below. Liquid flows from one tank to the next, with the outflow of each tank assumed to be proportional to the level in that tank. The outlet of the second tank is split, with a fraction $F$ exiting and the remainder pumped back to the first tank. The modeling equations can be assumed as

$$\frac{dh_1}{dt} = u_1 - \alpha h_1 + \beta h_2$$

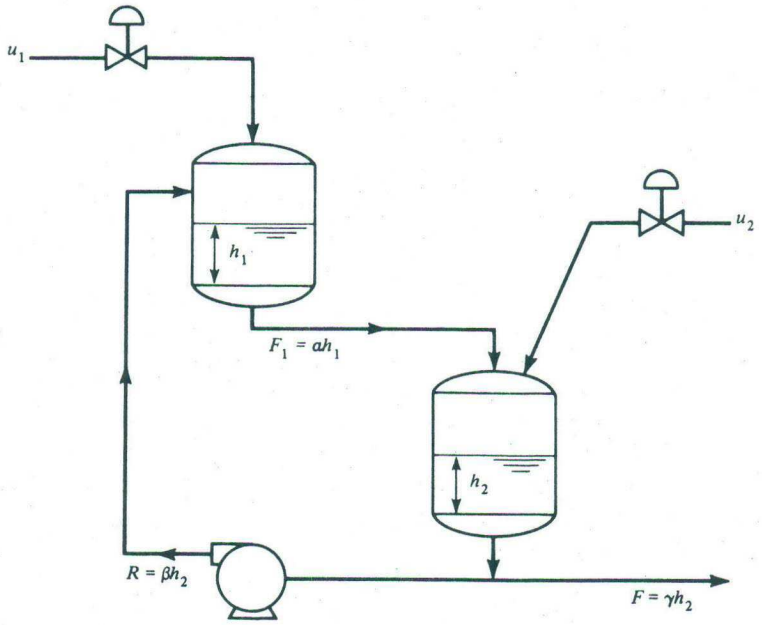
$$\frac{dh_2}{dt} = u_2 + \alpha h_1 - (\beta + \gamma)h_2$$

It is desired to control the liquid levels in these tanks by choosing a controlled inlet flow position (either $u_1$ or $u_2$) as shown. As a means of aiding your choice,

(*a*) Determine if the controllability conditions are satisfied for $u_1$ alone (valve $u_2$ at constant flow rate) and for $u_2$ along (valve $u_1$ at constant flow rate).

(*b*) Are there other practical conditions to be considered which are not treated by formal controllability conditions, but which may influence your choice? Discuss these.

(*c*) If the recycle loop were removed, modify the modeling equations, and reevaluate controllability as in part (*a*).




**3.5** For the vehicle suspension problem discussed in Prob. 3.2, develop a transform-domain representation between the measured vertical height $y = x_1$ and control force $u$, i.e.,

$$\bar{y}(s) = g(s)\bar{u}(s)$$

Determine $g(s)$. [Hint: See Eq. (3.3.50) and the accompanying discussion.]

**3.6** You have been given the multivariable control system

$$\bar{\mathbf{y}}(s) = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} \qquad \bar{\mathbf{u}}(s) = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} \qquad \mathbf{G}(s) = \begin{bmatrix} \dfrac{3}{1+10s} & \dfrac{1}{1+20s} \\ \dfrac{2}{1+25s} & \dfrac{4}{1+5s} \end{bmatrix}$$

in the Laplace domain. Convert this to an equivalent set of ordinary differential equations of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

**3.7** Consider the water tank system of Prob. 3.4 with both control valves $u_1$, $u_2$ available. Carry out the following types of controller designs:

(*a*) Two single-loop proportional controllers.

(*b*) Modal feedback control.

(*c*) Set-point compensation for the controller in part (*a*).

(*d*) Complete dynamic noninteracting control. Compare the controller responses (interaction, offset, etc.) and the relative ease of implementation of the controllers.

As test disturbances, consider a unit step change in the $h_1$ set point while endeavoring to keep $h_2$ constant. Conversely, consider a unit step change in $h_2$ while $h_1$ is held constant. Make use of the following parameters: $\alpha = 3$, $\beta = 2$, $\gamma = 2$. For each design, choose the more convenient system representation (i.e., time domain or transform domain).

**3.8** Pulse testing of a distillation column yields the following dynamic model between product concentrations $y_1, y_2$ and product drawoff rates $u_1, u_2$:

$$\bar{\mathbf{y}}(s) = \mathbf{G}(s)\bar{\mathbf{u}}(s)$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \mathbf{G}(s) = \begin{bmatrix} \dfrac{1.0}{1+10s} & \dfrac{0.3}{1+8s} \\ \dfrac{0.2}{1+7s} & \dfrac{0.5}{1+5s} \end{bmatrix}$$

(*a*) Convert the above *transform state space model* to a *time-domain model* in the form

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx}$$

Is this a minimal realization in state space?

(*b*) Is the distillation column *output controllable*? Why?

(*c*) Write down the block diagram and the design equations for a noninteracting proportional feedback controller which eliminates *steady-state* interactions. What is the closed-loop transfer function between y and set point $\mathbf{y}_d$?

**3.9** Extend the linear quadratic optimal feedback control law of Sec. 3.3 to include *optimal feedforward control* when measured disturbances are included in the model; i.e., find the optimal feedback-feedforward control law for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{Dd} \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

where **D** is an $n \times k$ matrix and **d** is a $k$ vector of disturbances. (*Hint:* see Ref. [15]).

**3.10** Consider the nonisothermal CSTR in which the exothermic reaction $A \rightarrow B$ is being carried out. The modeling equations in dimensionless form may be written

$$\frac{dx_1}{dt} = -x_1 + \mathrm{Da}(1 - x_1)e^{x_2} + u_1$$

$$\frac{dx_2}{dt} = -(1 + \beta)x_2 + B\,\mathrm{Da}(1 - x_1)e^{x_2} + \beta u_2$$

(*a*) Determine if this system is controllable when operated around the steady state $x_{1s}, x_{2s}$ resulting from $u_{1s} = u_{2s} = 0$.

(*b*) Design a proportional feedback control system which contains a steady-state noninteracting compensator. Begin by linearization around the steady-state noted in (*a*).

(*c*) Simulate the reactor for the parameters $\mathrm{Da} = 0.1$, $\beta = 3.0$, $B = 19$, and demonstrate your controller performance when applied to the nonlinear system.

**3.11** Consider the CSTR described in Example 3.2.5 with continuous modeling equations

$$\frac{dx_1}{dt} = -(1 + \mathrm{Da}_1)x_1 + u_1$$

$$\frac{dx_2}{dt} = \mathrm{Da}_1 x_1 - (1 + \mathrm{Da}_3)x_2 + u_2$$

It is planned to implement $P + I$ control relating $u_1$ to $x_{1d} - x_1$ and $u_2$ to $x_{2d} - x_2$ under DDC mode with sampling time $\Delta t$. Here $x_{1d}, x_{2d}$, are the state set points. Convert the continuous model to the equivalent discrete model necessary for DDC design. Draw the block diagram and specify final controller design equations.