Part IV

SAMPLED DATA MULTI-INPUT MULTI-OUTPUT SYSTEMS
Chapter 15

MIMO Sampled-Data Systems

We assume that the reader has mastered the chapters on SISO sampled-data systems as well the chapters on MIMO continuous systems. When the extension of results to MIMO sampled-data systems is straightforward the discussion will be brief and sometimes limited to simply defining the appropriate notation.

15.1 Fundamentals of MIMO Sampled-Data Systems

15.1.1 Sampled-Data Feedback

The block diagram of a typical sampled-data feedback loop is shown in Fig. 15.1-1A. Thick lines are used to represent the paths along which the signals are continuous. Equations (7.1-1) to (7.10-11) carry through to the MIMO case, with vectors instead of scalars. $C(z)$ denotes the discrete controller implemented through a digital computer. $H_0(s)$ models the D/A converter. We have

$$H_0(s) = h_0(s)I$$  \hspace{1cm} (15.1 - 1)

where $h_0(s)$ is the zero-order hold given by (7.1-12) and $I$ is the identity matrix with dimension equal to the number of controller outputs. The block $\Gamma(s)$ represents an anti-aliasing prefilter. The problem of aliasing was discussed in Sec. 7.1. Assuming that the the same sampling time is used for all the process outputs, it is reasonable to choose

$$\Gamma(s) = \gamma(s)I$$  \hspace{1cm} (15.1 - 2)

where $I$ has dimension equal to the number of the process outputs. $P(s)$ is the continuous system transfer matrix described in Sec. 10.1.1.

When the continuous output $y$ is not observed directly but after the prefilter and only at the sampling points, then Fig. 15.1-1A can be simplified to Fig. 15.1-1B, where

$$d^c_s(z) = \mathcal{Z}^{-1}\{\Gamma(s)d(s)\} = \mathcal{Z}^{-1}\{\gamma(s)d(s)\}$$  \hspace{1cm} (15.1 - 3)
Figure 15.1-1. Block diagram of computer controlled system: A: Sampled-data structure with thick lines indicating analog signals. B: Discrete structure with all signals discrete.
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\[ y^*_T(z) = \mathcal{ZL}^{-1}\{\Gamma(s)y(s)\} = \mathcal{ZL}^{-1}\{\gamma(s)y(s)\} \quad (15.1-4) \]

and all signals are discrete. Note that when the operator \( \mathcal{ZL}^{-1} \) is applied to a vector or matrix, it is simply applied to each element separately. We define

\[ P^*_T(z) = \mathcal{ZL}^{-1}\{\Gamma(s)P(s)H_0(s)\} = \mathcal{ZL}^{-1}\{h_0(s)P(s)\gamma(s)\} \quad (15.1-5) \]

\[ P^*(z) = \mathcal{ZL}^{-1}\{P(s)H_0(s)\} = \mathcal{ZL}^{-1}\{h_0(s)P(s)\} \quad (15.1-6) \]

All the elements of a vector or matrix pulse transfer function are always rational in \( z \), although the continuous transfer functions may include time delays. In order to be physically realizable the transfer matrices (or vectors) have to be proper or causal.

**Definition 15.1-1.** A vector or matrix \( G^*(z) \) is proper or causal if all its elements are proper and strictly proper if all its elements are strictly proper. All systems \( G^*(z) \) which are not proper are called improper or noncausal.

15.1.2 Poles and Zeros

Let

\[ G^*(z) = \mathcal{ZL}^{-1}\{h_0(s)G(s)\} \quad (15.1-7) \]

where \( G(s) \) is the transfer matrix representation of the system of differential and algebraic equations of Sec. 10.1.1. Then \( G^*(z) \) is the \( z \)-transfer matrix that describes the system of difference equations

\[ x(kT + T) = \Phi x(kT) + \Gamma u(kT) \quad (15.1-8) \]

\[ y(kT) = C x(kT) + D u(kT) \quad (15.1-9) \]

where \( T \) is the sampling time and

\[ \Phi = e^{AT} \quad (15.1-10) \]

\[ \Gamma = \int_0^T e^{AM} dt \begin{bmatrix} B 
\end{bmatrix} \quad (15.1-11) \]

Taking the \( z \)-transform of (15.1-8), (15.1-9) we get

\[ x(z) = (zI - \Phi)^{-1}\Gamma u(z) \quad (15.1-12) \]
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\[ y(z) = Cx(z) + Du(z) \]  \hspace{1cm} (15.1-13)

and substituting (15.1-12) into (15.1-13) yields

\[ y(z) = G^*(z)u(z) \]  \hspace{1cm} (15.1-14)

where

\[ G^*(z) \triangleq C(zI - \Phi)^{-1}\Gamma + D \]  \hspace{1cm} (15.1-15)

The matrix \( G^*(z) \) will be assumed to be of full normal rank. The poles and zeros of \( G^*(z) \) are defined in exactly the same way as those of \( G(s) \).

**Definition 15.1-2.** The eigenvalues \( \pi_i, i = 1, \ldots, n_p \), of the matrix \( \Phi \) are called the poles of the system (15.1-8), (15.1-9). The pole polynomial \( \pi(z) \) is defined as

\[ \pi(z) = \prod_{i=1}^{n_p} (z - \pi_i) \]  \hspace{1cm} (15.1-16)

**Definition 15.1-3.** \( \zeta \) is a zero of \( G^*(z) \) if the rank of \( G^*(\zeta) \) is less than the normal rank of \( G^*(z) \).

The zero polynomial is defined as

\[ \zeta(z) = \prod_{i=1}^{n_z} (z - \zeta_i) \]  \hspace{1cm} (15.1-17)

where \( n_z \) is the number of finite zeros of \( G^*(z) \).

**15.1.3 Internal Stability**

Assuming that no unstable poles of the continuous process have become unobservable after sampling, the internal stability of the system in Fig. 15.1-1A can be assessed from the internal stability of the system in Fig. 15.1-1B.

**Theorem 15.1-1.** The sampled-data system in Fig. 15.1-1A is internally stable if and only if the transfer matrix in (15.1-18)

\[
\begin{pmatrix}
y^*_r \\ u
\end{pmatrix} =
\begin{pmatrix}
P^*_r C(I + P^*_r C)^{-1} & (I + P^*_r C)^{-1} P^*_r \\ C(I + P^*_r C)^{-1} & -C(I + P^*_r C)^{-1} P^*_r
\end{pmatrix}
\begin{pmatrix}
y^*_r \\ u
\end{pmatrix} \hspace{1cm} (15.1-18)
\]

is stable — i.e. if and only if all its poles are strictly inside the unit circle.

Another test for internal stability is the Nyquist criterion, which was discussed for continuous systems in Sec. 10.2.2. The derivation follows exactly the same steps. The difference is that when we are dealing with \( z \)-transfer functions instead
of Laplace transfer functions, the Nyquist D-contour encircles the area outside the UC instead of the RHP.

**Theorem 15.1-2 (Nyquist Stability Criterion).** Let the map of the Nyquist D-contour under $\det(I + P^*(s)C(z))$ encircle the origin $n_F$ times in the clockwise direction. Let the number of open-loop unstable poles of $P^*_C$ be $n_{PC}$. Then the closed-loop system is stable if and only if

$$n_F = -n_{PC} \quad (15.1 - 19)$$

### 15.1.4 IMC Structure

The block diagram of the sampled-data MIMO IMC structure is shown in Fig. 15.1-2A, where

$$\hat{P}^*_k(z) = ZL^{-1}\{\Gamma(s)\hat{P}(s)H_0(s)\} = ZL^{-1}\{\gamma(s)\hat{P}(s)h_0(s)\} \quad (15.1 - 20)$$

$$\hat{P}^*(z) = ZL^{-1}\{\hat{P}(s)H_0(s)\} = ZL^{-1}\{\hat{P}(s)h_0(s)\} \quad (15.1 - 21)$$

When the IMC controller $Q$ and the feedback controller $C$ are related through

$$C = Q(I - \hat{P}^*_kQ)^{-1} \quad (15.1 - 22)$$

$$Q = C(I + \hat{P}^*_kC)^{-1} \quad (15.1 - 23)$$

then $u(z)$ and $y(s)$ react to inputs $r^*(z)$ and $d(s)$ in exactly the same way for both the classic feedback and the IMC structure.

Figure 15.1-2B is a different representation of the sampled-data IMC structure, which is equivalent to that in Fig. 15.1-2A, but not suitable for computer implementation because of the presence of the continuous model $\hat{P}(s)$. If only the sampled signals are of interest, then Fig. 15.1-2A is equivalent to Fig. 15.1-2C, where all signals are digital.

### 15.1.5 Model Uncertainty Description

In Sec. 7.3.2, we demonstrated how the modeling error in the description of the discretized plant is related to that in the continuous plant description. We pointed out that some conservativeness is introduced when the uncertainty bounds for the discrete plant are derived from those for the continuous plant. However, the conservativeness is quite small for the type of unstructured SISO system.
Figure 15.1-2. IMC structure: A: Sampled-data structure; B: Structure equivalent to (A) but not implementable; C: Discrete structure (all signals discrete).
uncertainty that was used in Chaps. 7 through 9. The same is true for a few
types of MIMO-system uncertainty. For example, let us assume that the additive
uncertainty for the continuous plant is bounded by \( \bar{\ell}_A \):

\[
\bar{\sigma}(P(i\omega) - \hat{P}(i\omega)) \leq \bar{\ell}_A(\omega)
\]  \hspace{1cm} (15.1-24)

For the discretized plant we have from (15.1-6), (15.1-21)

\[
P^*(z) - \hat{P}^*(z) = ZL^{-1}\{h_0(s)(P(s) - \hat{P}(s))\}
\]  \hspace{1cm} (15.1-25)

Then from the z-transform property (7.1-5), the singular value property
\( \bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \), and (15.1-24), (15.1-25), it follows that

\[
\bar{\sigma}(P^*(e^{i\omega T}) - \hat{P}^*(e^{i\omega T})) \leq \frac{1}{T} \sum_{k=-\infty}^{\infty} |h_0(i\omega + ik2\pi/T)|\bar{\ell}_A(i\omega + ik2\pi/T) \triangleq \bar{\ell}_0(\omega)
\]  \hspace{1cm} (15.1-26)

The ZOH \( h_0(s) \) is small at frequencies higher than \( \pi/T \) and goes to 0 as fast as
1/\( \omega \) as \( \omega \rightarrow \infty \). Therefore only a few terms around \( k = 0 \) are important in the
infinite sum. Also note that for a physical system, \( \bar{\ell}_A(\omega) \rightarrow 0 \) at least as fast as
1/\( \omega \) as \( \omega \rightarrow \infty \), and hence the sum converges.

However, it is not always possible to obtain a mathematical description for
the uncertainty in the z-domain in a non-conservative way, starting from the
uncertainty in the s-domain. In the absence of first-principles models, these
descriptions may be the result of experiments conducted with different sampling
rates, one of which may be small enough to approximate the continuous system.
A discussion of identification techniques is beyond the scope of this book. We
will assume in this chapter that non-conservative uncertainty descriptions for the
discrete and the continuous plant are available.

15.2 Nominal Internal Stability

15.2.1 IMC Structure

The same arguments as in Sec. 12.2 imply that the following matrix must be
stable for internal stability of the IMC structure in Fig. 15.1-2C.

\[
S_1 = \begin{pmatrix}
P^*_\gamma Q & (I - P^*_\gamma Q)P^*_\gamma & P^*_\gamma \\
Q & -Q^*P^*_\gamma & 0 \\
P^*_\gamma Q & -P^*_\gamma QP^*_\gamma & P^*_\gamma
\end{pmatrix}
\]  \hspace{1cm} (15.2-1)

Note that stability of the structure in Fig. 15.1-2C implies stability of that in
Fig. 15.1-2A, provided that no open-loop unstable poles of the plant become
unobservable after sampling.
Theorem 15.2-1. For $P = \hat{P}$, the IMC system in Fig. 15.1-2A is internally stable if and only if both the plant $P$ and the controller $Q$ are stable.

Hence for open-loop unstable plants, the IMC structure cannot be implemented. In such cases, the IMC design procedure is used to design the controller, which is then implemented through the classic feedback structure.

15.2.2 Feedback Structure

When there is no modeling error, substitution of (15.1-22) into (15.1-18) yields for the internal stability matrix

$$S_2 = \begin{pmatrix} P_\gamma^* Q & (I - P_\gamma^* Q) P_\gamma^* \\ Q & -Q P_\gamma^* \end{pmatrix}$$  \hspace{1cm} (15.2 - 2)

All four transfer matrices in (15.2-2) have to be stable for nominal stability of the classic feedback structure in Fig. 15.1-1A.

Theorem 15.2-2 provides a parametrization of all proper stabilizing controllers in terms of a stable transfer matrix $Q_1$. The following assumptions are analogous to those made in Sec. 12.3.

Assumption A1. If $\pi$ is a pole of $\hat{P}^*$ outside the UC, then (a) The order of $\pi$ is equal to 1 and (b) $\hat{P}$ has no zeros at $z = \pi$.

Assumption A2. Any poles of $\hat{P}^*$ or $P^*$ on the UC are at $z = 1$. Also $\hat{P}^*$ has no zeros on the UC.

Theorem 15.2-2. Assume that Assns. A1 and A2 hold and that $Q_0(z)$ is a proper transfer matrix that stabilizes $\hat{P}^*$ — i.e., it yields a stable $S_2$. Then all proper $Q$'s that make $S_2$ stable are given by

$$Q(z) = Q_0(z) + Q_1(z)$$  \hspace{1cm} (15.2 - 3)

where $Q_1(z)$ is any proper and stable transfer matrix such that $P^*(z) Q_1(z) P^*(z)$ is stable.

Proof. The fact that $Q_1$ has to be proper in order for $Q$ to be proper and vice versa, follows from the properness of $Q_0$. For the following part of the proof we will use the fact that $P^*$ and $P_\gamma^*$ have the same unstable poles.

⇒ We shall show that any $Q$ given by (15.2-3) makes $S_2$ stable. From substitution of (15.2-3) into (15.2-2) it follows that all that is required is that $(P^* Q_1 \quad Q_1 P^* \quad P_\gamma^* Q_1 P^*)$ be stable. From the properties of $Q_1$, it follows that the third element in the above matrix is stable. Stability of the other
two elements follows by pre- and post-multiplication of $P^*Q_1P^*$ by $(P^*)^{-1}$, since according to assumptions A1 and A2, $P^*$ has no zeros at the location of its unstable poles and these are the only possible unstable poles of $S_2$.

\[ \Delta S_2 = S_2(Q) - S_2(Q_0) = \begin{pmatrix} P^*(Q - Q_0) \\ (Q - Q_0) \\ P^*(Q - Q_0)P^* \\ (Q - Q_0)P^* \end{pmatrix} \]  \hspace{1cm} (15.2 - 4)

is stable. This implies that $(Q - Q_0) = Q_1$ and $P^*Q_1P^*$ are stable. \(\square\)

### 15.3 Nominal Performance

#### 15.3.1 Sensitivity and Complementary Sensitivity Function

The development in this section follows that in Sec. 7.5.1. Therefore we shall limit ourselves to simply setting the appropriate notation for the MIMO systems.

From Fig. 15.1-2A we get for $P = \hat{P}$

\[ y(s) = h_0(s)P(s)Q(e^{sT})(r^*(e^{sT}) - d^*_1(e^{sT})) + d(s) \]  \hspace{1cm} (15.3 - 1)

Define

\[ e(s) = y(s) - r(s) \]  \hspace{1cm} (15.3 - 2)

Then for an $r(s)$ that remains constant between sampling points, we have $r(s) = h_0(s)r^*(e^{sT})$ and we can define the sensitivity and complementary sensitivity operators that relate $r(s)$ to $-e(s)$ and $y(s)$, correspondingly as

\[ \bar{E}_r(s) \triangleq I - P(s)Q(e^{sT}) \]  \hspace{1cm} (15.3 - 3)

\[ \bar{H}_r(s) \triangleq P(s)Q(e^{sT}) \]  \hspace{1cm} (15.3 - 4)

An approximate sensitivity function for the relation between $y(s)$ and $d(s)$ can only be obtained when the assumption is made that the disturbance is limited to the frequency band up to $\pi/T$.

\[ y(i\omega) \cong \bar{E}_d(i\omega)d(i\omega) \]  \hspace{1cm} (15.3 - 5)

where

\[ \bar{E}_d(s) = I - P(s)\hat{Q}(s) \]  \hspace{1cm} (15.3 - 6)
\[
\hat{Q}(s) = \frac{1}{T}h_0(s)Q(e^{sT})\gamma(s) \tag{15.3-7}
\]

Sampling of (15.3–1) yields

\[
y^*(z) = P^*(z)Q(z)(r^*(z) - d^*_\gamma(z)) + d^*(z) \tag{15.3–8}
\]

from which we can obtain the pulse sensitivity and complementary sensitivity functions, relating \(e^\gamma(z)\) to \(r^*(z)\) and \(d^*(z)\) (for \(\gamma(s) = 1\)) or \(d^*_\gamma(z)\) (when \(d^*_\gamma\) is substituted for \(d^*\) in (15.3–8)):

\[
\tilde{E}^*(z) \triangleq I - P^*(z)Q(z) \tag{15.3–9}
\]

\[
\tilde{H}^*(z) \triangleq P^*(z)Q(z) \tag{15.3–10}
\]

15.3.2 \(H_2^*\) Performance Objectives

We define as \(L_2^{*n}\) the Hilbert space of complex valued vector functions \(y(z)\) with \(n\) elements, defined on the unit circle and square integrable with respect to \(\theta\) — i.e., for which the following quantity is finite:

\[
\|y\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} y(e^{i\theta})^H y(e^{i\theta}) \, d\theta\right)^{1/2} \tag{15.3–11}
\]

Note that (15.3–11) defines a norm on \(L_2^{*n}\). In the case where \(y(z)\) has no poles outside the UC, Parseval’s theorem yields a time domain expression for \(\|y\|_2\):

\[
\|y\|_2 = \left(\sum_{k=0}^{\infty} y_k^* y_k \right)^{1/2} \tag{15.3–12}
\]

For matrix valued functions \(G(z)\) of dimensions \(n \times m\), the space \(L_2^{*n \times m}\) is defined similarly with norm

\[
\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}[G(e^{i\theta})^H G(e^{i\theta})] \, d\theta\right)^{1/2} \tag{15.3–13}
\]

The spaces \(H_2^{*n}\) and \(H_2^{*n \times m}\) are defined as subspaces of the corresponding \(L_2\) spaces as in the scalar case.

The \(H_2^*\) performance objective is to minimize over all stabilizing \(\hat{Q}\) the weighted sum of squared errors for the response to an input or a set of inputs of interest. Several \(H_2^*\)–type objective functions will be considered. For a specific external input \(v^*\) (\(r^*\) or \(d^*\) or \(d^*_\gamma\)) define by using (15.3-9)
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\[
\Phi(v^*) \triangleq \| We^* \|^2 \| W \hat{E}^* v^* \|^2 \| W(I - P^* \hat{Q}) v^* \|_2^2 = (15.3 - 14)
\]

where \( W \) is a frequency dependent matrix or scalar weight. One objective could be

**Objective O1:**

\[
\min_{\hat{Q}} \Phi(v^*)
\]

for a particular input \( v^* = (v_1 \ v_2 \ \ldots \ v_n)^T \).

A more meaningful objective would be to minimize \( \Phi(v) \) not just for one input vector \( v^* \), but for every input in a set \( \mathcal{V} \):

\[
\mathcal{V} = \{v^i(z) | i = 1, \ldots, n\} \quad (15.3 - 15)
\]

where \( v^i(z), \ldots, v^n(z) \) are vectors that describe the directions and the frequency content of the expected external system inputs and \( n \) is the dimension of \( P \). Thus, the objective is

**Objective O2:**

\[
\min_{\hat{Q}} \Phi(v^*) \quad \forall v^* \in \mathcal{V}
\]

However a linear time invariant \( \hat{Q}(z) \) that solves O2 does not always exist. The conditions necessary for its existence are expressed in Thm. 15.6-3. An alternative is

**Objective O3:**

\[
\min_{\hat{Q}} [ \Phi(v^1) + \ldots + \Phi(v^n) ]
\]

In this case the objective is the sum of the squared errors caused by each of the \( v^i \)'s, when applied separately.

For every external input \( v^* \) that will be considered in this chapter the following assumptions will be made. They are analogous to those discussed in Sec. 12.6.1 and their physical meaning is identical.

**Assumption A3.** Every nonzero element of \( v^* \) includes all the poles of \( \hat{P} \) outside the UC, each with degree 1, and those are the only poles of \( v^* \) outside the UC.

**Assumption A4.** Let \( \ell_i \) be the maximum number of poles at \( z = 1 \) of any element in the \( i \)th row of \( P \). Then the \( i \)th element of \( v^* \), \( v_i \), has at least \( \ell_i \) poles at \( z = 1 \). Also \( v^* \) has no other poles on the UC and its elements have no zeros on the UC.

For the case, where a set \( \mathcal{V} \) of inputs is considered, define
\[ V \triangleq (v^1 \ v^2 \ \ldots \ v^n) \]  \hspace{1cm} (15.3 - 16)

where \( v^1, \ldots, v^n \) satisfy Assn. A3. An additional assumption on \( V \) is needed:

**Assumption A5.** \( V \) has no zeros at the location of its unstable poles or on the UC and \( V^{-1} \) cancels the unstable poles of \( \tilde{P} \) in \( V^{-1}\tilde{P} \).

### 15.3.3 \( H_\infty \) Performance Objective

The \( H_\infty \) objective discussed in Sec. 10.4.4 can now be extended to discrete systems with band-limited disturbances. The development is similar to that for the SISO case (Sec. 7.5.5). The objective can be written as

\[ \| W \tilde{E}_v \|_\infty < 1 \]  \hspace{1cm} (15.3 - 17)

where \( W \) is the frequency weight and \( \tilde{E}_d(s) \) is either the approximate disturbance sensitivity function \( \tilde{E}_d \) given by (15.3-6) or the setpoint sensitivity function \( \tilde{E}_r \) given by (15.3-3). Since the disturbance is assumed to be limited to the frequency band up to \( \pi/T \) and \( h_0(s) r^*(e^{sT}) \) is also limited because of \( h_0 \), the weight should satisfy

\[ \bar{\sigma}(W(\omega)) << 1, \quad \omega > \pi/T \]  \hspace{1cm} (15.3 - 18)

Hence (15.3-17) can be written as

\[ \bar{\sigma}(W(\omega)\tilde{E}_v(i\omega)) < 1, \quad 0 \leq \omega \leq \pi/T \]  \hspace{1cm} (15.3 - 19)

If \( W \) is a scalar, (15.3-19) becomes

\[ \bar{\sigma}(\tilde{E}_v(i\omega)) < |w(\omega)|^{-1}, \quad 0 \leq \omega \leq \pi/T \]  \hspace{1cm} (15.3 - 20)

### 15.4 Robust Stability

In Sec. 15.1 we explained that if no open-loop unstable poles of the plant or the model become unobservable after sampling, then stability of the structures in Figs. 15.1-1A and 15.1-2A is equivalent to stability of those in Figs. 15.1-1B and 15.1-2C correspondingly. In Sec. 15.2 we developed the nominal internal stability conditions. We shall now concentrate on the robust stability of completely discrete structures like the ones in Figs. 15.1-1B and 15.1-2C. The development of robustness conditions follows exactly the same steps as those in Chap. 11.

First the \( M - \Delta \) structure (Fig. 11.1-2) which is needed in the structured singular value theory, has to be generated from the given discrete control structure. For this purpose the same type of block manipulations have to be carried
out as were demonstrated in Chap. 11. Then, if $M$ and $\Delta$ are stable, the condition for robust stability is that the map of the discrete Nyquist D-contour under \(\det(I - M\Delta)\) does not encircle the origin. Recall that for discrete systems the D-contour encircles the area outside the UC. We can now use the SSV to obtain the following theorem.

**Theorem 15.4-1.** Assume that the nominal systems $M$ is stable and that the perturbation $\Delta$ is such that the perturbed closed-loop system is stable if and only if the map of the Nyquist contour under $\det(I - M\Delta)$ does not encircle the origin. Then the system in Fig. 11.1-2 is stable for all $\Delta \in X_1$ if and only if

\[
\mu(M(e^{i\omega T})) < 1 \quad 0 \leq \omega \leq \pi/T \quad (15.4 - 1)
\]

Note that because of the periodicity of the $z$-transforms and the property described by (7.1-8), only the frequencies up to $\pi/T$ need to be considered.

### 15.5 Robust Performance

#### 15.5.1 Sensitivity Function Approximation

First, we shall obtain an approximate sensitivity function in a similar way as in Sec. 15.3.1. Then we shall use this function to assess robust performance. From Fig. 15.1-2A it follows that

\[
e(s) \triangleq y(s) - r(s) = (d(s) - r(s)) - P(s)H_0(s)Q(e^{sT})
\]

\[
(I + (P^*_s(e^{sT}) - \tilde{P}^*_1(e^{sT}))Q(e^{sT}))^{-1}(d^*_1(z) - r^*(z)) \quad (15.5 - 1)
\]

We shall now obtain an approximation to (15.5-1) by considering the frequencies $0 \leq \omega \leq \pi/T$. Note that because of the periodicity of $Q(z)$, these are the only frequencies which one can influence independently by using a digital controller. It follows from (7.1-5) that if $a(s)$ is small for $\omega > \pi/T$, then

\[
a^*(e^{i\omega T}) \approx \frac{1}{T}a(i\omega), \quad 0 \leq \omega \leq \pi/T \quad (15.5 - 2)
\]

Use of (15.5-2) for all the $z$-transforms in (15.5-1) except for $r^*$ for which we assume $r(s) = h_0(s)r^*(e^{sT})$, yields the approximation

\[
e(i\omega) \approx E_d(i\omega)d(i\omega) - E_r(i\omega)r(i\omega), \quad 0 \leq \omega \leq \pi/T \quad (15.5 - 3)
\]
where

\[ E_r(i\omega) \triangleq I - P(i\omega)Q(e^{i\omega T}). \]

\[ [I + (P(i\omega) - \tilde{P}(i\omega))Q(e^{i\omega T})\gamma(i\omega)h_0(i\omega)/T]\]

\[ E_d(i\omega) \triangleq I - P(i\omega)Q(e^{i\omega T})\gamma(i\omega)h_0(i\omega)/T. \]

\[ [I + (P(i\omega) - \tilde{P}(i\omega))Q(e^{i\omega T})\gamma(i\omega)h_0(i\omega)/T]\]

Note that the above approximation is valid when the input signals \( r \) and \( d \) are small for \( \omega > \pi/T \). If we assume that \( r(t) \) is a staircase function then it has the desired property. If one expects disturbances with high frequency content at \( \omega > \pi/T \) then one should reduce \( T \) or use the anti-aliasing prefilter whose function is to cut off signals with frequencies higher than \( \pi/T \).

### 15.5.2 \( H_\infty \) Performance Objective

We require that the objective defined in Sec. 15.3.3 be satisfied for all plants \( P(s) \) in the uncertainty set \( \Pi \) (note that \( E_v(\tilde{P}) = E_v \)).

\[ \max_{0 \leq \omega \leq \pi/T} \| W(\omega)E_v(i\omega) \| < 1 \quad \forall P \in \Pi \]  \hspace{1cm} (15.5 - 6)

From this point on the treatment of the problem is identical to that presented in Sec. 11.3.1. Note that only the continuous plant \( P(s) \) appears in \( E_v(s) \) and therefore all the uncertain \( \Delta \)'s are continuous transfer functions. Hence the need mentioned in Sec. 15.1.4 for continuous as well as discrete (used for test of robust stability) uncertainty bounds.

### 15.6 IMC Design: Step 1 (\( \hat{Q} \))

#### 15.6.1 \( H_\infty \)-Optimal Control

The plant \( P^* \) can be factored into an allpass portion \( P_A^* \) and a minimum phase portion \( P_M^* \):

\[ P^* = P_A^*P_M^* \]  \hspace{1cm} (15.6 - 1)

Here \( P_A^* \) is stable and such that \( P_A^*(e^{i\theta})^HP_A^*(e^{i\theta}) = I \). Also \( (P_M^*)^{-1} \) is stable. \( P_M^* \) has the additional property that both \( P_M^* \) and \( (P_M^*)^{-1} \) are proper. In the case where \( P^* \) is scalar, this factorization can be easily accomplished as described by (8.1-2). In the general multivariable case, this "inner-outer factorization" can be accomplished by using the bilinear transformation \( z = (1 + s)/(1 - s) \), to reduce
the problem to the one for the s-domain, which was discussed in Sec. 12.6.4. The steps involved in this procedure are explained in Sec. 15.6.4.

**Objective O1: Specific Input**

Let $v_0(z)$ be the scalar allpass with the property $v_0(1) = 1$, which includes the common zeros outside the UC and the common delays of the elements of $v^*(z)$. Write

$$v^*(z) = v_0(z)\hat{v}(z) \quad (15.6 - 2)$$

where $\hat{v}(z)$ is a vector. Hence $\hat{v}$ is proper with at least one element semi-proper and there is no point $z$ outside the UC where $\hat{v}$ becomes identically zero.

**Theorem 15.6-1.** Assume that Assns. A1-A4 hold. Any stabilizing $\hat{Q}$ that solves Obj. O1 satisfies

$$\hat{Q}\hat{v} = z(WP_A^*)^{-1}\{z^{-1}W(P_A^*)^{-1}\hat{v}\} \quad (15.6 - 3)$$

where the operator $\{\cdot\}$ denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained except those corresponding to poles of $(P_A^*)^{-1}$. Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (15.6-3) is infinite. Guidelines for the construction of such a controller are given in the proof.

Note that not every $\hat{Q}$ satisfying (15.6-3) is necessarily a stabilizing controller. Equation (15.6-3) should be compared to (9.2-4) for SISO systems. If we assume that the disturbance and the plant have the same open-loop poles outside the UC, then the two equations are identical.

**Proof of Theorem 15.6-1.** We shall assume $W = I$. The proof of the weighted case is left as an exercise. Let $V_0$ be a diagonal matrix where each column satisfies Assn. A3 and every element has $\ell_v$ poles at $z = 1$, where $\ell_v$ is the maximum number of such poles in any element of $v$. Assume that there exists $Q_0$, which stabilizes $P^*$ in the sense of Thm. 15.2.2 and also makes $(I - P^*Q_0)V_0$ stable. Its existence will be proven by construction. Substitution of (15.2-3) into (15.3-14) and use of the fact that pre- or post-multiplication of a function with an allpass does not change its $L_2$-norm, yields:

$$\Phi(v^*) = \|z^{-1}(P_A^*)^{-1}(I - P^*Q_0)\hat{v} - z^{-1}P_M^*Q_1\hat{v}\|_2^2$$

$$\triangleq \|f_1 - f_2Q_1\hat{v}\|_2^2 \quad (15.6 - 4)$$

The term $f_1$ has no poles at $z = 1$ because $(I - P^*Q_0)V_0$ has no such poles. Any rational function $f_1(z)$ with no poles on the UC, can be uniquely decomposed
into a strictly proper, stable part \( \{ f_1 \}_+ \) in \( H^*_2 \) and a strictly unstable part \( \{ f_1 \}_- \) in \( (H^*_2)^\perp \):

\[
 f_1 = \{ f_1 \}_+ + \{ f_1 \}_-
\]

(15.6-5)

Note that according to the definition of \( H^*_2, (H^*_2)^\perp \), any improper terms as well as the constant term in a partial fraction expansion of \( f_1 \), belong in \( \{ f_1 \}_- \). Next we want to show that \( f_2Q_1\hat{v} \) has to be stable. The fact that \( (I - P^*Q_0) \hat{v} \) is stable implies that \( (I - P^*Q_0) \hat{v} \) is stable. We require that \( (I - P^*Q) \hat{v} \) has no poles outside the UC and therefore that \( (I - P^*Q) \hat{v} = (I - P^*Q_0) \hat{v} - P^*Q_1 \hat{v} \) have no poles outside the UC. But since \( (I - P^*Q_0) \hat{v} \) is stable, this requirement reduces to \( P^*Q_1 \hat{v} \) having no poles outside the UC. Also in order for \( \Phi(v^*) \) to be finite, \( Q_1 \) must be such that \( (I - P^*Q) \hat{v} \) has no poles on the UC. But since \( (I - P^*Q_0) \hat{v} \) is stable, this is equivalent to \( P^*Q_1 \hat{v} \) having no poles on the UC. Hence the optimal \( Q_1 \) must be such that \( P^*Q_1 \hat{v} \) is stable. Then the only possible unstable poles of \( f_2Q_1\hat{v} = z^{-1}(P^*_A)^{-1}P^*Q_1\hat{v} \) are the poles of \( (P^*_A)^{-1} \). But Assns. A1, A2 imply that the poles of \( (P^*_A)^{-1} \) are not among those of \( f_2Q_1\hat{v} \) and therefore \( f_2Q_1\hat{v} \) has to be stable. To proceed we will assume that \( Q_1 \) has this property. We will verify later that the solution indeed has this property.

Hence we can write

\[
\Phi(v^*) = \|\{ f_1 \}_- \|^2 + \|\{ f_1 \}_+ - f_2Q_1\hat{v} \|^2
\]

(15.6-6)

The first term on the RHS of (15.6-6) does not depend on \( Q_1 \). Hence for solving O1 we only have to look at the second term. The obvious solution is

\[
Q_1 \hat{v} = f_2^{-1}\{ f_1 \}_+
\]

(15.6-7)

Clearly such a \( Q_1 \) produces a stable \( f_2Q_1\hat{v} \) as was assumed. It should now be proved that \( Q_1 \)'s that satisfy the internal stability requirements exist among those described by (15.6-7), so that the obvious solution is a true solution. For \( n = 1 \), (15.6-7) yields a unique \( Q_1 \), which can be shown to satisfy the requirements by following the arguments in the proof of Thm. 15.6-2. For \( n \geq 2 \) write

\[
\hat{v} \triangleq (\hat{v}_1 \ \hat{v}_2 \ \ldots \ \hat{v}_n)^T
\]

(15.6-8)

\[
\hat{V}_2 \triangleq (\hat{v}_2 \ \ldots \ \hat{v}_n)^T
\]

(15.6-9)

\[
Q_1 \triangleq (q_1 \ q_2)
\]

(15.6-10)
where without loss of generality the first element of \( v^* \), and thus \( \hat{v}_1 \), is assumed to be nonzero. Also \( q_1 \) is \( n \times 1 \) and \( q_2 \) is \( n \times (n-1) \). Then from (15.6-7) it follows that

\[
Q_1 = (\hat{v}_1^{-1}(f_2^{-1}\{f_1\}_+ - q_2 \hat{V}_2) \quad q_2) \quad (15.6-11)
\]

We now need to show that a proper, stable \( q_2 \) exists such that \( Q_1 \) is proper, stable and produces a stable \( P^*Q_1P^* \). Select a \( q_2 \) of the form:

\[
q_2(z) = \hat{q}_2(z)(1 - z^{-1})^{3k} \prod_{i=1}^{k} (1 - \pi_i z^{-1})^3
\]

(15.6-12)

where \( \hat{q}_2 \) is proper and stable and \( \{\pi_1, \ldots, \pi_k\} \) are the poles of \( P^* \) outside the UC. Then from (15.6-11) it follows that in order for \( P^*Q_1P^* \) to be stable it is sufficient that \( P^*\hat{v}_1^{-1}f_2^{-1}\{f_1\}_+ \{P^*\}_{1 \times \text{row}} \) has no poles on or outside the UC. But \( P^*f_2^{-1} = zP_A^{-1} \) is stable and the only possible poles of \( \hat{v}_1^{-1}\{P^*\}_{1 \times \text{row}} \) on or outside the UC are poles of \( \hat{v}_1^{-1} \) outside the UC, because of Assns. A3 and A4. These are also the only possible unstable poles of \( Q_1 \). Let \( \alpha \) be such a pole (zero of \( \hat{v}_1 \)). Then for stability we need to find \( \hat{q}_2 \) such that

\[
\hat{q}_2(\alpha)\hat{V}_2(\alpha) = (1 - \alpha^{-1})^{-3k} \prod_{i=1}^{k} (1 - \pi_i \alpha^{-1})^{-3}f_2^{-1}(\alpha)\{f_1\}_+\bigg|_{z=\alpha} \quad (15.6-13)
\]

The above equation always has a solution because the vector \( \hat{V}_2(\alpha) \) is not identically zero since any common zeros in \( v^* \) outside the UC were factored out in \( v_0 \).

We now need to examine the properness of \( Q_1 \). Since \( (P^*_M)^{-1} \) is proper and \( \{f_1\}_+ \) is strictly proper, \( f_2^{-1}\{f_1\}_+ \) is proper. Then if \( \hat{v}_1^{-1} \) is improper (\( \hat{v}_1 \) strictly proper) there exists at least one element in \( \hat{V}_2 \) that is semi-proper. Hence by solving a system of linear equations we can always select a \( \hat{q}_2(z) \) such that of the first impulse response coefficients of \( f_2^{-1}\{f_1\}_+ - q_2 \hat{V}_2 \), as many are zero as needed to make the first element of the matrix in (15.6-11) proper.

We shall now proceed to obtain an expression for \( Q\hat{v} \). (15.2-3) and (15.6-11) yield

\[
Q\hat{v} = z(P^*_M)^{-1}\left[ z^{-1}(P^*_A)^{-1}P^*Q_0\hat{v} - \left\{ z^{-1}(P^*_A)^{-1}P^*Q_0\hat{v}\right\}_+ + \left\{ z^{-1}(P^*_A)^{-1}\hat{v}\right\}_+ \right]
\]

\[
= z(P^*_M)^{-1}\left[ \left\{ z^{-1}(P^*_A)^{-1}P^*Q_0\hat{v}\right\}_{0-} + \left\{ z^{-1}(P^*_A)^{-1}\hat{v}\right\}_+ \right] \quad (15.6-14)
\]

where \( \{\cdot\}_0 \) indicates that in the partial fraction expansion all poles on or outside the UC are retained. For (15.6-14), these poles are the poles of \( \hat{v} \) on or outside
the UC; \((P^*_M)^{-1}P^*Q_0 = P^*_M Q_0\) is strictly stable and proper because of Assn. A1 and the fact that \(Q_0\) is a stabilizing controller. The fact that \((I - P^*Q_0)V_0\) has no poles at \(z = 1\) imply that \((I - P^*Q_0)\) and its derivatives up to and including the \((\ell_v - 1)^{th}\) are equal to zero at \(z = 1\). Also, the fact that \((I - P^*Q_0)V_0\) is stable and that the columns of this diagonal \(V_0\) satisfy Assn. A3, imply that \((I - P^*Q_0) = 0\) at 1, \(\pi_1, \ldots, \pi_k\). Thus (15.6-14) simplifies to (15.6-3).

We now need to establish that a stabilizing controller \(Q_0\) exists with the property that \((I - P^*Q_0)V_0\) is stable. The selection of a \(V_0\) with the properties mentioned at the beginning of this section and its use instead of \(V\) in (15.6-16) yields such a controller.

\(\square\)

Objectives O2 and O3: Set of \(v^*\)'s.

Factor \(V\) similarly to \(P^*\) (see Sec. 15.6.4):

\[ V = V_M V_A \quad (15.6 - 15) \]

**Theorem 15.6-2.** Assume that Assns. A1-A5 hold. The controller

\[ \tilde{Q} = z(W P^*_M)^{-1}(z^{-1}W(P^*_M)^{-1}V_M)\cdot V^{-1}_M \quad (15.6 - 16) \]

is the unique solution to O3. Here the operator \(\{\cdot\}_\ast\) denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained except those corresponding to poles of \((P^*_M)^{-1}\).

**Proof.** Again we assume \(W = I\) and leave the weighted case as an exercise. From (15.3-13), (15.3-14), and (15.3-16) it follows that

\[ \Phi(v^1) + \Phi(v^2) + \ldots + \Phi(v^n) = \|(I - P\tilde{Q})V\|_2^2 \triangleq \Phi(V) \quad (15.6 - 17) \]

The minimization of \(\Phi(V)\) follows the steps in the proof of Thm. 15.6-1 up to (15.6-7), with \(V_M\) used instead of \(\tilde{v}\). In this case \(\ell_v\) is the maximum number of poles at \(z = 1\) in any element of \(V\). From the equivalent to (15.6-7) we obtain

\[ Q_1 = f_2^{-1}\{f_1\} + V^{-1}_M \quad (15.6 - 18) \]

We now have to establish that \(Q_1\) is stable, proper and produces a stable \(P^*Q_1P^*\). In \(P^*Q_1P^*\) the unstable poles of the \(P^*\) on the left cancel with those of \((P^*_M)^{-1}\) in \(f_2^{-1}\). As for the \(P^*\) on the right, cancellation follows from Assn. A5. Then in the same way that (15.6-3) follows from (15.6-14), (15.6-16) follows from (15.6-18).

\(\square\)

A more meaningful objective would be to solve Obj. O2. However a \(\tilde{Q}\) that solves Obj. O2 will also solve Obj. O3. Then from Thm. 15.6-2 it follows that
if a solution to O2 exists, it is given by (15.6–16). Factor each of the \( v^i \) in the same way as in (15.6–2):

\[
v^i(z) = v_0^i(z) \hat{v}^i(z)
\]

(15.6 – 19)

Define

\[
\hat{V} \triangleq (\hat{v}^1 \hat{v}^2 \ldots \hat{v}^n)
\]

(15.6 – 20)

**Theorem 15.6-3.** Assume that Assns. A1-A5 hold.

(i) If \( \hat{V}(z) \) is non-minimum phase (i.e., \( \hat{V}^{-1} \) is unstable or improper), then there exists no solution to Obj. O2.

(ii) If \( \hat{V}(z) \) is minimum phase, then use of \( \hat{V} \) instead of \( V_M \) in (15.6–16) yields exactly the same \( \hat{Q} \), which also solves Obj. O2. In addition \( \hat{Q} \) minimizes \( \Phi(v^*) \) for any \( v^*(z) \) that is a linear combination of \( v^i \)'s that have the same \( v_0^i \)'s.

**Proof.** (\( W = I \)). A stabilizing controller that solves Obj. O2 has to solve Obj. O1 for all \( v^i, i = 1, \ldots, n \). Satisfying (15.6–3) for every \( v^i \) is equivalent to

\[
\hat{Q} = z(P_M^*)^{-1}(z^{-1}(P_A^*)^{-1}\hat{V})_* \hat{V}^{-1}
\]

(15.6 – 21)

Hence the above \( \hat{Q} \) is the only potential solution for Obj. O2. However, it is not necessarily a stabilizing controller since not only stabilizing \( \hat{Q} \)'s satisfy (15.6–3) for some \( v^* \). Indeed, if \( \hat{V} \) is non-minimum phase, \( \hat{V}^{-1} \) is unstable and/or improper and this results in an unstable and/or improper \( \hat{Q} \), which is therefore unacceptable. Hence in such a case, there exists no solution for Obj. O2, which completes the proof of part (i) of the theorem.

In the case where \( \hat{V}^{-1} \) is stable and proper (\( \hat{V} \) minimum phase), the controller given by (15.6–21) is stable and proper and therefore it is the same as the one given by (15.6–16). This fact can be explained as follows. We have

\[
V = \hat{V}V_0
\]

(15.6 – 22)

where

\[
V_0 = \text{diag}\{v_0^1, v_0^2, \ldots, v_0^n\}
\]

(15.6 – 23)

Since \( \hat{V}^{-1} \) is stable and proper, (15.6–22) represents a factorization of \( V \) similar to that in (15.6–15). From spectral factorization theory it follows that
\[ \hat{V}(z) = V_M(z)A \]  

(15.6 - 24)

where \( A \) is a constant matrix such that \( AA^H = I \). Then (15.6-16) is not altered when \( \hat{V} \) is used instead of \( V_M \) because \( A \) cancels.

Let us now assume without loss of generality that the first \( j \) \( v^i \)'s have the same \( v_0^i \)'s. Consider a \( v^* \) that is a linear combination of \( v^1, \ldots, v^j \):

\[ v^*(z) = \alpha_1 v^1(z) + \ldots + \alpha_j v^j(z) \]  

(15.6 - 25)

Then it follows that

\[ v_0(z) = v_0^1(z) = \ldots = v_0^j(z) \]  

(15.6 - 26)

\[ \hat{v}(z) = \alpha_1 \hat{v}^1(z) + \ldots + \alpha_j \hat{v}^j(z) \]  

(15.6 - 27)

One can easily check that a \( \bar{Q} \) that satisfies (15.6-3) for \( \hat{v}^1, \ldots, \hat{v}^j \), will also satisfy (15.6-3) for the \( \hat{v} \) given by (15.6-27) because of the property

\[ \{\alpha_1 f_1(z) + \ldots + \alpha_j f_j(z)\}_* = \alpha_1 \{f_1(z)\}_* + \ldots + \alpha_j \{f_j(z)\}_* \]  

(15.6 - 28)

But then from Thm. 15.6-1 it follows that if a stabilizing controller \( \bar{Q} \) satisfies (15.6-3) for \( \hat{v} \), then it minimizes the \( L_2 \) error \( \Phi(v^*) \).

The following corollary to Thm. 15.6-3 holds for a specific choice of \( V \).

**Corollary 15.6-1.** Let

\[ V = \text{diag}\{v_1, v_2, \ldots, v_n\} \]  

(15.6 - 29)

where \( v_1(z), \ldots, v_n(z) \) are scalars. Then use of \( \hat{V} \) instead of \( V_M \) in (15.6-16) yields exactly the same \( \bar{Q} \), which minimizes \( \Phi(v^*) \) for the following \( n \) vectors:

\[ v^* = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} \]  

(15.6 - 30)

and their multiples, as well as for the linear combinations of those directions that correspond to \( v_i \)'s with the same zeros outside the UC with the same degree and the same time delays.

**Example 15.6-1 (Minimum phase \( P \)).** \( P^*(z) \) cannot be truly MP for a physical system. Even if the Laplace transfer matrix representing the continuous
plant is MP but strictly proper, the discretized plant \( P^*(z) \) will still have a delay of one unit because of sampling. Hence \( P_A^* = z^{-1}I \), \( P_M^* = zP^* \) and (15.6-16) yields for \( W = \text{constant} \)

\[
\hat{Q} = (P^*)^{-1}(I - KV_M^{-1})
\]  
(15.6 - 31)

where \( K \) is the constant term in a partial fraction expansion of \( V_M \). This is equal to the first non-zero matrix in the impulse response description of \( V(z) \), which can be obtained by long division.

### 15.6.2 Setpoint Prediction

In the case of setpoint tracking, future values of \( r^* \) are often known and supplied to the computer ahead of time. If at time \( t \) the setpoint value that is provided to the control algorithm as \( Z^{-1}\{r^*(z)\} \) is the one we wish the plant output to reach at time \( t + NT \), then the objective function has to be modified to:

\[
\Phi_N(r^*) = \|W(z^{-N}I - P^*\hat{Q})r^*\|^2_2
\]  
(15.6 - 32)

If the above objective function is used for Objs. O1, O2, O3, then the resulting expressions for the \( H_2^* \)-optimal controller are the same as in Thms. 15.6-1, 15.6-2, and 15.6-3, but with the term \( z^{-N-1} \) instead of \( z^{-1} \) inside \( \{\cdot\} \). All the steps in the proofs remain the same when (15.6-32) is used rather than (15.3-14).

### 15.6.3 Intersample Rippling

The \( H_2^* \)-optimal controller minimizes the sum of squared errors and completely disregards the plant’s output behavior between the sample points. Therefore the performance of the \( H_2^* \)-optimal controller may be excellent at the sample points but may suffer from severe intersample rippling. This problem was demonstrated in Sec. 7.5.3. A modification was introduced in Secs. 8.1.2 and 9.2.2 to substitute poles in \( \tilde{Q} \) close to (-1,0) with poles at \( z = 0 \). The new \( \tilde{Q} \) was shown to be free of the problem of intersample rippling and to combine desirable deadbeat type characteristics with those of the \( H_2^* \)-optimal controller. This section extends the modification to MIMO systems and general open-loop stable and unstable plants. It should be pointed out that this modification is sufficient only if there are no open-loop oscillatory poles in the continuous plant transfer function, which have become unobservable after sampling.

Let \( \hat{Q}_H(z) \) be the \( H_2^* \)-optimal \( \hat{Q} \) obtained according to the previous sections. Also let \( \delta(z) \) be the least common denominator of the elements of \( P^*(z) \), and \( \kappa_i; \) \( i = 1, \ldots, \rho \) be the roots of \( \delta(z) \) close to (-1,0) (or in general with negative real
part). Define
\[ \tilde{q}_-(z) = z^{-\rho} \prod_{j=1}^{\rho} \frac{z - \kappa_j}{1 - \kappa_j} \] (15.6 - 33)

Then \( \tilde{Q}_H \) is modified as follows:
\[ \tilde{Q}(z) = \tilde{Q}_H(z)\tilde{q}_-(z)B(z) \] (15.6 - 34)

where the scalar \( B(z) \) is selected so that the matrix \( S_2 \) (15.2-2) and \( (I - P^*\tilde{Q})V \) remain stable. Let \( \pi_i, i = 1, \ldots, \xi \) be the unstable roots (including \( \pi_1 = 1 \)) of the least common denominator of \( P^*(z), V(z) \). Let the multiplicity of each of them be \( m_i \). Note that the poles outside the UC have multiplicity one, according to Assns. A1 and A3. Remember also that according to Assns. A3 and A4, \( V \) has at least as many poles at \( z = 1 \) as \( P^* \) and that each pole of \( V \) outside the UC is also a pole of \( P^* \). Then, since \( \tilde{Q}_H \) makes \( S_2 \) and \( (I - P^*\tilde{Q}_H)V \) stable, it follows that the requirements on \( B(z) \) are:

\[ \frac{d^k}{dz^k}(1 - \tilde{q}_-(z)B(z)) \bigg|_{z=\pi_i} = 0, \quad k = 0, \ldots, m_i - 1; \quad i = 1, \ldots, \xi \] (15.6 - 35)

We can write
\[ B(z) = \sum_{j=0}^{M-1} b_j z^{-j} \] (15.6 - 36)

where
\[ M = \sum_{i=0}^{\xi} m_i \] (15.6 - 37)

and then compute the coefficients \( b_j, j = 0, \ldots, M - 1 \) from (15.6-35). Note that since none of the \( \pi_i \)'s is 0 or \( \infty \), (15.6-35) is equivalent to

\[ \frac{d^k}{d\lambda^k}(1 - \tilde{q}_-(\lambda^{-1})B(\lambda^{-1})) \bigg|_{\lambda=\pi_i^{-1}} = 0, \quad k = 0, \ldots, m_i - 1; \quad i = 1, \ldots, \xi \] (15.6 - 38)

Both \( \tilde{q}_-(\lambda^{-1}) \) and \( B(\lambda^{-1}) \) are polynomials in \( \lambda \) and therefore their derivatives with respect to \( \lambda \) can be computed easily. Then (15.6-38) yields a system of \( M \) linear equations with \( M \) unknowns \( (b_0, b_1, \ldots, b_{M-1}) \). The resulting controller \( \hat{Q} \) combines the desirable properties of the \( H_\infty^2 \)-optimal controller and deadbeat type controllers.
Example 15.6-2. This example is presented to demonstrate the problem of intersample rippling in the $H_2^*$-optimal controller and the modification discussed above. Consider the continuous system

$$P(s) = \begin{pmatrix} 0.50 & 1.42 \\ 1.00 & 0.00 \end{pmatrix} \begin{pmatrix} s+1 \\ 2s+1 \end{pmatrix} \begin{pmatrix} 0.6s+1 \\ 1.00 \end{pmatrix}$$

(15.6 - 39)

The discretized system (zero order hold included) for a sampling time of $T = 1$, is

$$P^*(z) = \begin{pmatrix} 0.316 & 0.218 \\ 0.303 & 0.221 \end{pmatrix} \begin{pmatrix} z-0.306 & z-0.846 \\ z-0.607 & z-0.779 \end{pmatrix}$$

(15.6 - 40)

Computation of the roots of $\det P(z) = 0$ shows that the system in (15.6-40) has two finite zeros, at $a_1 = -0.95$ and $a_2 = 0.75$. The first zero is close to (-1,0) and is expected to cause intersample rippling when the $H_2^*$-optimal controller is used.

We find from (15.6-40) that $P_A^* = z^{-1}I$, $P_M^* = zP$. We shall consider step setpoint changes as external inputs - i.e.,

$$V(z) = \frac{z}{z-1}I$$

(15.6 - 41)

Then (15.6-16) yields

$$\tilde{Q}_H(z) = z^{-1}P^{-1}$$

(15.6 - 42)

Figure 15.6-1A shows the time response of this control system for a unit step change in the setpoint of output 1:

$$v^*(z) = r^*(z) = \begin{pmatrix} z/(z-1) \\ 0 \end{pmatrix}$$

(15.6 - 43)

The prediction of intersample rippling is verified. Note that at the sample points the outputs are indeed exactly at the setpoints yielding the minimum sum of squared errors.

The IMC controller is now obtained from (15.6-34) with $B(z) = 1$ and

$$\hat{q}_-(z) = \frac{z + 0.95}{1.95z}$$

(15.6 - 44)

The response for this control system is shown in Fig. 15.6-1B. Clearly the rippling problem has disappeared. Note the inverse responses caused by the RHP zero of the continuous system $P(s)$. □
15.6.4 Inner-Outer Factorization

As mentioned in Sec. 15.6.1, the factorization (15.6–1) is accomplished by employing the bilinear transformation

\[ z = \frac{1 + s}{1 - s} \quad (15.6 - 45) \]

to reduce the problem into the one discussed in detail in Sec. 12.6.4. The theorems below provide the formulas for the transformation of state space descriptions implied by (15.6–45) or its inverse

\[ s = \frac{-1 + z}{1 + z} \quad (15.6 - 46) \]

The following lemma is used in the proofs of the theorems.

**Lemma 15.6-1.** Let \( G(x) = C(xI - A)^{-1}B + xD \). Then
15.6. IMC DESIGN: STEP 1 ($\bar{Q}$)

\[ xG(x) = C(xI - A)^{-1}AB + CB + xD \]  \hspace{2cm} (15.6-47)

**Proof.** We have

\[ xG(x) = C(xI - A)^{-1}(A + xI - A)B + xD \]
\[ = C(xI - A)^{-1}AB + CB + xD \]

\[ \square \]

**Theorem 15.6-4.** Let $G^*(z) = C(zI - \Phi)^{-1}\Gamma + D$ have no poles at $z = -1$. Then

\[ \hat{G}(s) \triangleq G^* \left( \frac{1+s}{1-s} \right) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \]  \hspace{2cm} (15.6-48)

where

\[ \hat{A} = (\Phi + I)^{-1}(\Phi - I) \]  \hspace{2cm} (15.6-49)
\[ \hat{B} = 2(\Phi + I)^{-1}\Gamma \]  \hspace{2cm} (15.6-50)
\[ \hat{C} = C \]  \hspace{2cm} (15.6-51)
\[ \hat{D} = D - C(\Phi + I)^{-1}\Gamma \]  \hspace{2cm} (15.6-52)

**Proof.** Since $P^*(z)$ is assumed to have no poles at $z = -1$, $\Phi + I$ is nonsingular. We can then write

\[ \hat{G}(s) = C \left( \frac{1+s}{1-s} I - \Phi \right)^{-1}\Gamma + D \]
\[ = (1-s)C(s(\Phi + I) + I - \Phi)^{-1}\Gamma + D \]
\[ = (1-s)C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(\Phi + I)^{-1}\Gamma + D \]  \hspace{2cm} (15.6-53)

Use of Lem. 15.6-1 in (15.6-53) yields

\[ \hat{G}(s) = C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(\Phi + I)^{-1}\Gamma \]
\[ - [C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(\Phi + I)^{-1}(\Phi - I)(\Phi + I)^{-1}\Gamma + C(\Phi + I)^{-1}\Gamma] + D \]
\[ = C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(I - (\Phi + I)^{-1}(\Phi - I))\Phi(I + I)^{-1} + D - C(\Phi + I)^{-1}\Gamma \]

\[ = C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}2(\Phi + I)^{-2}\Gamma + D - C(\Phi + I)^{-1}\Gamma \]

\[ \square \]

**Theorem 15.6.5.** Let \( \hat{G}(s) = \hat{C}(sI - \hat{\Delta})^{-1}\hat{B} + \hat{D} \) have no poles at \( s = 1 \). Then

\[ G^*(z) \triangleq \hat{G}(\frac{-1+z}{1+z}) = C(zI - \Phi)^{-1}\Gamma + D \]  \hspace{1cm} (15.6 - 54)

where

\[ \Phi = (I - \hat{\Delta})^{-1}(I + \hat{\Delta}) \]  \hspace{1cm} (15.6 - 55)

\[ \Gamma = 2(I - \hat{\Delta})^{-2}\hat{B} \]  \hspace{1cm} (15.6 - 56)

\[ C = \hat{C} \]  \hspace{1cm} (15.6 - 57)

\[ D = \hat{D} + \hat{C}(I - \hat{\Delta})^{-1}\hat{B} \]  \hspace{1cm} (15.6 - 58)

**Proof.** \( I - \hat{\Delta} \) is nonsingular because \( \hat{G}(s) \) is assumed to have no poles at \( s = 1 \).

We have

\[ G^*(z) = \hat{C}(\frac{-1+z}{1+z}I - \hat{\Delta})^{-1}\hat{B} + \hat{D} \]

\[ = (1+z)\hat{C}(zI - \hat{\Delta}) - (I + \hat{\Delta}))^{-1}\hat{B} + \hat{D} \]

\[ = (1 + z)\hat{C}(zI - (I - \hat{\Delta})^{-1}(I + \hat{\Delta}))^{-1}(I - \hat{\Delta})^{-1}\hat{B} + \hat{D} \]  \hspace{1cm} (15.6 - 59)

Application of Lem. 15.6-1 to (15.6-59) yields

\[ G^*(z) = \hat{C}(zI - (I - \hat{\Delta})^{-1}(I + \hat{\Delta}))^{-1}(I - \hat{\Delta})^{-1}\hat{B} \]

\[ + \left[ \hat{C}(zI - (I - \hat{\Delta})^{-1}(I + \hat{\Delta}))^{-1}(I - \hat{\Delta})^{-1}(I + \hat{\Delta})(I - \hat{\Delta})^{-1}\hat{B} + \hat{C}(I - \hat{\Delta})^{-1}\hat{B} \right] + \hat{D} \]

\[ = \hat{C}(zI - (I - \hat{\Delta})^{-1}(I + \hat{\Delta}))^{-1}(I + (I - \hat{\Delta})^{-1}(I + \hat{\Delta}))\hat{B} + \hat{D} + \hat{C}(I - \hat{\Delta})^{-1}\hat{B} \]
\[ (15.6 - 45) \]

The factorization

\[ P^*(z) = P^*_A(z)P^*_M(z) \]  \hspace{1cm} (15.6 - 1)

involves the following steps:

**Step 1:** Use the variable transformation (15.6–45) on \( P^*(z) \) to obtain \( \hat{P}(s) \). Note that the assumption of Thm. 15.6-4 that \( P^*(z) \) has no poles at \( z = -1 \) holds for the \( P^* \)'s under consideration in this chapter because of Assn. A2.

**Step 2:** Apply Thm. 12.6-4 on \( \hat{P}(s) \) to obtain the factorization

\[ \hat{P}(s) = \hat{P}_A(s)\hat{P}_M(s) \]  \hspace{1cm} (15.6 - 60)

Note that for a strictly proper system \( D = 0 \) and therefore from (15.6–52) we have \( \hat{D} = -C(\Phi + I)^{-1}\Gamma = P^*(-1) \). According to Assn. A2, \( P^*(z) \) has no zeros on the UC and therefore \( P^*(-1) \) has full rank. Hence, the assumption of no zeros on the imaginary axis including infinity in Thm. 12.6-4, holds for \( \hat{P}(s) \).

**Step 3:** Use the variable transformation (15.6–46) on \( \hat{P}_A(s) \) and \( \hat{P}_M(s) \) to obtain \( P^*_A(z) \) and \( P^*_M(z) \) correspondingly. Note that \( \hat{P}_A(s) \) satisfies the assumption of no poles at \( s = 1 \), since by construction all its poles are in the LHP. Also, \( \hat{P}_M(s) \) has the poles of \( \hat{P}(s) \),which do not include a pole at \( s = 1 \), since \( P^*(z) \) has no poles at \( z = \infty \).

The result of the above steps is a stable, all-pass \( P^*_A \) and a minimum phase \( P^*_M \). Both \( P^*_A \) and \( P^*_M \) are proper because \( \hat{P}_A \) and \( \hat{P}_M \) have no poles at \( s = 1 \). Also note that since \( \hat{P}_M(s) \) is minimum phase, it does not have a zero at \( s = 1 \) and therefore \( P^*_M(z) \) has no zero at \( z = \infty \), which means that \( (P^*_M(z))^{-1} \) is proper.

To obtain the factorization

\[ V = V_MV_A \]  \hspace{1cm} (15.6 - 15)

one should follow the same steps as above with the difference that in Step 2, Thm. 12.6-4 is applied on \( \hat{V}(s)^T \) as described in Sec. 12.6-4.

### 15.7 IMC Design: Step 2 \((F)\)

The controller \( \hat{Q} \) obtained in Step 1 of the IMC design procedure is detuned in Step 2 to satisfy the robustness conditions by augmenting it with the IMC filter \( F(z) \):
\[ Q = Q(\hat{Q}, F) \]  
(15.7 - 1)

First we will postulate reasonable filter structures. Then we will define appropriate minimization problems to be solved over the filter parameters and discuss the computational issues involved.

### 15.7.1 Filter Structure

Some structure has to be assumed for \( F \), which can be as general as the designer wishes. However, in order to keep the number of variables in the optimization problems small, a rather simple structure like a diagonal \( F \) with first- or second-order terms would be recommended. In most cases this is not restrictive because the controller \( \hat{Q} \) that was designed in the first step of the IMC procedure is in general a full high-order transfer matrix. More complex filter structures may be necessary in cases of ill-conditioned systems (\( \hat{\sigma}(\hat{P}^*)/\hat{\sigma}(P^*) \) very large). For such systems a two-filter structure was discussed in detail in Sec. 12.7.1. The elements of each of the two filters in that structure can be designed as described below.

The filter \( F(z) \) is chosen to be a diagonal rational function that satisfies the following requirements.

a. Internal Stability. \( S_1 \) in (15.2-1) must be stable.

b. Asymptotic setpoint tracking and/or disturbance rejection. \( (I - \hat{P}^* \hat{Q} F) v^* \) must be stable.

Write

\[ F(z) = \text{diag}\{f_1(z), \ldots, f_n(z)\} \]  
(15.7 - 2)

Then, Assns. A1-A5 and the fact that by construction \( \hat{Q}(z) \) makes \( S_1 \) and \( (I - \hat{P}^* \hat{Q}) V \) stable, imply that the requirements on an element \( f_\ell \) of \( F \) are:

\[
\frac{d^j}{dz^j}(1 - f_\ell(z)) \bigg|_{z = \pi_1} = 1, \quad j = 0, \ldots, m_{1\ell} - 1 
\]  
(15.7 - 3)

\[
f_\ell(\pi_i) = 1, \quad i = 2, \ldots, \xi 
\]  
(15.7 - 4)

where \( \pi_1 = 1 \) and \( m_{1\ell} \) is the highest multiplicity of such a pole in any element of the \( \ell^{th} \) row of \( V \) and \( \pi_i, i = 2, \ldots, \xi \) are the poles of \( P^* \) outside the UC, each with multiplicity 1, according to Assn. A1.

One can now select the filter elements to be of the form discussed in Sec. 9.3

\[ f(z) = \phi(z)f_1(z) \]  
(9.3 - 3)
where

\begin{align*}
  f_1(z) &= \frac{(1 - \alpha)z}{z - \alpha} \\
  \phi(z) &= \sum_{j=0}^{w} \beta_j z^{-j}
\end{align*}

and the coefficients $\beta_0, \ldots, \beta_w$ are computed so that (15.7-3), (15.7-4) are satisfied for some specified $\alpha$. The parameter $\alpha$ can be used as a tuning parameter.

Note that for $\xi = 1$, $\pi_1 = 1$, $m_{1\ell} = 1$, we only need $\phi(z) = 1$. For the general case, (15.7-3) and (15.7-4) a system of $M_\ell$ linear equations with $\beta_0, \ldots, \beta_w$ as unknowns where $M_\ell$ is given by

\begin{equation}
  M_\ell = m_{1\ell} + \xi \tag{15.7 - 5}
\end{equation}

The procedure for solving these equations is identical to the one described in Sec. 9.3, with $m_{1\ell}$ and $M_\ell$ replacing $m_1$ and $M$. Also, when the two-filter structure of Sec. 12.7.1 is used in the case of ill-conditioned plants, $\hat{m}_1 = \max_\ell m_{1\ell}$ should be used for all $\ell$ in the place of $m_{1\ell}$ in $F_1(z)$. This is required for internal stability and no steady-state offset.

15.7.2 Robust Stability Interconnection Structure

Consider the block diagram in Fig. 15.1-2C. The same block manipulations that were used to obtain Fig. 12.7-1A from Fig. 12.1-1B, can be used on Fig. 15.1-2C to obtain Fig. 15.7-1. The only difference is that $P^*_\gamma$ and $P^*_\gamma$ take the place of $P$ and $\hat{P}$. All the transfer matrices in Fig. 15.7-1 are discrete.
The development described in Sec. 12.7.2 can now be applied to the block structure of Fig. 15.7-1 to put it in the form shown in Figs. 12.7-1B, C. The difference is that the uncertainty block \( \Delta \) represents a discrete transfer function. If simple uncertainty descriptions of the form discussed in Sec. 12.7.2 are available for the discrete plant \( P^* \), then the corresponding formulas carry over to the discrete case, where \( \bar{L}_A, \bar{L}_O, \bar{L}_I, W_{s1}, W_{s2} \) are now discrete. Therefore only their values up to \( \pi/T \) need be considered.

Theorem 15.4-1 provides the robust stability condition. The matrix \( M \) in (15.4-1) is \( G_{11}^F \) and therefore for robust stability the filter has to be designed such that

\[
\mu_\Delta(G_{11}^F(i\omega)) < 1, \quad 0 \leq \omega \leq \pi/T
\]  

(15.7 - 6)

The superscript * is used to indicate that in this case \( G^F \) is a discrete transfer matrix.

### 15.7.3 Robust Performance Interconnection Structure

If one only cared about the performance at the sample points, then one could use Fig. 15.7-1 to state the robust performance conditions. However, because of the intersample rippling problem, one has to consider the continuous output of the plant and express the robust performance requirements in terms of the approximate sensitivity functions \( E_r(s) \) or \( E_d(s) \) given by (15.5-4) and (15.5-5).

One can obtain the appropriate interconnection structures of Fig. 12.7-1 by starting from Fig. 15.1-2B. The use of (15.5-2) in the derivation of (15.5-4) and (15.5-5), is equivalent to approximating the function of the sampling operator by \( 1/T \) for \( 0 \leq \omega \leq \pi/T \). This approximation is reasonable for signals with small power for \( \omega > \pi/T \). Use of \( 1/T \) in the place of the sampling switch in Fig. 15.1-2B, allows us to derive the matrix \( G \) in the block diagram in Fig. 12.7-1A, which is slightly different from the one given by (12.7-13) for the continuous controller.

For \( v = d, e = y \), we have

\[
G_d = \begin{pmatrix}
0 & 0 & h_0 \bar{Q}
\end{pmatrix}
\begin{pmatrix}
I & -\frac{2}{T}I & 0
\end{pmatrix}
\begin{pmatrix}
I & I & h_0 \bar{P}\bar{Q}
\end{pmatrix}
\]  

(15.7 - 7)

For \( v = -r, e = y - r \), with \( r(s) = h_0(s)r^*(e^{sT}) \):

\[
G_r = \begin{pmatrix}
0 & 0 & h_0 \bar{Q}
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0
\end{pmatrix}
\begin{pmatrix}
I & h_0 \bar{P}\bar{Q}
\end{pmatrix}
\]  

(15.7 - 8)

Note that in this case the uncertainty block \( \Delta \) represents continuous transfer functions as in Sec. 12.7.2.
15.7. **IMC DESIGN: STEP 2 (F)**

### 15.7.4 Robust $H_\infty$ Performance Objective

The next step is to transform (15.5–6) into an equivalent SSV condition. By using the equations in Sec. 12.7.2, but with the appropriate $G_v$ instead of $G$ and with $W_2 = W$, $W_1 = I$, we can obtain the corresponding $G_v^F$. Then (15.5–6) is satisfied if and only if

$$
\mu_\Delta(G_v^F(i\omega)) < 1, \quad 0 \leq \omega \leq \pi/T \tag{15.7-9}
$$

where $\Delta = \text{diag}\{\Delta_u, \Delta_p\}$, with $\Delta$ representing the uncertainty block $\Delta$ of Fig. 12.7-1 and $\Delta_p$ the additional block introduced for performance.

We can now write

$$
F \triangleq F(z; \Lambda) \tag{15.7-10}
$$

where $\Lambda$ is an array with the adjustable filter parameters. The filter design problem can be formulated as an minimization problem over the elements of $\Lambda$. In the filter structure proposed in Sec. 15.7.1, there is one adjustable parameter $\alpha$ for each element of the diagonal filter, or of each of the two diagonal filters, if two are used. Each one of these real parameters, say $\alpha_j$, has to be inside the UC for $F$ to be stable. The stability constraints can be removed from the minimization problem by setting

$$
\alpha_j = e^{-T/\lambda_j^2} \tag{15.7-11}
$$

where $\lambda_j$ is an element of $\Lambda$. Then any $\lambda_j$ in $(-\infty, \infty)$ produces an $\alpha_j$ in $[0, 1)$. Note that if the parametrization (15.7–11) is used, then it is $\lambda_j^2$ and not $\lambda_j$ that corresponds to a time constant. If one wishes to use a higher order $f_1(z)$ with more parameters in (8.2–1), one can write the denominator of each element of $F$ as the product of polynomials of degree 2 and one of degree 1 if the order is odd. A polynomial of degree 2 with roots inside the UC can be written as $z^2 - (e^{T_{p_1}} + e^{T_{p_2}})z + e^{T_{p_1} + T_{p_2}}$, where $p_1, p_2$ are the roots of $\lambda_2^2x^2 + \lambda_1^2x + 1 = 0$ for some value of $\lambda_1, \lambda_2$. In this way, the optimization problem is unconstrained in the optimization variables $\lambda_1, \lambda_2$, which can take any value in $(-\infty, \infty)$.

Our goal is to satisfy (15.7–9). The filter parameters can be obtained by solving

**Objective O4:**

$$
\min_{\Lambda} \max_{0 \leq \omega \leq \pi/T} \mu_\Delta(G_v^F) \tag{15.7-12}
$$

It should be noted however that the optimal solution for Obj. O4, may still not satisfy (15.7–9). The reason is usually that the performance requirements set by the selection of $W$ in (15.5–6) are too tight be to satisfied in the presence of the
model-plant mismatch. In this case one should choose a less tight \( W \) and resolve Obj. O4.

Another important point is that satisfaction of the robust performance condition (15.7–9) does not necessarily imply satisfaction of the robust stability condition (15.7–6), which was the case in the continuous controller design. This is so even if the uncertainty descriptions for the continuous plant [used in (15.7–9)] and the discretized plant [used in (15.7–6)] correspond to exactly the same sets of possible plants. The reason is that (15.7–9) was obtained by using the approximations discussed in detail in Sec. 15.5.1, while there are no approximations involved in the derivation of (15.7–6). Note however, that if the uncertainty descriptions for the continuous and the discrete plant are equivalent in the sense discussed in Sec. 15.1.5, then satisfaction of (15.7–9) is usually sufficient for satisfaction of (15.7–6), although this is not guaranteed. As a result of the above discussed possibility, when a solution to Obj. O4 is found, one should check if (15.7–6) holds. If this does not happen, then one can always substitute the robust stability \( \mu \) (15.7–6) in Obj. O4 and proceed with the minimization until (15.7–6) becomes less than one.

The type of problem defined by (15.7–12) is nearly identical to that defined by (12.7–25). The only difference is that the search over \( \omega \) is limited to \( 0 \leq \omega \leq \pi/T \) in (15.7–12). This difference disappears, when only a finite number of frequencies is considered, as described in Obj. O4' in Sec. 12.7.3. Hence the entire procedure and equations of Sec. 12.7.3 carry over to this case.

### 15.8 Illustration of the Design Procedure

The purpose of this section is to demonstrate the IMC design procedure by applying it to a \( 2 \times 2 \) open-loop unstable system.

#### 15.8.1 System Description

Let the continuous system be modeled by

\[
\dot{x} = Ax + Bu
\]

\[
y = Cx + Du
\]

where

\[
A = \begin{pmatrix}
2.375 & 0.857 & 1.000 \\
-17.719 & -5.500 & -5.250 \\
-14.766 & -6.750 & -7.375 \\
\end{pmatrix}
\]
15.8. ILLUSTRATION OF THE DESIGN PROCEDURE

\[ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \quad (15.8 - 2) \]

\[ C = \begin{pmatrix} 0 & 0.3 & 1.8 \\ 0 & 0 & -4.0 \end{pmatrix} \quad (15.8 - 3) \]

\[ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (15.8 - 4) \]

The eigenvalues of \( A \), which are the poles of the system (see Def. 10.1-3), are located at -1, -10, +0.5. Hence the open-loop system has an unstable pole of multiplicity 1 at 0.5.

From (10.1-7) we obtain the transfer matrix of the system:

\[ \hat{P}(s) = \begin{pmatrix} \frac{-1.5(s-0.2)}{(s-0.5)(s+1)} & \frac{0.3(s+7.5)}{(s-0.5)(s+10)} \\ \frac{4s-0.5}{(s-0.5)(s+1)} & \frac{-(4s+8.5)}{(s-0.5)(s+10)} \end{pmatrix} \quad (15.8 - 5) \]

Note that the unstable pole \((s = 0.5)\) appears in all elements of \( \hat{P}(s) \), though it has only multiplicity 1. This is not an artifact of the example but rather the generic case for systems described by equations of the type (10.1–1), (10.1–2).

Let us now compute the zero-order hold discrete equivalent of \( \hat{P}(s) \) for a sampling time of \( T = 0.1 \). This is a reasonable choice, equal to 1/10 of the dominant stable time constant and 1/20 of the unstable time constant of the system. From (15.1–6) we find \( \hat{P}^*(z) \), which can be written in the form (15.1–15):

\[ \hat{P}^*(z) = C(zI - \Phi)^{-1}\Gamma + D \quad (15.8 - 6) \]

where \( C \) and \( D \) are given by (15.8–3) and (15.8–4) and

\[ \Phi = \begin{pmatrix} 1.2757 & 1.1138 & 1.0 \\ -0.15462 & 0.44053 & -0.41687 \\ -0.079536 & -0.44598 & 0.60772 \end{pmatrix} \quad (15.8 - 7) \]

\[ \Gamma = \begin{pmatrix} 0 & 0 \\ 0.071429 & 0 \\ -0.094864 & 0.071429 \end{pmatrix} \quad (15.8 - 8) \]

For the design we need some information on the potential model error. We will assume a diagonal input multiplicative uncertainty

\[ P^*(z) = \hat{P}^*(z)(I + L^*_I(z)) \quad (15.8 - 9) \]

where

\[ L^*_I(z) = \text{diag}\{\ell^*_1(z), \ell^*_2(z)\} \quad (15.8 - 10) \]
and $\ell_1^v, \ell_2^v$ are bounded by

$$
|\ell_1^v(z)| \leq \hat{\ell}_1^v(z) = \left| 0.2 \frac{z - p_i}{z - p_i^{10}} \frac{1 - p_i^{10}}{1 - p_i} \right|
$$

(15.8 - 11)

with $p_i = e^{-T/\tau_i}$. We will also assume that all plants $P^*(z)$ have exactly one unstable pole. The bound (15.8-11) implies that the uncertainty starts to increase around $\omega = 1/\tau_i$ with slope 1 and flattens out after one decade. Also, the low frequency uncertainty can be as much as 20%. The $\tau_i$'s are selected here to correspond to the dominant stable time constants of $\hat{P}(s)$ associated with the respective inputs, i.e., $\tau_1 = 1$ and $\tau_2 = 0.1$. Bode plots of $\hat{\ell}_1^v, \hat{\ell}_2^v$ are shown in Fig. 15.8-1, for $0 \leq \omega \leq \pi/T$.

### 15.8.2 Design of $\tilde{Q}$

First one has to decide on the type of external input $v$ for which $\tilde{Q}$ will be designed. Here we will consider step-like disturbances entering at the plant inputs. The diagonal $V(z)$ is of the form described in Cor. 15.6-1 with

$$
v_1(z) = v_2(z) = v^*(z) = ZL^{-1}\{v(s)\}
$$

(15.8 - 12)

where $v(s)$ is an appropriate transfer function. Since the $v_i$'s represent the effect of step-like inputs on the plant outputs, $v(s)$ should include both an integrator and a pole at $s = 0.5$. The simplest choice would be $v(s) = s^{-1}(-s + 0.5)^{-1}$. However, such an input is “sluggish” and will result in poor robustness (see observation 3 in Sec. 4.1.2). To avoid this problem we select

$$
v(s) = \frac{s + 0.5}{s(-s + 0.5)}
$$
15.8. ILLUSTRATION OF THE DESIGN PROCEDURE

The next task is the factorization of \( \tilde{P}^* \) into \( P_A^* \) and \( P_M^* \) (15.6-1). We follow the steps described in Sec. 15.6.4. This procedure yields the matrices \( \Phi_A, \Gamma_A, C_A, D_A \) and \( \Phi_M, \Gamma_M, C_M, D_M \) that define \( P_A^*(z) \) and \( P_M^*(z) \) respectively through (15.1-15):

\[
\Phi_A = \begin{pmatrix} 1.54714 & 1.50513 & 1.41162 \\ -0.69253 & -0.67372 & -0.63186 \\ -0.098133 & -0.095468 & -0.089537 \end{pmatrix} \quad (15.8 - 13)
\]

\[
\Gamma_A = \begin{pmatrix} -8.27667 & -3.06852 \\ -15.61316 & -4.02293 \\ -3.78625 & -3.05041 \end{pmatrix} \quad (15.8 - 14)
\]

\[
C_A = \begin{pmatrix} -7.0645 \times 10^{-4} & -0.012260 & 0.20435 \\ 0.027830 & 0.13477 & -0.46555 \end{pmatrix} \quad (15.8 - 15)
\]

\[
D_A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (15.8 - 16)
\]

\[
\Phi_M = \begin{pmatrix} 1.27570 & 1.11380 & 1.0 \\ -0.15462 & 0.44053 & -0.41687 \\ -0.079536 & -0.44598 & 0.60772 \end{pmatrix} \quad (15.8 - 17)
\]

\[
\Gamma_M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1.32810 & 1 \end{pmatrix} \quad (15.8 - 18)
\]

\[
C_M = \begin{pmatrix} -0.017300 & -0.060253 & 0.050708 \\ 0.021810 & 0.12528 & -0.18355 \end{pmatrix} \quad (15.8 - 19)
\]

\[
D_M = \begin{pmatrix} -0.13652 & 0.086180 \\ 0.39111 & -0.30647 \end{pmatrix} \quad (15.8 - 20)
\]

We also need to factor \( V(z) \) according to (15.6-15), but this is trivial since \( V(z) \) is diagonal and \( v^*(z) \) can be factored as described by (8.1-3).

The final task is to determine \( \tilde{Q}(z) \) from (15.6-16). For \( W = I \) a state space description (15.1-15) of \( \tilde{Q}(z) \) is given by

\[
\Phi_Q = \begin{pmatrix} 1.27570 & 1.11380 & 1 & 0 & 0 \\ -0.57541 & -0.50239 & -0.45106 & -2.25465 & -0.80172 \\ 0.013486 & 0.011775 & 0.010572 & 0.058490 & 3.2232 \times 10^{-3} \\ 0 & 0 & 0 & 0.94873 & 0 \\ 0 & 0 & 0 & 0 & 0.94873 \end{pmatrix} \quad (15.8 - 21)
\]
Figure 15.8-2. Nominal response to a step change at the plant input. (No filter).

\[
\Gamma_Q = \begin{pmatrix} 0 & 0 \\ 81.6566 & 26.2327 \\ -3.09800 & 2.64970 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\] (15.8 - 22)

\[
C_Q = \begin{pmatrix} 0.42079 & 0.94292 & 0.034188 & 2.25465 & 0.80172 \\ 0.46584 & 0.79454 & 0.64255 & 2.93590 & 1.06154 \end{pmatrix}
\] (15.8 - 23)

\[
D_Q = \begin{pmatrix} -81.6556 & -26.2327 \\ -105.3488 & -37.4893 \end{pmatrix}
\] (15.8 - 24)

Figure 15.8-2 shows the response with this controller for the disturbance

\[
u'(s) = \begin{pmatrix} s^{-1} \\ \end{pmatrix}
\] (15.8 - 25)

entering at the plant inputs, when \( P = \hat{P} \). (The same disturbance will be used in all subsequent simulations).

### 15.8.3 Design of \( F \)

In this section we will design a filter \( F(z) \) that guarantees robust stability in the presence of the model-plant mismatch described by (15.8-9). The condition for robust stability is given by (15.7-6). Here \( \Delta \) consists of two scalar blocks and

\[
G_{11}^{F} = -\bar{Q} F \bar{P}^* \bar{L}_I^*
\] (15.8 - 26)
where
\[ L_i^* = \text{diag} \{ \ell_1^*, \ell_2^* \} \]  \hspace{1cm} (15.8 - 27)

The selection of the filter structure follows Sec. 15.7.1. A simple scalar filter will be used:
\[ F(z) = f(z)I \]  \hspace{1cm} (15.8 - 28)
where \( f(z) \) is given by (9.3–3) with \( w = 29 \). The tuning parameter \( \alpha \) in \( f_1(z) \) must be in \([0,1]\) and can be parame\textfrakening{\textfrak{r}}
educed as
\[ \alpha = e^{-T/\lambda} \]  \hspace{1cm} (15.8 - 29)
where \( \lambda \) is a positive time constant which becomes the new tuning parameter. We prefer \( \lambda \) over \( \alpha \) because \( \lambda \) has a clear physical meaning and effect as was illustrated in Sec. 9.3.2. Note that the coefficients of \( \phi(z) \) are functions of \( \lambda \) and are obtained from (9.3–6) and (9.3–9). If one wishes to remove the positivity constraint from the design parameter \( \lambda \), then one should use (15.7–11) instead of (15.8–29). In this example however, as in the SISO case, we only have a single design variable to search over, which is a simple optimization problem. Hence (15.8–29) is used here to maintain a clear physical meaning for the optimization variable \( \lambda \).

For \( F = I \) (\( \lambda = \alpha = 0 \)) we find \( \mu(G_{11}^f) = 3.75 \), which implies that there exist plants among those described by (15.8–9) for which the closed loop system is unstable. A plot of \( \mu \) is shown in Fig. 15.8-3. A search over the parameter \( \lambda \) shows that one has to increase \( \lambda \) to at least 0.5 to get \( \mu = 1.0 \) so that robust stability is guaranteed. Further increase of \( \lambda \) can reduce \( \mu(G_{11}^f) \) even further. Plots of \( \mu \) for \( \lambda = 0.5 \) and \( \lambda = 1 \) can be seen in Fig. 15.8-3.

Note, however, that the lower \( \mu \) for \( \lambda = 1 \) does not necessarily mean that the performance of the system is superior because \( \mu(G_{11}^f) \) is not the robust performance index. For determining robust performance, one has to select an appropriate performance weight \( W \) and compute \( \mu(G^f) \) (Sec. 15.7.4). For our particular example, \( \bar{P}(s) \) has an unstable pole at \( s = 0.5 \) and the uncertainty becomes significant for \( \omega > 1 \). Therefore there is not much room for performance improvement. The question of robust performance will not be examined any further in this section. The reader is referred to Sec. 12.8 for a detailed example on the design of a filter for robust performance.

Let us now look at some simulations to examine the behavior of the control system when there is model-plant mismatch. The following transfer function was chosen for the “real” continuous plant \( P(s) \):
\[ P(s) = \bar{P}(s)(I + L_1(s)) \]  \hspace{1cm} (15.8 - 30)
where

\[
L_I(s) = \begin{pmatrix}
-0.2 & 0 \\
0.1s+1 & -0.2 \\
0 & 0.1s+1
\end{pmatrix}
\]

(15.8 - 31)

Note that this \(L_I(s)\) does not generate a plant that falls exactly in the class described by (15.8-9, 10, 11), although the steady-state gains and time constants of \(L_I(s)\) match those used in (15.8-11) exactly. The reason is that no simple and non-conservative method is available for translating a type of uncertainty description (input multiplicative in this case) from the \(s\)-domain to exactly the same type in the \(z\)-domain. As explained in Sec. 15.1.5, such descriptions may be obtained either from first – principles models or via experiments conducted with different sampling rates. For the purposes of this example, (15.8–31) yields a plant sufficiently close to the class described by (15.8–9) to serve our illustration goals.

The responses to the input disturbance (15.8–22) are shown in Fig. 15.8-4 for \(\lambda = 0.5\) and in Fig. 15.8-5 for \(\lambda = 1\), for both the nominal case \((P = \hat{P})\) and the case of model-plant mismatch with \(P\) given by (15.8–30). Without the IMC filter, the system is unstable for the “real” plant \(P\) in (15.8–30) as expected from the large value of \(\mu(G_{11}^{*\hat{P}})\). The nominal response is shown in Fig. 15.8-1. The responses for \(\lambda = 1\) are not significantly better than that for \(\lambda = 0.5\), although the robust stability \(\mu\) is smaller for \(\lambda = 1\). This is not surprising because \(\mu(G_{11}^{*\hat{P}})\) is an indicator of stability only.
Figure 15.8.4. Responses (A) for nominal system and (B) the plant given by (15.8–30) for IMC filter time constant $\lambda = 0.5$. 
Figure 15.8-5. Responses (A) for nominal system and (B) the plant given by (15.8-30) for IMC filter time constant $\lambda = 1.0$. 
15.9 Summary

For internal stability of the IMC structure, both the plant $P$ and the IMC controller $Q$ have to be stable. For open-loop unstable plants, it is convenient to use the IMC design procedure to design $Q$ and then obtain the classic feedback controller $C$ from (15.1-22) for implementation. Under some mild assumptions about pole-zero cancellations (Sec. 15.2.2), all stabilizing controllers $Q$ for the plant $P^*$ are parametrized by

$$ Q(z) = Q_0(z) + Q_1(z) $$  \hfill (15.2 - 3)

where $Q_0$ is an arbitrary proper controller that stabilizes $P^*$ and $Q_1$ is any stable, proper transfer matrix such that $P^*Q_1P^*$ is stable.

**Design Procedure**

**Step 1: Nominal Performance**

First, the stabilizing $H_2^*$-optimal controller $\tilde{Q}_H(z)$ is determined which minimizes the sum of the Sums of Squared Errors that each of the inputs $v^i$ in a set $\mathcal{V} = \{v^i(z) : i = 1, \ldots, n\}$ would cause, when applied to the system separately.

**Objective O3:**

$$ \min_{\tilde{Q}} [\Phi(v^1) + \ldots + \Phi(v^n)] $$

where

$$ \Phi(v^i) \triangleq \| W e^i \|_2^2 = \| W \tilde{E}^* v^i \|_2^2 = \| W(I - P^*\tilde{Q})v^i \|_2^2 $$  \hfill (15.3 - 14)

The unique controller which meets Obj. O3 is given by

$$ \tilde{Q}_H = z(P_{M}^*)^{-1}\{z^{-1}(P_{A}^*)^{-1}V_{M}\},V_{M}^{-1} $$  \hfill (15.6 - 16)

where the operator $\{\cdot\}$ denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained, except for those that correspond to poles of $(P_{A}^*)^{-1}$. The factorization of the plant

$$ P^* = P_{A}^*P_{M}^* $$  \hfill (15.6 - 1)

into an allpass portion $P_{A}^*$ and a minimum phase portion $P_{M}^*$ can be accomplished through "inner-outer" factorization (Sec. 15.6.4). The input matrix $V = (v^1 \ v^2 \ \ldots \ v^n)$ can be factored similarly

$$ V = V_{M}V_{A} $$  \hfill (15.6 - 15)
In some special cases (Thm. 15.6-3), $\tilde{Q}_H$ is also $H_2^*$-optimal for each of the inputs $v^i$ separately, as well as their linear combinations.

Next, $\tilde{Q}_H$ is modified as described in Sec. 15.6.3 to eliminate the potential problem of intersample rippling:

$$\tilde{Q}(z) = \tilde{Q}_H(z)\tilde{q}_-(z)B(z)$$  \hspace{1cm} (15.6 - 33)

**Step 2: Robust Stability and Robust Performance**

In this step, the controller $\tilde{Q}$ is augmented by a low-pass filter $F$ such that for the detuned controller $Q = Q(\tilde{Q}, F)$ both the robust stability

$$\mu_\Delta(G_{ii}^F(\tilde{P}^*, Q)) < 1, \quad 0 \leq \omega \leq \pi/T, \quad \Delta = \Delta_u$$  \hspace{1cm} (15.7 - 6)

and the robust performance

$$\mu_\Delta(G_{ii}^F(\tilde{P}, Q)) < 1, \quad 0 \leq \omega \leq \pi/T, \quad \Delta = \text{diag}\{\Delta_u, \Delta_p\}$$  \hspace{1cm} (15.7 - 9)

conditions are satisfied. A nonlinear program was formulated (Sec. 15.7-4) to minimize $\mu_\Delta(G^F(\tilde{P}, Q))$ as a function of the filter parameters for a filter with a fixed diagonal structure. For unstable plants the filter has to be identity at the unstable poles of $\tilde{P}^*$. For ill-conditioned plants, two diagonal filters may be necessary to meet the requirement (15.7-9). It should be noted that satisfaction of (15.7-9) does not guarantee satisfaction of (15.7-6), although this is usually so. Hence it should be verified that the optimal solution to (15.7-9) also satisfies (15.7-6).

**15.10 References**

15.1.2. For a discussion on the computation of the matrices $\Phi$ and $\Gamma$ see Åström & Wittenmark (1984).

15.1.3. See the same reference for more details on the Nyquist D-contour for discrete systems.

15.2. For modeling and identification methods for discrete systems see Åström & Wittenmark (1984). Jenkins and Watts (1969) is an excellent reference for identification techniques that result in norm uncertainty bounds for each element or a whole row of the system transfer matrix.
Part V

CASE STUDY
Chapter 16

LV-CONTROL OF A HIGH-PURITY DISTILLATION COLUMN

In this chapter the high-purity distillation column described in the Appendix will be studied, when reflux (L) and boilup (V) are manipulated to control the top (yD) and bottom (xB) compositions. This column was used as an example on several occasions earlier in this book. The LV-configuration is selected because this choice of manipulated inputs is most common in industrial practice. This does not mean that this is necessarily the best configuration; for example, the \( L/V \) configuration may be preferrable.

The distillation column investigated here was chosen to be representative of a large class of moderately high-purity distillation columns. The goal of this chapter is to provide a realistic control design and simulation study for the column. To be realistic at least the issues of uncertainty and nonlinearity must be addressed.

The reader is assumed to be thoroughly familiar with the material in Part III of this book.

16.1 Features

16.1.1 Uncertainty

We showed in Sec. 13.3.4 that the closed-loop system may be extremely sensitive to input uncertainty when the LV-configuration is used. In particular, inverse-based controllers were found to display severe robustness problems. In a similar manner as in Secs. 11.3.5 and 12.8 we will take uncertainty explicitly into account here when designing and analyzing the controllers via the structured singular value (\( \mu \)). We will demonstrate that \( \mu \) provides a much more efficient tool for comparing and analyzing the effect of various combinations of controllers, uncertainty and disturbances than the traditional simulation approach.
16.1.2 Nonlinearity

High-purity distillation columns are known to be strongly nonlinear (see Appendix), and any realistic study should take this into account. Our approach is to base the controller design on a linear model. The effect of nonlinearity is taken care of by analyzing this controller for linearized models at different operating points. Furthermore, all simulations will be based on the full nonlinear model.

16.1.3 Logarithmic Compositions

Several authors found that the high-frequency behavior of distillation columns is only weakly affected by operating conditions when the scaled transfer matrix is considered

\[
\begin{pmatrix}
\frac{d y_D^s}{d x_B^s} \\
\frac{d L}{d V}
\end{pmatrix} = G^s \begin{pmatrix}
\frac{1}{1 - y_D^s} \\
0 \\
\frac{1}{1 - x_B^s}
\end{pmatrix} G \tag{16.1 - 1}
\]

All plant models and controllers in this chapter are scaled in this manner. \(G^s\) is obtained by scaling the outputs with respect to the amount of impurity in each product

\[
y_D^s = \frac{y_D}{1 - y_D^N}, \quad x_B^s = \frac{x_B}{x_B^N} \tag{16.1 - 2}
\]

Here \(x_B^N\) and \(y_D^N\) are the compositions at the nominal operating point. This relative scaling is obtained automatically by using logarithmic compositions

\[
Y_D = \ln(1 - y_D) \tag{16.1 - 3}
\]

\[
X_B = \ln x_B
\]

because

\[
dY_D = -\frac{dy_D}{1 - y_D}, \quad dX_B = \frac{dx_B}{x_B} \tag{16.1 - 4}
\]

Furthermore, the use of logarithmic compositions \((Y_D\) and \(X_B\)) effectively eliminates the effect of nonlinearity at high frequency and also reduces its effect at steady-state. For control purposes the high-frequency behavior is of principal importance. Consequently, if logarithmic compositions are used we expect a linear controller to perform satisfactorily even when the operating conditions are far removed from the nominal operating point for which the controller was designed. Another objective of this chapter is to confirm that this is indeed true.

In most cases the column is operated close to its nominal operating point and there is hardly any advantage in using logarithmic compositions which merely corresponds to a rescaling of the outputs in this case. However, if, for some reason,
the column is taken far away from this nominal operating point, for example, during startup or due to a temporary loss of control, the use of logarithmic compositions may bring the column safely back to its nominal operating point, whereas a controller based on unscaled compositions ($y_D$ and $x_B$) may easily yield an unstable response.

16.1.4 Choice of Nominal Operating Point

The design approach suggested by the above discussion is to design a linear controller based on a linearized model for some nominal operating point. What operating point should be used? If an operating point corresponding to both products of high and equal purities is chosen (i.e., $1 - y_D = x_B$ is small), it is easily shown that the steady-state gains and the linearized time constant will change drastically for small perturbations from this operating point. We may therefore question if acceptable closed-loop control can be obtained by basing the controller design on a linearized model at such an operating point. Some authors indicate that this is not advisable, and that a model based on a perturbed operating point should be used. However, as we just discussed, the high-frequency behavior, which is of primary importance for feedback control, shows much less variation with operating conditions. Therefore, provided the model gives a good description of the high-frequency behavior, we expect to be able to design an acceptable controller also when the nominal operating point has both products of high purity. This is also confirmed by the results in this chapter.

Therefore, a main conclusion is that acceptable closed-loop performance may be obtained by designing a linear controller based on a linear model at any nominal operating point. If large perturbations from steady state are expected then logarithmic compositions should be used to reduce the effect of nonlinearity.

16.2 The Distillation Column

The column model is derived in the Appendix. The following simplifying assumptions are made: binary separation; constant relative volatility; constant molar flows; constant holdups on all trays; perfect pressure and level control. The last assumption results in immediate flow response, that is, flow dynamics are neglected. This is somewhat unrealistic, and in order to avoid unrealistic controllers, we will add “uncertainty” at high frequencies to include the effect of neglected flow dynamics when designing and analyzing the controllers.

We will investigate the column at two different operating points. At the nominal operating point, $A$, both products are high-purity and $1 - y_D^0 = x_B^0 = 0.01$. 
Operating point \( C \) is obtained by increasing \( D/F \) from 0.500 to 0.555 which yields a less pure top product and a purer bottom product; \( 1 - \psi_D^C = 0.10 \) and \( x_{BC}^C = 0.002 \) (subscript \( C \) denotes operating point \( C \) while no subscript denotes operating point \( A \)). We will study the column for the following three assumptions regarding reboiler and condenser holdup

**Case 1:** Almost negligible condenser and reboiler holdup \( (M_D/F = M_B/F = 0.5 \text{ min}) \).

**Case 2:** Large condenser and reboiler holdup \( (M_D/F = 32.1 \text{ min}, M_B/F = 11 \text{ min}) \).

**Case 3:** Same holdup as in Case 2, but the composition of the overhead vapor \( (y_T) \) is used as a controlled output instead of the composition in the condenser \( (y_D) \).

These three cases will be denoted by subscripts 1, 2 and 3, respectively. The holdup on each tray inside the column is \( M_i/F = 0.5 \text{ min} \) in all three cases.

### 16.2.1 Modelling

*Nominal operating point (A).* A 41st order linear model for the columns is easily derived

\[
\begin{pmatrix}
\frac{dy_D}{dx_B}
\end{pmatrix} = G(s) \begin{pmatrix}
\frac{dL}{dV}
\end{pmatrix}
\tag{16.2 - 1}
\]

The *scaled* steady-state gain matrix is

\[
G(0) = \begin{bmatrix}
87.8 & -86.4 \\
108.2 & -109.6
\end{bmatrix}
\tag{16.2 - 2}
\]

which yields the following values for the condition number and the 1,1-element in the RGA

\[
\gamma(G(0)) = \sigma(G(0))/\sigma(G(0)) = 141.7 \quad \lambda_{11}(G(0)) = 35.1
\]

However, \( \gamma(G) \) and \( \lambda_{11}(G) \) are much smaller at high frequencies as seen from Fig. 16.2-1. A very crude model used throughout the earlier chapters is

Model 0: \[
G_0(s) = \frac{1}{1 + 75s}G(0)
\tag{16.2 - 3}
\]

This model has the same \( \gamma(G) \) and \( \lambda_{11}(G) \) for all frequencies and is therefore a poor description of the actual plant at high frequency.
16.2. THE DISTILLATION COLUMN

Figure 16.2-1. Column A, Case 1 \((G = G_1)\). The condition number of the plant is about 10 times lower at high frequencies than at steady state. (Reprinted with permission from Chem. Eng. Sci. 43, 35 (1988), Pergamon Press, plc.)

**Case 1.** For the case of negligible reboiler and condenser holdup the following simple two time-constant model yields an **excellent** approximation of the 41st order linear model.

\[
G_1(s) = \begin{pmatrix}
\frac{87.8}{1+\tau_1 s} - \frac{87.8}{1+\tau_1 s} + \frac{14}{1+\tau_2 s} \\
\frac{108.2}{1+\tau_1 s} - \frac{108.2}{1+\tau_1 s} - \frac{14}{1+\tau_2 s}
\end{pmatrix} \quad \tau_1 = 194 \text{ min} \\
\tau_2 = 15 \text{ min} 
\]  

This two state model uses two time constants: \(\tau_1\) is the time constant for changes in the external flows and is dominant. \(\tau_2\) is the time constant for changes in internal flows (simultaneous change in \(L\) and \(V\) with constant product rates, \(D\) and \(B\)). The simple model (16.2-4) matches the observed variation of the condition number with frequency (Fig. 16.2-1).

The effect of the reboiler and condenser holdups (Case 2) can be partially accounted for with Model 1 by multiplying \(G_1(s)\) by \(\text{diag}\{1+\tau_D s, 1+\tau_B s\}^{-1}\), where in our case \(\tau_D = M_D/V_T = 10\) min and \(\tau_B = M_B/L_B = 3\) min. However, in practice the top composition is often measured in the overhead vapor line (Case 3), rather than in the condenser. \(G_1(s)\) provides a good approximation of the plant in such cases.

**Cases 2 and 3.** In order to obtain a low-order model for Cases 2 and 3, we performed a model reduction on the full 41st order model. These reduced order models are denoted by \(G_2(s)\) and \(G_3(s)\) respectively. A good approximation was obtained with a 5th-order model as illustrated in Fig. 16.2-2.

**Operating point C.** We will return with a discussion of the model for this case in Sec. 16.5 when we also discuss the control of the plant.
16.2.2 Simulations

The design and analysis of the controllers are based on the linear models $G_1(s), G_2(s),$ and $G_3(s)$. However, except for the five simplifying assumptions stated above, all simulations are carried out with the full nonlinear model. (In some cases the changes are so small, however, that the results are equivalent to linear simulations.) To get a realistic evaluation of the controllers input uncertainty must be included. Simulations are therefore shown both with and without 20% uncertainty with respect to the change of the two inputs. The following uncertainties are used:

$$\Delta L = (1 + \Delta_1) \Delta L_c, \quad \Delta_1 = 0.2 \quad (16.2 - 5a)$$

$$\Delta V = (1 + \Delta_2) \Delta V_c, \quad \Delta_2 = -0.2 \quad (16.2 - 5b)$$

Here $\Delta L$ and $\Delta V$ are the actual changes in manipulated flow rates, while $\Delta L_c$ and $\Delta V_c$ are the desired values as computed by the controller. $\Delta_1 = -\Delta_2$ was chosen to represent the worst combination of the uncertainties (Sec. 13.3.4).
16.3 Formulation of the Control Problem

16.3.1 Performance and Uncertainty Specifications

The uncertainty and performance specifications are the same as those used elsewhere in this book.

Uncertainty. The only source of uncertainty which is considered here is uncertainty on the manipulated inputs (L and V) with a magnitude bound:

\[ w_I(s) = 0.2 \frac{5s + 1}{0.5s + 1} \]  \hspace{1cm} (16.3 - 1)

The bound (16.3-1) allows for an input error of up to 20% at low frequency as was assumed for the simulations (16.2-5). The uncertainty bound (16.3-1) increases with frequency. This allows, for example, for a time delay of about 1 min in the response between the inputs, L and V, and the outputs, \( y_D \) and \( x_B \). In practice, such delays may be caused by the flow dynamics. Therefore, although flow dynamics are not included in the models or in the simulations, they are partially accounted for in the \( \mu \)-analysis and in the controller design.

Performance. Robust performance is satisfied if

\[ \tilde{\sigma}(E) = \tilde{\sigma}((I + GC)^{-1}) \leq \frac{1}{|w_p|} \]  \hspace{1cm} (16.3 - 2)

is satisfied for all possible plants \( G \). We use the performance weight

\[ w_p(s) = 0.5 \frac{10s + 1}{10s} \]  \hspace{1cm} (16.3 - 3)

A particular \( E \) which exactly matches the bound (16.3-2) at low frequencies and satisfies it easily at high frequencies is \( E = 20s/20s + 1 \). This corresponds to a first-order response with closed-loop time constant 20 min.

16.3.2 Analysis of Controllers

Comparison of controllers is based mainly on \( \mu \) for robust performance (\( \mu_{RP} \)). Simulations are used only to support conclusions found using the \( \mu \)-analysis. The main advantage of the \( \mu \)-analysis is that it provides a well-defined basis for comparison. On the other hand, simulations are strongly dependent on the choice of setpoints, uncertainty, etc.

The value of \( \mu_{RP} \) is indicative of the worst-case response. If \( \mu_{RP} > 1 \) then the “worst case” does not satisfy our performance objective, and if \( \mu_{RP} < 1 \) then the “worst case” is better than required by our performance objective. Similarly,
if \( \mu_{NP} < 1 \) then the performance objective is satisfied for the nominal case. However, this may not mean very much if the system is sensitive to uncertainty and \( \mu_{RP} \) is significantly larger than one. We will show below that this is the case, for example, if an inverse-based controller is used for our distillation column.

### 16.3.3 Controllers

We will study the distillation column with the following six controllers:

1. **Diagonal PI-controller.**

   \[
   C_{PI}(s) = \frac{0.01}{s} \begin{pmatrix} 2.4 & 0 \\ 0 & -2.4 \end{pmatrix} (1 + 75s)
   \]

   This controller was studied in Sec. 11.3.5 and was tuned to achieve as good a performance as possible while maintaining robust stability (see also Fig. 16.4-1).

2. **Steady-state decoupler plus two PI-controllers.**

   \[
   C_{oinv}(s) = 0.7 \frac{(1 + 75s)}{s} G(0)^{-1} = \frac{0.01(1 + 75s)}{s} \begin{pmatrix} 27.96 & -22.04 \\ 27.60 & -22.40 \end{pmatrix}
   \]

   This controller was tuned to achieve good nominal performance. However, the controller has large RGA-elements \( \lambda_{11}(C) = 35.1 \) at all frequencies and we expect the controller to be extremely sensitive to input uncertainty (see Sec. 13.3).

3. **Inverse-based controller based on the linear model \( G_1(s) \) for Case 1.**

   \[
   C_{1inv}(s) = \frac{0.7}{s} G_1(s)^{-1}
   \]

   At low frequency this controller is equal to \( C_{oinv}(s) \). Note that \( C_{1inv}(s) \) and \( G_1(s)^T \) have the same RGA-elements. Therefore from Fig. 16.2-1 we expect \( C_{1inv}(s) \) to be sensitive to input uncertainty at low frequency, but not at high frequency.

4, 5, and 6. \( \mu \)-optimal controllers based on the models \( G_0(s), G_1(s) \) and \( G_2(s) \). The controllers are denoted \( C_{0\mu}(s), C_{1\mu}(s) \) and \( C_{2\mu}(s) \), respectively.

These controllers were obtained by minimizing \( \sup_{\omega} \mu(N_{RP}) \) for each model using the input uncertainty and performance weights given above. The numerical procedure used for the minimization is the same as mentioned in Sec. 11.3.5. The \( \mu \)-plots for robust performance for the \( \mu \)-optimal controllers are of particular interest since they indicate the best achievable performance for the plant. Bode plots of the transfer matrix elements of \( C_{1\mu}(s) \) and \( C_{2\mu}(s) \) are shown in Fig. 16.3-1. Note the similarities between these controllers and the simple diagonal PI controller (16.3-4).
At low frequency ($s \to 0$) the six controllers are approximately

$$C_{PI} = \frac{0.01}{s} \begin{pmatrix} 2.4 & 0 \\ 0 & -2.4 \end{pmatrix}$$

$$C_{0_{inv}} = C_{1_{inv}} = \frac{0.01}{s} \begin{pmatrix} 27.96 & -22.04 \\ 27.80 & -22.40 \end{pmatrix}$$

$$C_{0_{\mu}} = \frac{0.01}{s} \begin{pmatrix} 3.82 & -0.92 \\ 1.73 & -3.52 \end{pmatrix}; C_{1_{\mu}} = \frac{0.01}{s} \begin{pmatrix} 6.07 & -0.90 \\ 2.80 & -2.93 \end{pmatrix}; C_{2_{\mu}} = \frac{0.01}{s} \begin{pmatrix} 4.06 & +0.15 \\ 2.85 & -2.93 \end{pmatrix}$$

$\|\Lambda(C)\|_1$ is shown as a function of frequency for the six controllers in Fig. 16.3-2. As expected, the $\mu$-optimal controllers have small RGA-elements, which make them insensitive to input uncertainty. For example, $C_{2_{\mu}}$ is nearly triangular at low frequency and consequently has $\Lambda \cong I$.

16.4 Results for Operating Point A

In this section we will study how the six controllers perform at the nominal operating point A for the three assumptions regarding condenser and reboiler holdup (corresponding to the models $G_1(s), G_2(s)$, and $G_3(s)$). The $\mu$-plots for the 18 possible combinations are given in Fig. 16.4-1. A number of interesting observations can be derived from these plots. These are presented below. In some cases the simulations in Figs. 16.4-2 to 16.4-4 are used to support the claims.
16.4.1 Discussion of Controllers

$C_{PI}(s)$. The simple diagonal PI-controller performs reasonably well in all cases. $\mu_{NP}$ is higher than one at low frequency, which indicates a slow return to steady state. This is confirmed by the simulations in Fig. 16.4-3 for a feed rate disturbance; after 200 min the column has still not settled. Operators are usually unhappy about this kind of response. The controller is insensitive to input uncertainty and to changes in reboiler and condenser holdup.

$C_{0inv}(s)$. This controller uses a steady-state decoupler. The nominal response is very good for Case 1 (Fig. 16.4-2), but the controller is extremely sensitive to input uncertainty. In practice, this controller will yield an unstable system.

$C_{1inv}(s)$. This controller gives an excellent nominal response for Case 1 (Fig. 16.4-1). This is also confirmed by the simulations in Fig. 16.4-2; the response is almost perfectly decoupled with a time constant of about 1.4 min. Since the simulations are performed with the full-order model, while the controller was designed based on the simple two time-constant model, $G_1(s)$ (16.2-4), this confirms that $G_1(s)$ yields a very good approximation of the linearized plant when the reboiler and condenser holdups are small. The controller is sensitive to the input uncertainty as expected from the RGA analysis. Also note that the controller performs very poorly when the condenser and reboiler holdups are increased. This shows that the controller is also very sensitive to other sources of model-plant mismatch.

$C_{0\mu}(s)$. This is the $\mu$-optimal controller from our previous study which was designed based on the very simplified model $G_0(s)$. The controller performs...
Figure 16.4-1. μ-plots for operating point. Upper solid line: μRp for robust performance; lower solid line: μNp for nominal performance; dotted line: μRs for robust stability. (Reprinted with permission from Chem. Eng. Sci., 43, 40 (1988), Pergamon Press, plc.)
16.4. RESULTS FOR OPERATING POINT A

*10^{-3}

\[ \Delta x_B \]

\[ \Delta y_D \]

A: \( C_{PI}(s) \)

*10^{-3}

\[ \Delta x_B \]

\[ \Delta y_D \]

B: \( C_{1\mu}(s) \)

Figure 16.4-3. Operating point A, Case 1. Closed-loop response to a 30% increase in feed rate. Solid lines: no uncertainty; dotted lines: 20% uncertainty on inputs \( L \) and \( V \) (16.2-5). (Reprinted with permission from Chem. Eng. Sci., 43, 41 (1988), Pergamon Press, plc.)

surprisingly well on the actual plant \( (G_1(s)) \) when the holdups are negligible. However, the controller is seen to perform very poorly when the holdups in the reboiler and condenser are increased, which shows that the controller is very sensitive to other sources of model inaccuracies (for which it was not designed).

\( C_{1\mu}(s) \). This is the \( \mu \)-optimal controller when there is negligible holdup \( (G_1(s)) \). The robust performance condition is satisfied for this case since \( \mu_{RP} \cong 0.95 \). The nominal performance is not as good as for the inverse-based controller \( C_{inv}(s) \); we have to sacrifice nominal performance to make the system robust with respect to uncertainty. The controller shows some performance deterioration when the reboiler and condenser holdups are increased (Case 2). This is not surprising since the added holdup makes the response of \( y_D \) and \( x_B \) more sluggish; the open-loop response for \( y_D \) changes from approximately \( (1+194s)^{-1} \) to \( ((1+194s)(1+10s))^{-1} \) [recall discussion following (16.2-4)]. As expected, the controller is much less sensitive to changes in condenser holdup if the overhead composition is measured in the vapor line (Case 3). Overall, this is the best of the six controllers.

\( C_{2\mu}(s) \). This is the \( \mu \)-optimal controller for the case with considerable reboiler and condenser holdup, and with \( y_D \) measured in the condenser \( (G_2(s)) \). \( \mu_{RP} \cong 1.00 \) for this case. The nominal response is good in all cases (Fig. 16.4-1), but the controller is very sensitive to uncertainty when the plant is \( G_1(s) \) or \( G_3(s) \) rather than \( G_2(s) \). This is clearly not desirable since changes in condenser and reboiler holdup are likely to occur during normal operation. The observed behavior is not
CHAPTER 16. LV-CONTROL OF A HIGH-PURITY DISTILLATION COLUMN

Case 1
(small reb./cond. holdup)

Case 2
(large reb./cond. holdup)

\[ C_{PI}(s) \]

\[ C_{1\mu}(s) \]

\[ C_{2\mu}(s) \]

Figure 16.4-4. Operating point A. Effect of reboiler and condenser holdup on closed-loop response. No uncertainty. (Reprinted with permission from Chem. Eng. Sci., 43, 42 (1988), Pergamon Press, plc.)
surprising since the controller includes lead elements at \( \omega \equiv 0.1 \) (Fig. 16.3-1B) to counteract the lags caused by the reboiler and condenser holdups. If these lags are not present in the plant \((G_1(s) \text{ or } G_3(s))\), the "derivative" action caused by the lead elements results in a system which is very sensitive to uncertainty.

### 16.4.2 Conclusions

- The \( \mu \)-optimal controller \( C_{0\mu}(s) \) for the plant \( G_0(s) \) has \( \mu_{RP} \approx 1.06 \) while the \( \mu \)-optimal controller \( C_{1\mu}(s) \) for the plant \( G_1(s) \) has \( \mu_{RP} \approx 0.95 \). Thus, somewhat surprisingly, the achievable performance is not much better for \( G_1(s) \) than for \( G_0(s) \), even though \( G_0(s) \) is ill-conditioned and has large RGA elements at all frequencies, while \( G_1(s) \) has large RGA elements only at low frequencies (Fig. 16.2-1). This seems to indicate that large RGA-elements at low frequency imply limitations on the achievable control performance and partially justifies the use of steady-state values of the RGA for selecting the best control configuration.

- However, the use of the more detailed model \( G_1(s) \), rather than \( G_0(s) \), is still justified since the resulting \( \mu \)-optimal controller is much less sensitive to changes in reboiler and condenser holdup (which will occur during operation).

- \( G_1(s) \) approximates the full-order model very closely as seen from Fig. 16.4-2C; the response is almost perfectly decoupled when there is no uncertainty.

- To avoid sensitivity to the amount of condenser and reboiler holdup, the overhead composition should be measured in the overhead vapor, rather than in the condenser. In practice, temperature measurements inside the column are often used to infer compositions, and the dynamic response of these measurements is similar to that when the condenser and reboiler holdup is neglected.

- The simple model \( G_2(s) \) is useful for controller design even when the reboiler and condenser holdup is large.

- The main advantage of the \( \mu \)-optimal controllers over the simple diagonal PI controller is a faster return to steady-state. This can be seen very clearly in Fig. 16.4-3 which shows the closed-loop response to a 30% increase in feed rate.
16.5 Effect of Nonlinearity (Results for Operating Point C)

We will not treat nonlinearity as uncertainty because this approach is not rigorous and is also very conservative due to the strong correlation between all the parameters in the model which is difficult to account for. Furthermore, we know from the data in the Appendix that the column is actually not as nonlinear as one might expect. Though the steady-state gains may change dramatically, the high frequency behavior, which is of principal importance for feedback control, is much less affected. In particular, this is the case if relative (logarithmic) compositions are used. To demonstrate this fact we compute $\mu$ and show simulations for some of the controllers when the "plant" is $G_C(s)$ rather than $G(s)$.

16.5.1 Modelling

The model $G_C(s)$ describes the same column as $G(s)$, but the distillate flow rate ($D$) has been increased from 0.5 to 0.555 such that $y_D = 0.9$ and $x_B = 0.002$. For Case 1 ($M_D/F = M_B/F = 0.5\text{ min}$), the following approximate model is derived when scaled compositions ($dy_D/0.1, dx_B/0.002$) are used:

$$G_{C1}(s) = \begin{pmatrix} 16.0 \\ 9.3 \end{pmatrix} \begin{pmatrix} 1 + \tau_1 s & 0.023 \\ 1 + \tau_2 s \end{pmatrix} \begin{pmatrix} \frac{16.0}{1 + \tau_1 s} & \frac{0.023}{1 + \tau_2 s} \\ \frac{9.3}{1 + \tau_1 s} & \frac{-9.3}{1 + \tau_2 s} \end{pmatrix} \begin{pmatrix} \tau_1 = 24.5\text{ min} \\ \tau_2 = 10\text{ min} \end{pmatrix}$$ (16.5 - 1)

The steady-state gains and time constants are entirely different from those at operating point A (16.2-4). Also note that at steady state $\lambda_{11}(G(0)) = 35.1$ for operating point A, but only 7.5 for operating point C. However, at high-frequency the scaled plants at operating points A and C are very similar. Equations (16.2-4) and (16.5-1) yield:

$$G_1(\infty) = \frac{1}{s} \begin{pmatrix} 0.45 & -0.36 \\ 0.56 & -0.65 \end{pmatrix} \quad \lambda_{11}(\infty) = 3.2 \quad (16.5 - 2a)$$

$$G_{C1}(\infty) = \frac{1}{s} \begin{pmatrix} 0.65 & -0.65 \\ 0.38 & -0.52 \end{pmatrix} \quad \lambda_{11}(\infty) = 3.7 \quad (16.5 - 2b)$$

Therefore, as we will show, controllers which were designed based on the model $G(s)$ (operating point A) remain satisfactory when the plant is $G_C(s)$ rather than $G(s)$. Recall that the use of a scaled plant is equivalent to using logarithmic compositions ($Y_D$ and $X_B$). The variation in gain with operating conditions is much larger if unscaled compositions are used – both at steady-state and at high frequencies:
16.5. EFFECT OF NONLINEARITY (RESULTS FOR OPERATING POINT C)

\[
G_1^{us}(\infty) = \frac{0.01}{s} \begin{pmatrix} 0.45 & -0.36 \\ 0.56 & -0.65 \end{pmatrix} \quad (16.5 - 3a)
\]

\[
G_{C1}^{us}(\infty) = \frac{0.01}{s} \begin{pmatrix} 6.5 & -6.5 \\ 0.08 & -0.10 \end{pmatrix} \quad (16.5 - 3b)
\]

16.5.2 \(\mu\)-Analysis

The \(\mu\)-plots with the model \(G_C(s)\) and four of the controllers are shown in Fig. 16.5-1 (all four controllers yield nominally stable closed-loop systems). For high frequencies the \(\mu\)-values are almost the same as those found at operating point A. The only exception is the inverse based controller \(C_{inv}(s)\) which is robustly stable at operating point A, but not at operating point C. Again, this confirms the sensitivity of this controller to model inaccuracies. Performance is clearly worse for low frequencies at operating point C (Fig. 16.5-1) than at operating point A (Fig. 16.4-1). This is expected; the controllers were designed based on model A, and the plants are quite different in the low frequency range.

The \(\mu\)-optimal controller \(C_{1\mu}(s)\) satisfies the robust performance requirements also at operating point C when the reboiler and condenser holdups are small. Consequently, with scaled (logarithmic) compositions, a single linear controller is able to give acceptable performance at these two operating points although the linear models are quite different. The main difference between \(C_{1\mu}(s)\) and the diagonal PI controller is again that the \(\mu\)-optimal controller gives a much faster return to steady-state. This is clearly seen from Fig. 16.5-2A.

16.5.3 Logarithmic Versus Unscaled Compositions

Figure 16.5-1 shows how controllers, designed based on the scaled plant \(G(s)\) at operating point A, perform for the scaled plant (different scaling factors!) at operating point C; this is equivalent to using logarithmic compositions (\(Y_D\) and \(X_B\)). We know from (16.5-3) that the plant model based on absolute compositions changes much more. Therefore we expect the closed-loop performance to be entirely different at operating points A and C when unscaled (absolute) compositions are used. This is indeed confirmed by Fig. 16.5-2B which shows the closed-loop response to a small setpoint change in \(x_B\) at operating point C. Fig. 16.5-2B should be compared to Fig. 16.5-2A which shows the same response, but with logarithmic compositions as controlled outputs. In Fig. 16.5-2B (absolute compositions) the response for \(x_B\) is significantly more sluggish, but the response for \(y_D\) is much faster than in Fig. 16.5-2A (logarithmic compositions). This is exactly what we would expect from a comparison of (16.5-3a) and (16.5-3b). The high-frequency gain for changes in \(y_D\) is increased by an order of magnitude and
Figure 16.5-1. $\mu$-plots for operating point C. Upper solid line: $\mu_{RP}$; lower solid line: $\mu_{NP}$; dotted line: $\mu_{RS}$. (Reprinted with permission from Chem. Eng. Sci., 43, 44 (1988), Pergamon Press, plc.)
16.5. EFFECT OF NONLINEARITY (RESULTS FOR OPERATING POINT C)

--- $C_{1\mu}(s)$ --- $C_{PI}(s)$

A: Logarithmic compositions

B: Absolute compositions

Figure 16.5-2. Operating point C, Case 1. Closed-loop response to small setpoint change in $x_B$ ($x_B$ increases from 0.002 to 0.0021) using diagonal PI controller (dotted line) and the $\mu$-optimal controller for operating point A (solid line). Left: logarithmic compositions as controlled outputs (equivalent to using scaled compositions); right: absolute (unscaled) compositions as controlled outputs. No uncertainty. (Reprinted with permission from Chem. Eng. Sci., 43, 45 (1988), Pergamon Press, plc.)

the gain for changes in $x_B$ is reduced by an order of magnitude. However, recall from (16.5–2) that the gain changes very little when logarithmic compositions are used.

16.5.4 Transition from Operating Point A to C

Figure 16.5-3 shows a transition from operating point A ($Y_D = X_B = 4.605$) to operating point C ($Y_D = 2.303, X_B = 6.215$) using logarithmic compositions as controlled outputs. The desired setpoint change is a first order response with time constant 10 min:

$$\Delta Y_{Ds} = \frac{2.303}{1 + 10s} \quad , \quad \Delta X_{Bs} = \frac{-1.609}{1 + 10s}$$

The closed-loop response is seen to be very good. The diagonal controller $C_{PI}(s)$ and the $\mu$-optimal controller $C_{1\mu}(s)$ give very similar responses in this particular case. (However, the $\mu$-optimal controller generally performs better at operating point C as is evident from Figs. 16.5-1 and 16.5-2.) This illustrates that a linear controller, based on the nominal operating point A, can be satisfactory for a large
deviation from this operating point when logarithmic compositions are used.

16.6 Conclusions

A single linear controller is able to provide satisfactory control for this high-purity column at widely different operating conditions. To compensate for the plant nonlinearity it is advantageous to use "logarithmic compositions." For small deviations from steady state linear controllers using "absolute compositions" work also well.

The performance with a simple diagonal controller is robust with respect to model-plant mismatch but after an upset the return to steady state can be very sluggish. This particular deficiency can be removed by a $\mu$-optimal controller. Inverse-based controllers, and specifically those involving steady-state decouplers, were shown to be very sensitive to model-plant mismatch.

16.7 References

This chapter is abstracted from a paper by Skogestad & Morari (1988a) where the state-space description of the $\mu$-optimal controllers and the reduced order
models for Cases 2 and 3 are also provided. For a general discussion of distillation control the reader is referred to the book by Shinskey (1984) or the thesis by Skogestad (1987). Skogestad and Morari (1987a) have reviewed the current industrial understanding of distillation control from the viewpoint of modern robust control.

16.1.2. The nonlinear behavior was observed specifically by Moczek, Otto & Williams (1963) and Fuentes and Luyben (1983).

16.1.3. The use of “logarithmic compositions” seems to have been first suggested by Ryskamp (1981).

16.1.4. The change of gain and time constant with operating condition was analyzed by Kapoor, McAvoy & Marlin (1986) and Skogestad & Morari (1988d).

16.2.1. The model reduction was performed via “Balanced Realization” (Moore, 1981). The dominant time constant $\tau_1$ can be estimated, for example, from the inventory time constant introduced by Moczek, Otto & Williams (1963). The time constant $\tau_2$ can be obtained by matching the high-frequency behavior as shown by Skogestad & Morari (1988d).
Appendix

Dynamic Model of Distillation Column

On many occasions in this book a high purity distillation column is used as an example. In this Appendix all the necessary information is summarized to enable the reader to verify any of the results reported in this book and to use the distillation model as a test case for other analysis and design procedures.

A.1 Nomenclature and Assumptions

The column is shown in Fig. A.1-1 where most symbols are also defined.

Symbols:

\[ M \] hold up
\[ N \] number of equilibrium (theoretical) stages
\[ N + 1 \] total number of stages including total condenser
\[ N_F \] feed stage location
\[ F \] feed rate
\[ z_F \] mole fraction of light component in feed
\[ q_F \] fraction liquid in feed
\[ D \] distillate flow
\[ V \] boilup
\[ V_T \] top vapor flow
\[ L \] reflux flow
\[ B \] bottom flow
\[ p \] pressure
\[ q \] mole fraction of feed which is liquid
\[ x \] mole fraction of light component in liquid
\[ y \] mole fraction of light component in vapor
\[ \alpha \] relative volatility
\[ \kappa \] linearized VLE-constant

The unit of mass is kmol and the unit of time is minute.
APPENDIX A. DYNAMIC MODEL OF DISTILLATION COLUMN

Figure A.1-1. Two product distillation column with single feed and total condenser.

Subscripts:
- \( i \) tray \( i \) (trays are numbered from bottom with the reboiler as tray
- \( F \) feed
- \( D \) distillate
- \( B \) bottom
- \( T \) top

Assumptions:
- binary mixture
- constant pressure
- constant relative volatility \( \alpha \)
- constant molar flows
- no vapor holdup (immediate vapor response, \( dV_T = dV_B \))
- constant liquid holdup \( M_i \) on all trays (immediate liquid response, \( dL_T = dL_B \))
- Vapor-Liquid Equilibrium (VLE) and perfect mixing on all stages
A.2 Nonlinear Model

Material balances for change in holdup of light component on each tray $i = 2, \ldots, N (i \neq N_F, i \neq N_F + 1)$:

$$M_i \dot{x}_i = L_{i+1} x_{i+1} + V_{i-1} y_{i-1} - L_i x_i - V_i y_i$$

Above feed location $i = N_F + 1$:

$$M_i \dot{x}_i = L_{i+1} x_{i+1} + V_{i-1} y_{i-1} - L_i x_i - V_i y_i + F_V y_F$$

Below feed location, $i = N_F$:

$$M_i \dot{x}_i = L_{i+1} x_{i+1} + V_{i-1} y_{i-1} - L_i x_i - V_i y_i + F_L x_F$$

Reboiler, $i = 1$:

$$M_B \dot{x}_i = L_{i+1} x_{i+1} - V_i y_i - B x_i, \quad x_B = x_1$$

Total condenser, $i = N + 1$:

$$M_D \dot{x}_i = V_{i-1} y_{i-1} - L_i x_i - D x_i, \quad y_D = x_{N+1}$$

VLE on each tray ($i = 1, \ldots, N$), constant relative volatility:

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$

Flow rates assuming constant molar flows:

- $i > N_F$ (above feed): $L_i = L, \quad V_i = V + F_V$
- $i \leq N_F$ (below feed): $L_i = L + F_L, \quad V_i = V$

$$F_L = q_P F, \quad F_V = F - F_L$$

$$D = V_N - L = V + F_V - L \quad \text{(constant condenser holdup)}$$

$$B = L_2 - V_1 = L + F_L - V \quad \text{(constant reboiler holdup)}$$

Compositions $x_F$ and $y_F$ in the liquid and vapor phase of the feed are obtained by solving the flash equations:

$$F_{ZF} = F_L x_F + F_V y_F$$
Table A.2-1. Column Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative volatility</td>
<td>$\alpha = 1.5$</td>
</tr>
<tr>
<td>No. of theoretical trays</td>
<td>$N = 40$</td>
</tr>
<tr>
<td>Feed tray (1=reboiler)</td>
<td>$N_F = 21$</td>
</tr>
<tr>
<td>Feed composition</td>
<td>$z_F = 0.5$</td>
</tr>
<tr>
<td>Condenser time constant</td>
<td>$M_D/F$: see Ch. 16</td>
</tr>
<tr>
<td>Reboiler time constant</td>
<td>$M_B/F$: see Ch. 16</td>
</tr>
<tr>
<td>Tray time constant</td>
<td>$M_t/F = 0.5$ min</td>
</tr>
</tbody>
</table>

$$y_F = \frac{\alpha x_F}{1 + (\alpha - 1)x_F}$$

The column data and operating conditions used in the book are shown in Tables A.2-1 and A.2-2. The tray compositions are listed in Table A.2-3. The column behavior is highly nonlinear as Fig. A.2-1 illustrates.
### Table A.2.2. Operating Variables:

<table>
<thead>
<tr>
<th>Operating Point</th>
<th>A</th>
<th>C</th>
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<tbody>
<tr>
<td>$y_D$</td>
<td>0.99</td>
<td>0.90</td>
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<tr>
<td>$x_B$</td>
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<td>0.002</td>
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<tr>
<td>$(L/D)_{\text{min}}$</td>
<td>3.900</td>
<td>3.000</td>
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<tr>
<td>$L/D$</td>
<td>5.413</td>
<td>4.935</td>
</tr>
<tr>
<td>$D/F$</td>
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<td>0.555</td>
</tr>
<tr>
<td>$B/F$</td>
<td>0.500</td>
<td>0.445</td>
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<td>$V/F$</td>
<td>3.206</td>
<td>3.291</td>
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<tr>
<td>$L/F$</td>
<td>2.706</td>
<td>2.737</td>
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</table>

**Figure A.2.1.** (—) Nonlinear open loop responses $\Delta y_D$ and $\Delta x_B$ for changes in boilup $V$ (reflux $L$ constant). (---) Approximation with linear first order response. (Different time constants for A, B and C). A: $V + 6.2\%$, B: $V + 0.003\%$, C: $V - 6.2\%$. 
### Table A.2-3. Tray compositions for operating conditions A and C.

<table>
<thead>
<tr>
<th>Tray</th>
<th>x</th>
<th>A</th>
<th>y</th>
<th>x</th>
<th>C</th>
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</table>
A.3 Linearized Model

We linearize the material balance on each tray \((dL_i = dL, \ dV_i = dV)\):

\[
M_i \dot{x}_i = L_{i+1} dx_{i+1} - (L_i + K_i V_i) dx_i + K_{i-1} V_{i-1} dx_{i-1} + (x_{i+1} - x_i) dL - (y_i - y_{i-1}) dV
\]

Here \(K_i\) is the linearized VLE-constant:

\[
K_i = \frac{dy_i}{dx_i} = \frac{\alpha}{(1 + (\alpha - 1)x_i)^2}
\]

and \(y_i, x_i, L_i\) and \(V_i\) are the steady-state values at the nominal operating point. Written in the standard state variable form in terms of deviation variables the model becomes

\[
\dot{x} = Ax + Bu, \quad y = Cx
\]

where \(x = (dx_1, \ldots, dx_{N+1})^T\) are the tray compositions, \(u = (dL, dV)^T\) are the manipulated inputs and \(y = (dy_D, dx_B)^T\) are the controlled outputs. The state matrix \(A = \{a_{i,j}\}\) is tri-diagonal:

\[
i \neq N + 1: \quad a_{i,i+1} = L_{i+1}/M_i \\
i \neq 1: \quad a_{i,i} = -(L_i + K_i V_i)/M_i \\
i \neq 1: \quad a_{i,i-1} = K_{i-1} V_{i-1}/M_i
\]

Input matrix \(B = \{b_{i,j}\}\):

\[
i \neq N + 1: \quad b_{i,1} = (x_{i+1} - x_i)/M_i, \quad b_{N+1,1} = 0 \\
i \neq 1, i \neq n + 1: \quad b_{i,2} = -(y_i - y_{i-1})/M_i, \quad b_{N+1,2} = 0, \quad b_{1,2} = (y_1 - x_1)/M_1
\]

Output matrix \(C\):

\[
C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}
\]

The condenser drum is assumed to be under perfect level control

\[
V = L + D
\]

Thus, if a different set of manipulated variables, \(U = (dD, dV)^T\) is employed the new model is obtained via the linear transformation

\[
\begin{pmatrix} dD \\ dV \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix}
\]
### Table A.4-1. Gain Information

<table>
<thead>
<tr>
<th>Operating Point</th>
<th>A</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G_{LV}(0))</td>
<td>((0.878 \ -0.864))</td>
<td>((1.604 \ -1.602))</td>
</tr>
<tr>
<td>(G_{LV}(0)) (scaled compositions)</td>
<td>((87.8 \ -86.4))</td>
<td>((16.0 \ 16.023))</td>
</tr>
<tr>
<td>RGA (\lambda_{11}(0))</td>
<td>35.1</td>
<td>7.5</td>
</tr>
<tr>
<td>(G_{LV}(\infty))</td>
<td>(\frac{0.01}{s} (0.45 \ -0.36))</td>
<td>(\frac{0.01}{s} (6.5 \ -6.5))</td>
</tr>
<tr>
<td>(G'_{LV}(\infty))</td>
<td>(\frac{1}{s} (0.56 \ -0.65))</td>
<td>(\frac{1}{s} (0.65 \ -0.65))</td>
</tr>
<tr>
<td>(\lambda_{11}(\infty))</td>
<td>3.2</td>
<td>3.7</td>
</tr>
</tbody>
</table>

### A.4 Gain Information

From the gain information in Tables A.4-1 through A.4-3 one can see that the non-linearity appears mostly in the low-frequency range and is much less pronounced for high frequencies. The time constant \((\tau = 75\text{min})\) used in the simulations represents an average value based on the simulations shown in Fig. A.2-1.
### Table A.4-2. Singular Value Decomposition of Gain Matrices

<table>
<thead>
<tr>
<th>Configuration</th>
<th>LV</th>
<th>DV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(0)$</td>
<td>\begin{pmatrix} 0.878 &amp; -0.864 \ 1.082 &amp; -1.096 \end{pmatrix}</td>
<td>\begin{pmatrix} -0.878 &amp; 0.014 \ -1.082 &amp; -0.014 \end{pmatrix}</td>
</tr>
<tr>
<td>RGA $\lambda_{11}$</td>
<td>35.1</td>
<td>0.45</td>
</tr>
<tr>
<td>Condition number $\kappa$</td>
<td>141.7</td>
<td>70.8</td>
</tr>
<tr>
<td>SVD: $G = U\Sigma V^H$</td>
<td>\begin{pmatrix} -0.625 &amp; 0.781 \ -0.781 &amp; -0.625 \end{pmatrix}</td>
<td>\begin{pmatrix} -0.630 &amp; 0.777 \ -0.777 &amp; -0.630 \end{pmatrix}</td>
</tr>
<tr>
<td>$U$</td>
<td>\begin{pmatrix} 1.972 &amp; 0 \ 0 &amp; 0.0139 \end{pmatrix}</td>
<td>\begin{pmatrix} 1.393 &amp; 0 \ 0 &amp; 0.0197 \end{pmatrix}</td>
</tr>
<tr>
<td>$V$</td>
<td>\begin{pmatrix} -0.707 &amp; 0.708 \ 0.708 &amp; 0.707 \end{pmatrix}</td>
<td>\begin{pmatrix} 1.000 &amp; -0.001 \ 0.001 &amp; 1.000 \end{pmatrix}</td>
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</table>

### Table A.4-3. Disturbance gains for LV-configuration.

<table>
<thead>
<tr>
<th></th>
<th>$x_F$</th>
<th>$F$</th>
<th>$q_F$</th>
<th>$V_d$</th>
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<tbody>
<tr>
<td>$dy_D$</td>
<td>0.881</td>
<td>0.394</td>
<td>0.868</td>
<td>0.864</td>
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<tr>
<td>$dx_B$</td>
<td>1.119</td>
<td>0.586</td>
<td>1.092</td>
<td>1.096</td>
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REFERENCES


REFERENCES


REFERENCES


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