Chapter 3

Nonlinear Systems Theory

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In this chapter, we introduce the machinery of differential geometry and related concepts as analysis tools for nonlinear process control systems. This chapter will also serve as background for the nonlinear controller synthesis material that is discussed in Chapter 5. There are several texts and review articles which pursue the details of the differential geometric approach in greater depth than is presented here. Notable are the texts by Isidori [18] and Nijmeijer and Van der Schaft [25]. In the process control area, there is a thorough treatment of the material in the tutorial article by Kravaris and Kantor [21]; in addition, there is a good overview of general nonlinear approaches to process control system design by Bequette [1].

3.1 Introduction

Throughout this chapter, the discussion is restricted to the class of nonlinear systems which are linear with respect to the manipulated input (control-affine systems); more general nonlinear systems are discussed, for example, in [25]. In the single-input single-output, time invariant case, the nonlinear state-space model can be written as follows:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,\ldots,x_n) + g_1(x_1,\ldots,x_n)u \\ &\vdots \\ \dot{x}_n &= f_n(x_1,\ldots,x_n) + g_n(x_1,\ldots,x_n)u \\ y &= h(x_1,\ldots,x_n)\end{aligned}$$

where $x_1,\ldots,x_n$ are state variables, u is the manipulated input, and y is the controlled output. For simplicity, the nonlinear functions $f_1,\ldots,f_n$, $g_1,\ldots,g_n$, and h are assumed to C^∞ functions; *i.e.*, their partial derivatives of any order exist and are continuous. This equation can be written in the following, more compact, vector form:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned} \tag{3.1}$$

We will make frequent comparisons to the linear approximation of this dynamical system, for which we will employ the standard state-space notation [4]:

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned} \tag{3.2}$$

3.2 Concepts from Differential Geometry

The material presented in this chapter draws largely from the field of differential geometry. There has been an explosive growth of research in this area over the past decade, and in this chapter we will attempt to lay the groundwork to enable an understanding of those results. In effect, these tools can provide both a means for nonlinear system *analysis* as well as nonlinear controller *synthesis* (the latter is the subject of Chapter 5). In the analysis context, these tools can be used to address questions such as: (i) does the system have a stable inverse?; (ii) are particular states reachable from the initial conditions?; and (iii) are outputs which result from different inputs distinguishable? In the synthesis context, the primary issue will be the design of a nonlinear change of coordinates and nonlinear feedback that make the system behave in a linear manner.

3.2.1 Manifolds

A manifold is a topological space, usually denoted M, which has special properties [27] that are useful for the results that follow. Most notably, a manifold is locally Euclidean. Consider the mapping of a point p in some neighborhood U of M to a point $\phi(p)$ in some open subset of $\mathcal{R}^n$. The mapping ϕ and its inverse (ϕ^{-1}) are assumed to be C^∞ functions. We can define a *coordinate chart* as the pair (U, ϕ). It is often useful to represent ϕ as a set $(\phi_1, \ldots, \phi_n)$, where $\phi_i : U \rightarrow \mathcal{R}$ is called the ith *coordinate function.* The set of real numbers $(\phi_1(p), \ldots, \phi_n(p))$ is called the set of *local coordinates* of p in the coordinate chart (U, ϕ).

3.2.2 Vector Fields

A *vector map* associates a point $x = (x_1, \ldots, x_n)$ on an open subset of $\mathcal{R}^n$ with the vector:

$$f(x_1, \ldots, x_n) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{pmatrix} \tag{3.3}$$

in $\mathcal{R}^m$. A scalar map or *function* merely maps some open subset of $\mathcal{R}^n$ to $\mathcal{R}$. A *vector field,* $f(x)$, on $\mathcal{R}^n$ is the mapping which assigns to every point $p \in M$ a tangent vector $f(p)$ in the tangent space to M. A vector mapping, f, is a *global diffeomorphism* if [18]: (i) f is invertible for all $x \in \mathcal{R}^n$, and (ii) f and its inverse (f^{-1}) are C^∞ functions. If these properties are only locally valid in the neighborhood of an equilibrium point, then f is called a *local diffeomorphism.*

3.2.3 Inverse and Implicit Function Theorems

The following two theorems will prove useful in the derivations that follow. In particular, they will find application in guaranteeing the existence of solutions to nonlinear algebraic equations.

Theorem 1 (Implicit Function Theorem) *[18] Let $A \subset \mathcal{R}^m$ and $B \subset \mathcal{R}^n$ be open sets. Let $F : A \times B \to \mathcal{R}^n$ be a C^∞ mapping. Let $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ denote a point of $A \times B$. Suppose that for some $(x_0, y_0) \in A \times B$*

$$F(x_0, y_0) = 0$$

and the matrix

$$\frac{\partial F}{\partial y} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

is nonsingular at (x_0, y_0). Then there exists open neighborhoods $A_0 \subset A$ of x_0 and $B_0 \subset B$ of y_0 and a unique C^∞ mapping $G : A_0 \to B_0$ such that

$$F(x, G(x)) = 0$$

for all $x \in A_0$.

Thus, the implicit function theorem provides sufficient conditions that guarantee the existence of a local solution $y = G(x)$ to the nonlinear algebraic equation $F(x, y) = 0$.

Theorem 2 (Inverse Function Theorem) *[18] Let A be an open set of $\mathcal{R}^n$ and $F : A \to \mathcal{R}^n$ a C^∞ mapping. If $\left[\frac{\partial F}{\partial x}\right]_{x_0}$ is nonsingular at some $x_0 \in A$, then there exists an open neighborhood $U \subset A$ of x_0 such that $V = F(U)$ is open in $\mathcal{R}^n$ and the restriction of F to U is a diffeomorphism onto V.*

The inverse function theorem provides sufficient conditions which guarantee the existence of a locally-defined inverse function F^{-1} for the function F.

3.2.4 The Lie Algebra

The derivative of a smooth (*i.e.*, C^∞) function $\lambda(x)$ along a smooth vector field f is defined to be the *Lie Derivative.* The notation we use is $(L_f\lambda)(p) = (f(p))(\lambda)$; *i.e.*, the Lie Derivative is equal to the value of the tangent vector

$f(p)$ at the point p. In local coordinates, one can represent this operator in the following manner:

$$L_f\lambda(x_1,\ldots,x_n) = \begin{pmatrix} \frac{\partial\lambda}{\partial x_1} & \cdots & \frac{\partial\lambda}{\partial x_n} \end{pmatrix} \begin{pmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_n(x_1,\ldots,x_n) \end{pmatrix}$$

The following notation is used for repeated Lie Derivatives: $L_f(L_f\lambda(x)) = L_f^2\lambda(x)$.

Consider the vector space $V(M)$ of all smooth vector fields on a manifold M. In addition, consider the following binary operation (product) on V : $[V,V] \equiv V \times V \to V$. Assume that the binary operator satisfies the following properties:

1. The operator is skew commutative: $[v_1, v_2] = -[v_2, v_1]$.
2. The operator is bilinear over $\mathcal{R}$: $[\alpha_1 v_1 + \alpha_2 v_2, v_3] = \alpha_1[v_1, v_3] + \alpha_2[v_2, v_3]$, where $\alpha_1, \alpha_2 \in \mathcal{R}$.
3. The operator satisfies the Jacobi Identity: $([v_1,[v_2,v_3]] + [v_2,[v_3,v_1]] + [v_3,[v_1,v_2]]) = 0$.

In this case, the vector space $V(M)$ is a *Lie Algebra* with some very interesting structural properties. This binary operator on $V(M)$ is defined as the *Lie Bracket*. For two vector fields $f(x), g(x) \in V(M)$:

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \tag{3.4}$$

In local coordinates, this is given by:

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \tag{3.5}$$

The notation that is typically employed for this operator is as follows:

$$\begin{aligned} ad_f^0 g(x) &\equiv g(x) \\ ad_f^1 g(x) &\equiv [f,g](x) \\ ad_f^2 g(x) &\equiv [f,[f,g]](x) = [f, ad_f^1 g](x) \\ &\vdots \end{aligned}$$

It is interesting to note the interpretation of the Lie Derivative and the Lie Bracket for a linear dynamical system. In this case, $f(x) = Ax$, $g(x) = b$, and $h(x) = cx$. The repeated Lie Derivative of the output function $h(x)$ along the $f(x)$ vector is given by $L_f^k h(x) = cA^k x$. Thus, one obtains the rows of the observability matrix of the linear system. If one applies the Lie Derivative to the output function along $g(x)$, preceded by repeated derivatives along $f(x)$, one obtains $L_g L_f^k h(x) = cA^{k-1}b$; in other words, the Markov parameters for the linear system. Finally, consider the Lie Bracket of the $f(x)$ and $g(x)$ vector fields: $ad_f^k g(x) = (-1)^k A^k b$. In this case, we obtain the columns of the controllability matrix of the linear system multiplied by $(-1)^k$.

3.2.5 Coordinate Transformations

A general objective in the synthesis of feedback linearizing controllers (see Chapter 5) is the derivation of coordinate transformations which convert the original nonlinear system into a system that is "simpler" in the sense that controller synthesis is more straightforward. For example, a linear system is "simpler" than a nonlinear system. Before presenting the main result for this problem (the Frobenius Theorem), a few background comments on nonlinear coordinate transformations will be given.

Using the inverse function theorem in Section 3.2.3, one can show that a nonlinear coordinate transformation, $z = \Phi(x)$ from $\mathcal{R}^n$ to $\mathcal{R}^n$, is invertible if and only if the Jacobian matrix:

$$\begin{pmatrix} \frac{\partial \Phi_1}{\partial x_1}(x) & \cdots & \frac{\partial \Phi_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial x_1}(x) & \cdots & \frac{\partial \Phi_n}{\partial x_n}(x) \end{pmatrix} \tag{3.6}$$

is invertible. Under this invertible coordinate transformation, a scalar function $h(x)$ is mapped to a new function:

$$\tilde{h}(z) = [h(x)]_{x=\Phi^{-1}(z)} \tag{3.7}$$

and a vector field $g(x)$ is mapped to the new vector field:

$$\begin{aligned} \tilde{g}_1(z) &= [< d\Phi_1(x), g(x) >]_{x=\Phi^{-1}(z)} \\ &\vdots \\ \tilde{g}_n(z) &= [< d\Phi_n(x), g(x) >]_{x=\Phi^{-1}(z)} \end{aligned} \tag{3.8}$$

where $< \cdot, \cdot >$ denotes the inner product.

For nonlinear systems, the following interpretation of the Lie Bracket is useful. Recall that the integral curve σ of f is the solution to the equation: $\frac{d\sigma}{dt} = f(\sigma(t))$. The Lie Bracket "measures" the extent to which the integral curves of f and g can be used to form the coordinates of a system. For example [27], consider the integral curve of f starting from a point p for some time δt. From the new location, follow the integral curve of g for time δt. From the new location, reverse the integral curve of f, and follow this for time δt. Finally, follow the reverse of the integral curve of g for time δt. If we denote the integral curve of f by ϕ and the integral curve of g by ψ, then the final location of the curve described above is denoted by $c(t) = \psi_{-\delta t} \circ \phi_{-\delta t} \circ \psi_{\delta t} \circ \phi_{\delta t}(p)$. The Lie Bracket provides insight into the ability to close this curve under the operations on the integral curves of f and g, thus validating their usefulness as coordinate lines for the system. One can show [27] that $\ddot{c}(0)$ is equal to $2[f, g]$ evaluated at p. Furthermore, if $[f, g] = 0$ in a neighborhood of p, then $c(t) = p \; \forall \; t$, *i.e.*, the curve is closed [27].

3.2.6 Distributions and Frobenius Theorem

Before introducing the main result of this section, a few definitions are required. First, the concept of a distribution is formalized:

Definition 1 (Distribution) *A distribution D is the vector space which consists of the span of some vectors f on some open set U of $\mathcal{R}^n$:*

$$D(x) \;=\; \text{span } \{f_1(x), \ldots, f_d(x)\}$$

A distribution D is non-singular if there exists an integer d such that $dim(D(X)) = d$ for all $x \in \mathcal{R}^n$

An involutive distibution is defined as follows:

Definition 2 (Involutivity) *A distribution D is involutive if for all vector fields $\tau_1, \tau_2 \in D$ the following holds: $[\tau_1, \tau_2] \in D$.*

A related property used later in this chapter is the notion of invariance for a distribution:

Definition 3 (Invariant Distribution) *A distribution D is invariant under a vector field f if for all vector fields $\tau \in D$ the following holds: $[\tau, f] \in D$.*

The Frobenius Theorem is concerned with solvability of a particular set of partial differential equations that arise in the derivation of coordinate transformations that "simplify" or "straighten out" a given nonlinear system. To see this, consider the transformation of the vector fields given in the previous section in (3.8). In order for the transformed vector field $\tilde{g}$ to align with e_n, the nth coordinate of the natural basis for linear coordinate systems, one requires that the first $n-1$ elements of $\tilde{g}$ vanish. In other words:

$$\frac{\partial \Phi_i(x)}{\partial x} g(x) = 0 \quad \text{for } i = 1, \ldots, n-1 \tag{3.9}$$

Thus one seeks $n-1$ independent solutions to this set of partial differential equation. When this is possible, one says that the distribution spanned by the single vector field, $\{g(x)\}$, is *completely integrable.* In a more general setting (the state feedback linearization problem), one is interested in the guaranteeing the *integrabilty* of a general distribution, $\{g_1(x), g_2(x), \ldots, g_d(x)\}$. For that problem, one seeks $n-d$ independent solutions of the following set of partial differential equations:

$$\frac{\partial \Phi_i(x)}{\partial x} [g_1(x), g_2(x), \ldots, g_d(x)] = 0 \quad \text{for } i = 1, \ldots, n-d \tag{3.10}$$

The key result which links this problem to the coordinate transformation construction results in the last section is given by the Frobenius Theorem [18]:

Theorem 3 (Frobenius Theorem) *A nonsingular distribution is completely integrable if and only if it is involutive.*

The proof of this theorem is rather involved, and the interested reader is referred to [18, 27] for the details.

3.2.7 Example – Continuous Fermentor

Many continuous fermentators can be described by the following unstructured model [14]:

$$\dot{X} = -DX + \mu X \tag{3.11}$$

$$\dot{S} = D(S_F - S) - \frac{1}{Y_{X/S}} \mu X \tag{3.12}$$

$$\dot{P} = -DP + (\alpha\mu + \beta)X \tag{3.13}$$

where X is the biomass concentration, S is the substrate concentration, P is the product concentration, D is the dilution rate, S_F is the feed substrate concentration, μ is the specific growth rate, $Y_{X/S}$ is the cell-mass yield, and α and β are kinetic parameters. The specific growth rate is modeled as:

$$\mu = \frac{\mu_m(1-\frac{P}{P_m})S}{K_m + S + \frac{S^2}{K_i}} \tag{3.14}$$

where μ_m is the maximum specific growth rate, P_m is the product saturation constant, K_m is the substrate saturation constant, and K_i is the substrate inhibition constant.

Consider the problem of controlling the productivity ($y = DP$) using the feed substrate concentration as the manipulated variable ($u = S_f$). In order to check the involutivity condition for this system, we will define the state vector $x^T = [X \;\; S \;\; P]$. Thus, the vector fields of interest are:

$$f(x) = \begin{pmatrix} -Dx_1 + \mu(x)x_1 \\ -Dx_2 - \frac{1}{Y_{X/S}}\mu(x)x_1 \\ -Dx_3 + [\alpha\mu(x) + \beta]x_1 \end{pmatrix} \tag{3.15}$$

$$g(x) = \begin{pmatrix} 0 \\ D \\ 0 \end{pmatrix} \tag{3.16}$$

where:

$$\mu = \frac{\mu_m(1-\frac{x_3}{P_m})x_2}{K_m + x_2 + \frac{x_2^2}{K_i}} \tag{3.17}$$

Furthermore, we can check that:

$$ad_f^0 g(x) = g(x) = \begin{pmatrix} 0 \\ D \\ 0 \end{pmatrix} \tag{3.18}$$

$$ad_f^1 g(x) = \frac{\partial g}{\partial x}f(x) - \frac{\partial f}{\partial x}g(x) = \begin{pmatrix} \frac{\partial\mu(x)}{\partial x_2}x_1 \\ -D - \frac{1}{Y_{X/S}}\frac{\partial\mu(x)}{\partial x_2}x_1 \\ \alpha\frac{\partial\mu(x)}{\partial x_2} \end{pmatrix} \tag{3.19}$$

To demonstrate involutivity, we need to show that $[ad_f^0 g(x), ad_f^1 g(x)]$ lies in the distribution spanned by $\left\{ad_f^0 g(x), ad_f^1 g(x)\right\}$:

$$[ad_f^0 g(x), ad_f^1 g(x)] = \begin{pmatrix} \frac{\partial^2 \mu(x)}{\partial x_2^2} x_1 \\ -\frac{1}{Y_{X/S}} \frac{\partial^2 \mu(x)}{\partial x_2^2} x_1 \\ \alpha \frac{\partial^2 \mu(x)}{\partial x_2^2} \end{pmatrix} \quad (3.20)$$

Through straightforward, though tedious, algebra one can show that the involutivity condition holds for $\forall\, x \in M$ where:

$$M \equiv \left\{x : x_2 \neq \sqrt{K_m K_i}\right\} \quad (3.21)$$

It is interesting to note that the point $x_2 = \sqrt{K_m K_i}$ corresponds to the optimum substrate concentration (*i.e.*, the substrate concentration corresponding to the maximum productivity).

3.3 Nonlinear Inversion

3.3.1 Linear Systems

Initially, we turn to linear systems analysis to develop concepts that will be useful later. Consider the state-space representation of the single-input single-output, linear system in (3.2). An equivalent input-output representation for this system is the following transfer function:

$$\frac{y(s)}{u(s)} = c(sI - A)^{-1} b = \frac{c\text{Adj}(sI - A)^{-1} b}{\det(sI - A)} \quad (3.22)$$

The *pole-zero excess* for this operator is defined to be the order of the denominator polynomial minus the order of the numerator polynomial. This quantity is also called the *relative degree* and can range from 1 to n. The relative degree also can be calculated from the Markov parameters, which represent the coefficients of the terms in a series expansion of the transfer function:

$$c(sI - A)^{-1} b = \frac{cb}{s} + \frac{cAb}{s^2} + \frac{cA^2 b}{s^3} + \cdots$$

Definition 4 (Relative Degree) *The relative degree of the system (3.2) is the smallest integer r for which $cA^{r-1} b \neq 0$.*

The relative degree also can be interpreted in the context of successive derivatives of the linear system output (y):

$$\begin{aligned}
\dot{y} &= cAx \\
\ddot{y} &= cA^2x \\
&\vdots \\
\frac{d^{r-1}y}{dt^{r-1}} &= cA^{r-1}x \\
\frac{d^r y}{dt^r} &= cA^r x + cA^{r-1}bu
\end{aligned}$$

From this analysis, it is clear that the relative degree corresponds to the lowest order derivative of the output that depends explicitly on the input. This equivalent definition will be utilized in defining the relative degree for a nonlinear system of the form in (3.1).

Next, we define a *normal form* representation for the linear system given in (3.2). A linear state transformation is sought which represents the dynamics in an insightful and compact form. Consider the following transfer function:

$$G(s) = k\frac{\beta_0 + \beta_1 s + \cdots + \beta_{n-r-1}s^{n-r-1} + s^{n-r}}{\alpha_0 + \alpha_1 s + \cdots + \alpha_{n-1}s^{n-1} + s^n} \tag{3.23}$$

A minimal state-space realization of this system is represented by the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k \end{pmatrix}$$

$$c = (\beta_0 \quad \beta_1 \quad \cdots \quad \beta_{n-r-1} \quad 1 \quad 0 \quad \cdots 0) \tag{3.24}$$

To obtain a suitable normal form, the following transformation is proposed:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{pmatrix} = Tx = \begin{pmatrix} cx \\ cAx \\ cA^2x \\ \vdots \\ cA^{r-1}x \\ x_1 \\ \vdots \\ x_{n-r} \end{pmatrix} \tag{3.25}$$

It is straightforward to check that this coordinate transformation is nonsingular, and leads to a transformed system of the following form [18]:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= R\xi + S\eta + ku \\ \dot{\eta} &= P\xi + Q\eta \\ y &= \xi_1 \end{aligned} \tag{3.26}$$

where R and S are row vectors and P and Q are matrices. In this structure, the rth order subsystem representing the direct effect of the manipulated variable, u, on the output, y, is hierarchically decomposed. This structure will prove useful for nonlinear systems, and in that context is referred to as the Byrnes-Isidori Normal Form. The matrix Q has special significance: its eigenvalues are exactly equal to the zeros of the transfer function given in Equation (3.23).

It is relatively straightforward to construct an inverse for the linear system (3.2), such that the cascade of the system and the inverse system form the identity operator. Using the original state-space representation in (3.2), an inverse can be realized as:

$$\begin{aligned} \dot{Z} &= \left(A - \frac{bcA^r}{cA^{r-1}b}\right) Z + \frac{b}{cA^{r-1}b}\frac{d^r y}{dt^r} \\ u &= -\frac{cA^r}{cA^{r-1}b} Z + \frac{1}{cA^{r-1}b}\frac{d^r y}{dt^r} \end{aligned} \tag{3.27}$$

where Z is the state vector of the inverse system. Note that the inverse system takes the rth derivative of y as an input, and generates the value of u that originally was introduced to the plant. From the transfer function operator for the system, it is clear that the inverse system can have a minimal order of r. However, the previously derived operator has a dynamic order of n, and therefore it is not a minimal realization.

Now consider the derivation of an inverse system using the normal form representation in (3.26). In this case, it is easy to show that an inverse can be realized as:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= \frac{d^r y}{dt^r} \\
\begin{pmatrix} \dot{z}_{r+1} \\ \vdots \\ \dot{z}_n \end{pmatrix} &= P \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} + Q \begin{pmatrix} z_{r+1} \\ \vdots \\ z_n \end{pmatrix} \\
u &= \frac{1}{k} \left(\frac{d^r y}{dt^r} - R \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} - S \begin{pmatrix} z_{r+1} \\ \vdots \\ z_n \end{pmatrix} \right)
\end{aligned} \tag{3.28}$$

However, this system can be represented in a more compact form as:

$$\begin{aligned}
\begin{pmatrix} \dot{\zeta}_1 \\ \vdots \\ \dot{\zeta}_{n-r} \end{pmatrix} &= P \begin{pmatrix} y \\ \vdots \\ \frac{d^{r-1}y}{dt^{r-1}} \end{pmatrix} + Q \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n-r} \end{pmatrix} \\
u &= \frac{1}{k} \left(\frac{d^r y}{dt^r} - R \begin{pmatrix} y \\ \vdots \\ \frac{d^{r-1}y}{dt^{r-1}} \end{pmatrix} - S \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n-r} \end{pmatrix} \right)
\end{aligned} \tag{3.29}$$

where we have introduced the following change of variables: $[\zeta_1, \ldots, \zeta_{n-r}] = [z_{r+1}, \ldots, z_n]$. Note that the derived inverse system has order r, and thus it represents a *minimal* realization.

3.3.2 Nonlinear Normal Form

The notion of relative degree, introduced earlier for linear systems, can be extended in a natural way to include nonlinear systems [16]:

Definition 5 (Relative Degree) *The relative degree of the nonlinear system (3.1) is the smallest integer r for which $L_g L_f^{r-1} h(x) \neq 0$ and $L_g L_f^{r-2} h(x) = 0 \ \forall \ x$ in some neighborhood of the defined operating point x_0.*

A useful interpretation of the relative degree for nonlinear systems can be obtained by calculating derivatives of the output:

$$
\begin{aligned}
\dot{y} &= \frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} = L_f h(x) \\
\ddot{y} &= L_f^2 h(x) \\
&\vdots \\
\frac{d^{r-1}y}{dt^{r-1}} &= L_f^{r-1} h(x) \\
\frac{d^r y}{dt^r} &= L_f^r h(x) + L_g L_f^{r-1} h(x) u
\end{aligned}
$$

As with linear systems, the relative degree characterizes the lowest order derivative of the output, y, that is explicitly dependent on the input, u. It is important to note that there may exist points, x_0, at which the relative degree is not well defined. For example, consider a one-dimensional system for which $L_g L_f^{r-1} h(x) = x$. In this case, the relative degree is not well defined as the point $x_0 = 0$. The presence of such *singular points* complicates the subsequent analysis considerably [17, 18], and therefore is not considered here.

Using the definition of the relative degree, one can derive a nonlinear analog of the normal form in (3.26). This system, called the *Byrnes-Isidori* normal form [3], is derived using the following nonlinear coordinate transformation:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ t_1(x) \\ \vdots \\ t_{n-r}(x) \end{pmatrix} \tag{3.30}$$

The $t_i(x)$ functions are obtained as the solution to the partial differential equation:

$$L_g t(x) = 0$$

As long as the relative degree is well defined, the existence of $t_i(x)$ which satisfy this equation is guaranteed [18]. This coordinate transformation produces the Byrnes-Isidori normal form:

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= \alpha(\xi,\eta) + \beta(\xi,\eta)u \\ \dot{\eta}_1 &= q_1(\xi,\eta) \\ &\vdots \\ \dot{\eta}_{n-r} &= q_{n-r}(\xi,\eta) \\ \\ y &= \xi_1 \end{aligned} \tag{3.31}$$

where $\alpha = \left[L_f^r h(x)\right]_{x=\Phi^{-1}(\xi,\eta)}$, $\beta = \left[L_g L_f^{r-1} h(x)\right]_{x=\Phi^{-1}(\xi,\eta)}$, and $q_i = \left[L_f \Phi_i(x)\right]_{x=\Phi^{-1}(\xi,\eta)}$.

3.3.3 Hirschorn Inverse

For nonlinear systems, the order of the cascaded system consisting of the plant operator and an appropriately defined inverse is important, and it is necessary to make a distinction between the two possible cases [16]:

- The *left inverse* reconstructs the input from the plant output, its derivatives, and the state variables of the inverse (Figure 3.1).
- The *right inverse* produces the input history required to obtain a particular output (*i.e.*, an ideal feedforward controller) using the plant output and the state variables of the inverse (Figure 3.2).

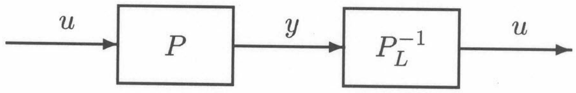

Figure 3.1: Left inverse of nonlinear operator

Following the derivation of the linear inverse, a nonminimal left inverse for the nonlinear system (3.1) can be written as:

$$\begin{aligned} \dot{Z} &= f(Z) + g(Z)\frac{\frac{d^r y}{dt^r} - L_f^r h(Z)}{L_g L_f^{r-1} h(x)} \\ u &= \frac{\frac{d^r y}{dt^r} - L_f^r h(Z)}{L_g L_f^{r-1} h(x)} \end{aligned} \tag{3.32}$$

In analogy to the linear case, a minimal realization can be achieved by exploiting the Byrnes-Isidori normal form:

$$\dot{\zeta}_1 = q_1\left(y, \dot{y}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}, \zeta_1, \ldots, \zeta_{n-r}\right)$$

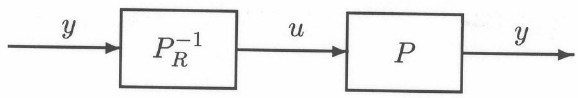

Figure 3.2: Right inverse of nonlinear operator

$$
\begin{aligned}
&\vdots \\
\dot{\zeta}_{n-r} &= q_{n-r}\left(y, \dot{y}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}, \zeta_1, \ldots, \zeta_{n-r}\right) \\
u &= \frac{\frac{d^r y}{dt^r} - \alpha\left(y, \dot{y}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}, \zeta_1, \ldots, \zeta_{n-r}\right)}{\beta\left(y, \dot{y}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}, \zeta_1, \ldots, \zeta_{n-r}\right)}
\end{aligned}
\tag{3.33}
$$

where the following change of variables has been introduced: $\zeta = \eta$. Note that the inverse operator is driven by the output y, and its first r derivatives and reconstructs the manipulated input u that produced y.

3.3.4 Example – Exothermic Chemical Reactor

Consider the problem of regulating the temperature in a nonisothermal CSTR in which an exothermic, irreversible, first-order reaction takes place. The mass and energy balances are given, in dimensionless form, as [30]:

$$
\begin{aligned}
\dot{x_1} &= -x_1 + \mathcal{D}a(1 - x_1)e^{\frac{x_2}{1+x_2/\gamma}} \\
\dot{x_2} &= -x_2 + B\mathcal{D}a(1 - x_1)e^{\frac{x_2}{1+x_2/\gamma}} + \beta(u - x_2) \\
y &= x_2
\end{aligned}
\tag{3.34}
$$

where x_1 is the reactant concentration, x_2 is the reactor temperature, u is the cooling jacket temperature, and $\mathcal{D}a, B, \gamma$, and β are dimensionless parameters. Consider the problem of deriving the Hirschorn inverse for this system. The relative degree $r = 1$ since $L_g h(x) = \beta \neq 0$. Thus, the Hirschorn left inverse (equation 3.32) can be realized by the following nonlinear dynamic system:

$$
\begin{aligned}
\dot{z}_1 &= -z_1 + \mathcal{D}a(1 - z_1)e^{\frac{z_2}{1+z_2/\gamma}} \\
\dot{z}_2 &= \frac{dy}{dt} \\
u &= \frac{\frac{dy}{dt} + z_2 - B\mathcal{D}a(1 - z_1)e^{\frac{z_2}{1+z_2/\gamma}} + \beta z_2}{\beta}
\end{aligned}
\tag{3.35}
$$

It is straightforward to verify that if the inverse system in equation 3.35 is driven by the output of the CSTR in equation 3.34, then the original input, u, from the CSTR is reconstructed.

3.3.5 Zero Dynamics

In the previous section, the concept of a nonlinear system inverse was introduced. It is apparent that there is no direct method for quantifying the behavior of the inverse dynamics. In other words, there are no simple quantities, such as the transmission zeros of a linear system, which characterize the stability of the inverse. Instead, one must analyze the dynamical system (3.33). Another complication which arises is that a nonlinear extension of transmission "zeros" yields multiple interpretations in the multivariable case [19]:

- The dynamics corresponding to the maximally unobservable system state variables.
- The invariant dynamics under which the system evolves when the output is constrained to be zero for all times.
- The dynamics corresponding a minimal realization of the process inverse.

Note that the three dynamic systems are equivalent for a linear system and a single-input single output, nonlinear system.

A question that arises naturally in the control context is whether the inverse of a dynamical system is stable. For example, in IMC design approach one utilizes the inverse of the process model directly in the controller synthesis [24]. In the linear context, this question is readily resolved through an analysis of the system zeros: systems with all zeros lying in the left-half plane are called *minimum-phase*, while systems with some zeros lying in the right-half plane are called *nonminimum-phase.* Referring to the system inverse derived in (3.33), it is clear that the nonlinear analog of the minimum-phase property is related to the stability of the dynamic system:

$$\begin{aligned}\dot{\zeta}_1 &= q_1\left(y, \dot{y}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}, \zeta_1, \ldots, \zeta_{n-r}\right)\\ &\vdots\\ \dot{\zeta}_{n-r} &= q_{n-r}\left(y, \dot{y}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}, \zeta_1, \ldots, \zeta_{n-r}\right)\end{aligned} \tag{3.36}$$

One approach to determine the stability of the inverse system is to analyze the unforced inverse dynamics, which are usually called the *zero dynamics*:

$$\begin{aligned}\dot{\zeta}_1 &= q_1(0, 0, \ldots, 0, \zeta_1, \ldots, \zeta_{n-r})\\ &\vdots\\ \dot{\zeta}_{n-r} &= q_{n-r}(0, 0, \ldots, 0, \zeta_1, \ldots, \zeta_{n-r})\end{aligned} \tag{3.37}$$

For example, in the case where the system output is constrained to a constant value (setpoint) — which can be assumed to be zero without loss of generality — the stability of the closed-loop system in which the inverse is employed as the controller is completely determined by the stability of these internal dynamics. In the case where the system output must follow a trajectory, the inverse dynamics are driven by the system output and its first $r-1$ derivatives. In this case, the stability of the forced zero dynamics (3.36) must be evaluated to determine the internal stability.

In the general case, one requires stronger conditions than stability of the zero dynamics to guarantee internal stability in order to avoid, for example, the peaking phenomena described in [29]. However, this is only a limitation on global or semi-global stabilization, and will not preclude local stabilization which can be guaranteed from an analysis of the local properties of the zero dynamics.

3.3.6 Example – Cyclopentenol Synthesis

Consider the synthesis of cyclopentenol from cyclopentadien [10], which follows the same reaction sequence as the van de Vusse reaction [31]: there is a primary reaction, a further reaction of the desired product, and a side reaction of the initial reactant. The reaction scheme looks like:

$$A \rightarrow B \rightarrow C$$

$$2A \rightarrow D$$

In this case, A is cyclopentadien, B is cyclopentenol, C is cyclopentandiol, and D is Dicyclopentadien. Assuming constant volume isothermal conditions, and mass action kinetics, the mass balances for A and B can be written as:

$$\begin{aligned} \dot{c}_A &= \frac{q}{V_R}(c_{A0} - c_A) - k_1 c_A - k_3 c_A^2 \\ \dot{c}_B &= -\frac{q}{V_R} c_B + k_1 c_A - k_2 c_B \end{aligned}$$

where c_A and c_B are the concentrations of A and B, respectively, c_{A0} is the inlet concentration of A, q is the inlet flow rate, V_R is the reactor volume, and k_1, k_2, and k_3 are reaction rate constants.

We are interested in determining the stability of the zero dynamics when the manipulated variable is the dilution rate ($u = \frac{q}{V_R}$), and the desired output is the concentration of B ($y = c_B$) [20]. The nonlinear state-space equations become:

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1 - k_3 x_1^2 + u(x_{10} - x_1) \\ \dot{x}_2 &= k_1 x_1 - k_2 x_2 + u(-x_2) \\ y &= x_2 \end{aligned} \tag{3.38}$$

where $x_1 = c_A$, $x_2 = c_B$, and $x_{10} = c_{A0}$. The system has relative degree $r = 1$ since $L_g h(x) = -x_2 \neq 0$. Hence, a minimal realization of the process inverse is a first-order nonlinear system. The change of coordinates (equation 3.30):

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} h(x) \\ t_1(x) \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{x_{10} - x_1}{x_2} \end{pmatrix} \tag{3.39}$$

and its inverse:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_{10} - \xi\eta \\ \xi \end{pmatrix} \tag{3.40}$$

can be used to realize the Byrnes-Isidori normal form:

$$\begin{aligned}\dot{\xi} &= \alpha(\xi,\eta) + \beta(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi\end{aligned} \tag{3.41}$$

where:

$$\begin{aligned}\alpha(\xi,\eta) &= k_1(x_{10} - \eta\xi) - k_2\xi \\ \beta(\xi,\eta) &= -\xi \\ q(\xi,\eta) &= \frac{k_1(1-\eta)(x_{10}-\eta\xi) + k_2\xi\eta + k_3(x_{10}-\xi\eta)^2}{\xi}\end{aligned}$$

Thus, the zero dynamics are:

$$\dot{\eta} = q(\xi_0,\eta) = \frac{k_1(1-\eta)(x_{10}-\eta\xi_0) + k_2\xi_0\eta + k_3(x_{10}-\xi_0\eta)^2}{\xi_0} \tag{3.42}$$

where ξ_0 is the (nonzero) equilibrium value of the output. If we convert the zero dynamics back to the original coordinates and analyze the Jacobian linearization, we find that the sign of the quantity:

$$\kappa = k_2 - k_1 - 2k_3x_1 + k_1\frac{x_{10} - 2x_1}{x_2}$$

will determine whether the nonlinear system is locally minimum phase ($\kappa < 0$) or nonminimum phase ($\kappa > 0$). In fact, at the point where $\kappa = 0$, a singularity occurs and the system relative degree is not well defined. It is interesting to note that the locus of points where $\kappa = 0$ cross the conversion locus at the point of maximum conversion.

3.3.7 Example – Isothermal CSTR

This example involves the following multicomponent isothermal kinetic sequence carried out in a CSTR [5]:

$$A \rightleftharpoons B \rightarrow C$$

The desired product is component C, and the manipulated input is the feed flowrate of component B. The dimensionless mass balances for A, B, and C

are given below:

$$\begin{aligned} \dot{x_1} &= 1 - x_1 - Da_1x_1 + Da_2x_2^2 \\ \dot{x_2} &= -x_2 + Da_1x_1 - Da_2x_2^2 - Da_3x_2^2 + u \\ \dot{x_3} &= -x_3 + Da_3x_2^2 \\ y &= x_3 \end{aligned} \tag{3.43}$$

where the Da_i terms are the respective Damköhler terms for the reactions.

The system has relative degree $r = 2$ since $L_gh(x) = 0$ and $L_gL_fh(x) = 2Da_3x_2 \neq 0$. Hence, a minimal realization of the process inverse is a second-order nonlinear system. The change of coordinates (equation 3.30):

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta \end{pmatrix} = \begin{pmatrix} x_3 - x_{3,ss} \\ -x_3 + Da_3x_2^2 \\ x_1 - x_{1,ss} \end{pmatrix} \tag{3.44}$$

and its inverse:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \eta + x_{1,ss} \\ \sqrt{\frac{\xi_2+\xi_1+x_{3,ss}}{Da_3}} \\ \xi_1 + x_{3,ss} \end{pmatrix} \tag{3.45}$$

can be used to realize the Byrnes-Isidori normal form. The objective in this example is a derivation of the zero dynamics for this reactor. They can be constructed from the dynamics for η:

$$\dot{\eta} = -(1 + Da_1)\eta + \frac{Da_2}{Da_3}(\xi_1 + \xi_2) \tag{3.46}$$

Thus, the unforced zero dynamics are:

$$\dot{\eta} = -(1 + Da_1)\eta \tag{3.47}$$

Therefore, the unforced zero dynamics are globally stable, and this CSTR is globally minimum phase.

3.4 Controllability and Observability

3.4.1 Linear Systems

We briefly discuss the controllability and observability of the single-input, single-output linear system (3.2). A more complete presentation of these concepts is available elsewhere [4]. A dynamic system is said to be *controllable* if there exists an input $u(t)$ such that any initial state $x(t_0)$ can be driven to any other state $x(t_1)$ in finite time $t_1 > t_0$. A necessary and sufficient condition for the linear system (3.2) to be controllable is that the n x n controllability matrix:

$$W_c = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} \tag{3.48}$$

has rank n. The system is said to be *uncontrollable* if the rank$(W_c) < n$.

A dynamic system is said to be *observable* if there exists a time t_1 such that any initial state $x(t_0)$ can be distinguished from any other state x_0 using the input $u(t)$ and output $y(t)$ over the time interval $t_0 \leq t \leq t_1$. A necessary and sufficient condition for the linear system (3.2) to be observable is that the n x n observability matrix:

$$W_o = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} \tag{3.49}$$

has rank n. If the rank$(W_c) < n$, the system is said to be *unobservable*.

The *Kalman decomposition* allows a linear system to be decomposed into controllable/uncontrollable and observable/unobservable parts. More specifically, there exists a linear change of coordinates $z = Tx$ such that (3.2) is transformed into:

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} &= \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} c_1 & 0 & c_3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \end{aligned} \tag{3.50}$$

It follows directly from (3.50) that: (1) z_1 represents the controllable and observable modes; (2) z_2 represents the controllable and unobservable modes;

(3) z_3 represents the uncontrollable and observable modes; and (4) z_4 represents the uncontrollable and unobservable modes.

3.4.2 Local Controllability

We now consider the controllability of the single-input, single-output nonlinear system (3.1). For simplicity, the subsequent analysis is *local*; *i.e.* the results are valid only in a neighborhood of the operating point. Global results are available elsewhere [15, 18, 25]. The nonlinear system (3.1) is said to be *weakly controllable* at x_0 if there exists an input $u(t)$ such that any initial state $x(t_0)$ in a neighborhood X_0 of x_0 can be driven to any other state $x(t_1) \in X_0$ in finite time $t_1 > t_0$ [15, 28]. It is important to note that this definition does not ensure that the state trajectory $x(t)$ will remain near x_0. For this reason, a local version of weak controllability is introduced. The nonlinear system (3.1) is said to be *locally weakly controllable* at x_0 if it is weakly controllable and there exists a neighborhood $X_1 \subset X_0$ such that $x(t) \in X_1$ for $t_0 \leq t \leq t_1$ [15, 28].

An important advantage of local controllability is that it can be determined by examining the rank of a matrix which is analogous to the linear controllability matrix. In particular, the nonlinear system (3.1) is locally weakly controllable at x_0 if the n x n controllability matrix [15, 28],

$$W_c(x) = \left[\begin{array}{cccc} g(x) & ad_f g(x) & \cdots & ad_f^{n-1} g(x) \end{array} \right] \tag{3.51}$$

has rank n at x_0. In this case, the system is said to satisfy the *controllability rank condition*. The system is said to be *uncontrollable* if this condition does not hold. It is illustrative to evaluate the controllability matrix $W_c(x)$ for the linear system (3.2), for which the Lie brackets can be expressed as:

$$ad_f^{k-1} g(x) = (-1)^{k-1} A^{k-1} b, \quad 1 \leq k \leq n \tag{3.52}$$

Hence, for a linear system the nonlinear controllability matrix $W_c(x)$ is identical to the linear controllability matrix W_c, modulo the $(-1)^{k-1}$ term which appears in $W_c(x)$.

In analogy to the linear case, the nonlinear system (3.1) can be locally decomposed into controllable/uncontrollable parts. If the system satisfies certain technical assumptions [18, 22], there exists a local change of coordinates $z = \Phi(x)$ defined in a neighborhood of x_0 such that the system is transformed into:

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2) + g_1(z_1, z_2)u \\ \dot{z}_2 &= f_2(z_2) \end{aligned} \qquad (3.53)$$

The state z_1 represents the locally weakly controllable dynamics and has dimension $d_c = \text{rank}(W_c(x_0))$. By contrast, the state z_2, which has dimension $n - d_c$, represents the uncontrollable dynamics.

3.4.3 Local Observability

We now consider the observability of the nonlinear system (3.1). As before, only local results will be presented; the interested reader is referred elsewhere [15, 18, 25] for global considerations. The nonlinear system (3.1) is said to be ***weakly observable*** at x_0 if there exists a time t_1 such that any initial state $x(t_0)$ in a neighborhood X_0 of x_0 can be distinguished from any other state $x_1 \in X_0$ using the input $u(t)$ and output $y(t)$ over the time interval $t_0 \leq t \leq t_1$ [15, 18]. It is useful to introduce a local version of weak observability to ensure that the state trajectory $x(t)$ will remain near x_0 and, therefore, that t_1 is reasonably small. The nonlinear system (3.1) is said to be ***locally weakly observable*** if it is weakly observable and there exists a neighborhood $X_1 \subset X_0$ of such that $x(t) \in X_1$ for $t_0 \leq t \leq t_1$ [15, 28].

Local observability can be checked using a matrix which is analogous to the linear observability matrix. It can be shown [15, 18] that the nonlinear system (3.1) is locally weakly observable at x_0 if the n x n observability matrix:

$$W_o(x) = \begin{bmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{n-1} h(x) \end{bmatrix} \qquad (3.54)$$

has rank n at x_0. In this case, the system is said to satisfy the ***observability rank condition.*** Otherwise, the system is said to be ***unobservable.*** It is illustrative to evaluate the nonlinear observability matrix $W_o(x)$ for the linear system (3.2). In this case, the Lie derivatives have the form:

$$dL_f^{k-1} h(x) = dCA^{k-1}x = CA^{k-1}, \quad 1 \leq k \leq n \qquad (3.55)$$

Thus, $W_o(x)$ is identical to the linear observability matrix W_o when the system is linear.

In analogy to the linear case, the nonlinear system (3.1) can be locally decomposed into observable/unobservable parts. Assuming the system satisfies certain technical assumptions [18, 22], there exists a local change of coordinates $z = \Phi(x)$ defined in a neighborhood of x_0 such that the system is transformed into:

$$\begin{aligned} \dot{z}_1 &= f_1(z_1, z_2) + g_1(z_1, z_2)u \\ \dot{z}_2 &= f_2(z_2) + g_2(z_2)u \\ y &= h(z_2) \end{aligned} \tag{3.56}$$

The state z_2 represents the locally weakly observable dynamics and has dimension $d_o = \text{rank } W_o(x_0)$. By contrast, the state z_1 represents the unobservable dynamics and has dimension $n - d_o$.

3.4.4 Example – Cyclopentenol Synthesis

Consider again the cyclopentenol synthesis example from Section 3.3.6:

$$\begin{aligned} \dot{x}_1 &= -k_1x_1 - k_3x_1^2 + (x_{10} - x_1)u = f_1(x) + g_1(x)u \\ \dot{x}_2 &= k_1x_1 - k_2x_2 - x_2u = f_2(x) + g_2(x)u \\ y &= x_2 = h(x) \end{aligned} \tag{3.57}$$

In order for the system to be locally weakly controllable, the matrix $W_c(x)$ $= \begin{bmatrix} g(x) & ad_f g(x) \end{bmatrix}$ must have rank two at the operating point x_0. It is easy to show that:

$$\begin{aligned} g(x) &= \begin{bmatrix} x_{10} - x_1 \\ -x_2 \end{bmatrix} \\ ad_f g(x) &= \begin{bmatrix} -f_1(x) - (k_1 + 2k_3x_1)g_1(x) \\ -f_2(x) - k_1g_1(x) + k_2g_2(x) \end{bmatrix} \end{aligned} \tag{3.58}$$

Thus, the model is locally controllable if:

$$\begin{aligned}\det W_c(x) &= [-f_2(x) - k_1 g_1(x) + k_2 g_2(x)]g_1(x) + \\ &\quad [f_1(x) + (k_1 + 2k_3 x_1)g_1(x)]g_2(x) \neq 0 \end{aligned} \tag{3.59}$$

at x_0. Although the controllability matrix has a generic rank of two, there may exist points x_0 for which $W_c(x)$ is singular. This possibility must be checked for particular x_0.

The model (3.57) is locally weakly observable if the matrix:

$$W_o(x) = \begin{bmatrix} dh(x) \\ dL_f h(x) \end{bmatrix} \tag{3.60}$$

has rank two at the operating point x_0. It is easy to show that:

$$\begin{aligned} h(x) &= x_2 \\ L_f h(x) &= k_1 x_1 - k_2 x_2 \end{aligned} \tag{3.61}$$

hence,

$$W_o(x) = \begin{pmatrix} 0 & 1 \\ k_1 & -k_2 \end{pmatrix} \tag{3.62}$$

Thus, the model is locally observable for any x_0 because:

$$\det W_o(x) = -k_1 \neq 0 \tag{3.63}$$

3.5 Input-Output Representations

3.5.1 Linear Systems

The general problem of input-output modeling for nonlinear systems involves the determination of a nonlinear functional which maps the entire past input history to the value of the output at the present time. For linear systems, the linear convolution integral:

$$y(t) = \int_{-\infty}^{\infty} h_1(\sigma) u(t - \sigma) d\sigma \tag{3.64}$$

provides a functional which accomplishes this task. Here $h_1(\sigma)$, the "kernel" of the transformation, is the system's impulse response function. Causality requires that the lower limit on this integral be zero; it is therefore customary to write (3.64) as:

$$y(t) = \int_0^\infty h_1(\sigma)u(t-\sigma)d\sigma \tag{3.65}$$

3.5.2 Volterra Series

Volterra generalized this functional representation for nonlinear systems in the form of a power series [26]:

$$y(t) = y_1(t) + y_2(t) + y_3(t) + \cdots \tag{3.66}$$

where $y_1(t)$, the first order term, is defined as in (3.65), and the other terms in the series are defined as follows:

$$\begin{aligned} y_2(t) &= \int_0^\infty \int_0^\infty h_2(\sigma_1,\sigma_2)\ u(t-\sigma_1)u(t-\sigma_2)\ d\sigma_1 d\sigma_2 \\ y_3(t) &= \int_0^\infty \int_0^\infty \int_0^\infty h_3(\sigma_1,\sigma_2,\sigma_3)\ u(t-\sigma_1)u(t-\sigma_2)u(t-\sigma_3)\ d\sigma_1 d\sigma_2 d\sigma_3 \\ y_i(t) &= \int_0^\infty \cdots \int_0^\infty h_i(\sigma_1,\ldots,\sigma_i)\ u(t-\sigma_1)\cdots u(t-\sigma_i)\ d\sigma_1\cdots d\sigma_i \end{aligned}$$

A discrete time representation of the Volterra operator takes the form of the power series:

$$y(k) = y_1(k) + y_2(k) + y_3(k) + \cdots \tag{3.67}$$

where the first term given by:

$$y_1(k) = \sum_{i=1}^{\infty} h_1(i)u(k-i) \tag{3.68}$$

is the convolution model employed in linear MPC approaches. The higher order terms are given by:

$$y_2(k) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_2(i,j)\ u(k-i)u(k-j) \tag{3.69}$$

$$y_3(k) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} h_3(i,j,l)\ u(k-i)u(k-j)u(k-l) \tag{3.70}$$

Just as Taylor series are limited to approximating *analytic* functions [26], Volterra series — which are in fact temporal equivalents of Taylor series — are limited to approximating systems with fading memory [2]. Nevertheless, a wide variety of nonlinear chemical processes exhibit behavior which cannot be approximated adequately by linear models but for which a Volterra series models (even of only second order) provides a reasonable representation [8]. Such problems include chemical reactors which exhibit optima as well as high purity distillation columns which exhibit asymmetric behavior.

3.5.3 Fliess Canonical Form

For nonlinear systems, the input-output representation corresponding to the Byrnes-Isidori normal form is the Fliess canonical form [11] or Observability canonical form [23, 34]. A useful property of this normal form is that the pole dynamics and zero dynamics are structurally separated in the Jacobian approximation of the nonlinear system. As with the Byrnes-Isidori normal form, successive differentiations of the output are considered; however, in this case, the differentiations continue past the rth derivative until the nth derivative is obtained. Thus, the coordinate transformations are a function of both the original state, x, *and* the manipulated input, u. The resulting normal form is:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= F(y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-r)}) \\
y &= z_1
\end{aligned} \tag{3.71}$$

Consequently, the nonlinear system can be represented by a single nth order ordinary differential equation:

$$\frac{d^n y}{dt^n} = F(y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-r)}) \tag{3.72}$$

The system in (3.72) is also referred to as the *external differential form*, and a general algorithm for constructing it from a state-space representation is presented in [25].

By performing a Jacobian linearization of the system (3.71) one obtains:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= \alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_n z_n + \beta_0 u + \beta_1 \dot{u} + \ldots + \beta_{n-r} u^{(n-r)}
\end{aligned} \tag{3.73}$$

where:

$$\alpha_i = \left.\frac{\partial F}{\partial z_i}\right|_0, \qquad \beta_i = \left.\frac{\partial F}{\partial u^{(i)}}\right|_0$$

The index 0 stands for evaluation at the nominal operating point, which can be taken as the origin without loss of generality. It is straightforward to show that the transfer function corresponding to (3.73) is given by:

$$G(s) = \frac{y(s)}{u(s)} = \frac{\beta_0 + \beta_1 s + \ldots + \beta_{n-r} s^{n-r}}{\alpha_1 + \alpha_2 s + \ldots + \alpha_n s^n} \tag{3.74}$$

Thus, the poles and zeros of the linear system are fully characterized by the final state equation. Returning to the original nonlinear system, one can easily show that the zero dynamics can be represented by the $(n-r)$th order differential equation:

$$0 = F(0,0,0,\ldots,0,u,\dot{u},\ldots,u^{(n-r)}) \tag{3.75}$$

3.5.4 Example – Exothermic Chemical Reactor

Consider the CSTR example from Section 3.3.4:

$$\dot{x_1} = -x_1 + \mathcal{D}a(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} \tag{3.76}$$

$$\dot{x_2} = -x_2 + B\mathcal{D}a(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} + \beta(u - x_2) \tag{3.77}$$

$$y = x_2 \tag{3.78}$$

To obtain the Fliess canonical form, one differentiates the output two times:

$$\dot{y} = -x_2 + B\mathcal{D}a(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}} + \beta(u - x_2) \tag{3.79}$$

$$\begin{aligned}
\ddot{y} &= -\dot{x}_2 - B\mathcal{D}a\dot{x}_1 e^{\frac{x_2}{1+x_2/\gamma}} + B\mathcal{D}a(1-x_1)e^{\frac{x_2}{1+x_2/\gamma}}\frac{1}{(1+x/\gamma)^2} \\
&\quad +\beta(\dot{u} - \dot{x}_2)
\end{aligned} \tag{3.80}$$

Using the inverse transformation:

$$x_1 = 1 - \frac{\dot{y} - \beta(u - y) + y}{B\mathcal{D}ae^{\frac{y}{1+y/\gamma}}} \tag{3.81}$$

$$x_2 = y \tag{3.82}$$

one can obtain an external differential representation for the CSTR problem:

$$\begin{aligned} \ddot{y} &= (-2-\beta)\dot{y} + (-\beta - 1)y + \beta\dot{u} + \beta u \\ &+ \frac{1}{(1+y/\gamma)^2}(\dot{y} - \beta(u-y) + y) + \\ &\mathcal{D}ae^{\frac{y}{1+y/\gamma}}(B - \dot{y} + \beta(u - y) + y) \end{aligned} \tag{3.83}$$

(Note that for complex, higher-order state dynamics, the analytical derivation of the transformation from the inputs, outputs and their derivatives to the states may require symbolic manipulation software, or may not be globally solvable.)

The resultant structure is useful for several reasons: (i) the expression has an affine dependence on both u and $\dot{u}$, indicating that a dynamic feedback linearizing controller is easily constructed; and (ii) the structural form of the y nonlinearities suggests terms that might be included in a nonlinear input/output semi-empirical model structure for identification (*e.g.* terms such as $\frac{1}{(1+y/\gamma)^2}$ which are not immediately evident from the fundamental model.)

3.5.5 Realization Theory

The general problem of realizing a state-space representation of a nonlinear system from an external differential form is quite difficult. In particular, it often is not possible to completely eliminate derivatives of the input in the final structure, and it is even more difficult to obtain a control affine realization as in (3.1). Representative results in this area are given, for example, in the papers by Freedman and Willems [12], Glad [13], and Delaleau and Respondek [6]. They show that a system given in external differential form (3.72) can be represented, under certain technical conditions, by a general state space model:

$$\begin{aligned} \dot{z} &= g(z, u) \\ y &= h(z, u) \end{aligned} \tag{3.84}$$

A limited class of problems which admit such a solution include single-input, single-output systems with $n - r = 1$ [12].

3.5.6 Output Invariance

In developing input-output structures for multivariable dynamic systems, an important issue for controller synthesis is the case of individual inputs which do not affect a given output variable. In the linear case, this is straightforward to analyze – the corresponding transfer function element is identically zero. Consider the general (*i.e.*, possibly nonsquare) multivariable, state-space system:

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i(t) \\ y_j(x) &= h_j(x) \qquad 1 \leq j \leq p \end{aligned} \tag{3.85}$$

In this section, we describe the conditions under which a given output channel, y_j, is unaffected by a given input, u_i.

Lemma 1 (Output Invariance [25]) *For the system (3.85), y_j is unaffected by u_i on a open set $U \subset \mathcal{R}^n$ if and only if for all $\gamma \geq 1$ and for any $\tau_1, \cdots, \tau_\gamma$ in the set $\{f, g_1, \cdots, g_m\}$, the following relations hold:*

$$\begin{aligned} L_{g_i}h_j(x) &= 0 \\ L_{g_i}L_{\tau_1}\cdots L_{\tau_\gamma}h_j(x) &= 0 \qquad \forall\, x \in U \end{aligned}$$

Note that this result only holds for analytic systems as there exists counterexamples for non-analytic systems [25]. If this result is applied to a linear system, the conditions become:

$$c_jA^kb_i = 0 \qquad k = 0, 1, \ldots$$

which indicates that output invariance is obtained if and only if b_i lies in the unobservable subspace of the output c_j. This suggests an alternative formulation of output invariance using the unobservability properties of the nonlinear system:

Proposition 1 *[25] For the system (3.85), y_j is unaffected by u_i on a open set $U \subset \mathcal{R}^n$ if there exists an involutive distribution D of constant dimension defined on U such that:*

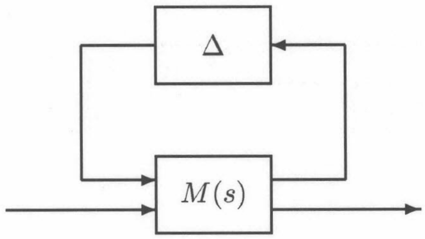

Figure 3.3: Uncertainty structure formulation

1. *D is invariant for (3.85)*
2. $g_i \in D$
3. $D \in ker \;\; dh_j$

where *ker* denotes the kernel (null-space).

3.5.7 Input-Output Stability

The results in this section rely on an input-output approach to nonlinear control system design and analysis. The question of closed-loop stability is analyzed for these systems in much the same manner as for linear systems; namely using the small gain theorem. We will briefly review this result, as well as the class of nonlinear systems which can be analyzed in this framework. Finally, a connection to linear robustness theory (μ analysis) is indicated.

The main result for stability of an interconnected block diagram such as that depicted in Figure 3.3 is given by the small gain theorem [33]. In this diagram, the uncertainty, Δ, is formulated as a separate element in a feedback loop about the nominal closed-loop system operator, M, which contains both the plant and the controller. This uncertainty can represent the effect of unmodeled dynamics and uncertain parameters in a process model. It is straightforward to show [24] that problems in which the uncertainty is of an additive, multiplicative, or inverse multiplicative nature can be cast in the structure in Figure 3.3.

Before stating the small gain result, we need to introduce the notion of an operator gain:

Definition 6 (Gain) *The incremental gain of an operator M, denoted by $g(M)$, is:*

$$g(M) \;=\; \sup \; \frac{||M(x_1) - M(x_2)||}{||x_1 - x_2||} \tag{3.86}$$

where the supremum is taken over all x_i in the domain of M, all $M(x_i)$ in the range of M, and all time for which $x \neq 0$.

This induced norm can be defined for various L_p norms (see for example, [32]). Having defined the gain of an operator, we can state the main result in [33]:

Theorem 4 (Small Gain Theorem) *If $g(M) \cdot g(\Delta) < 1$, then the closed-loop is internally input-output stable.*

Clearly the burden in this analysis is the calculation of the nonlinear operator norm. This remains an area of active research, and trends in computing algorithms may lead to more tractable solutions in the next few years.

The so-called M-Δ diagram in Figure 3.3 also represents a general framework for analysis of linear uncertain systems. In [9], there were extensions presented to include classes of nonlinear systems. It was shown that for norm bounded nonlinear operators, constant D-scalings could be used in the structured singular value analysis (see [7]) to give a conservative small gain condition for robust stability; an application to a CSTR was also presented. Furthermore, it is possible to calculate a Lyapunov function from the D-scalings, thus linking the input-output stability results to the Lyapunov stability.

A class of nonlinear systems is now described which fits into the proposed M-Δ structure framework. A conic sector is defined as:

$$\text{Cone}(C, R) \equiv \{(u, y) | ||y - Cu|| \leq ||Ru||\} \tag{3.87}$$

where (u, y) is the input/output pair for the operator. A nonlinear operator enveloped tightly by a conic sector is most accurately approximated linearly by the cone center C. In general, the cone center will not coincide with the plant described by the Jacobian of the nonlinear model evaluated at the operating point. Note that because we have replaced a potentially highly nonlinear function by two linear time-invariant operators, this simplification is likely to be conservative. The Cone(C, R) describes many input-output pairs, some of which may yield poorer performance than the original operator.

There is a direct correspondence between a nonlinear cone-bounded operator and a time-varying gain. From the conic sector definition, the plant can be interpreted as being equal to the nominal value (C) which is perturbed by a time varying gain of magnitude R. R and C can be absorbed into the system to arrive at the general uncertainty structure in Figure 3.3, where Δ is a time-varying gain of norm one. The construction of interconnection structures for general uncertainty descriptions can be found in [24].

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