Chapter 12

MIMO IMC DESIGN

We assume that the reader has mastered Chaps. 3 through 6 and is familiar with the IMC concept and its implications. Thus, the MIMO treatment of this topic will be much briefer. Some concepts which extend to the MIMO case in a straightforward manner (e.g., two-degree-of-freedom design) will not be covered at all. Issues for which analogies in the SISO case are lacking (e.g., structured uncertainty, ill-conditioned systems) will be treated in depth.

For SISO systems it was possible to design the IMC controllers analytically and to obtain insight into the effect of RHP zeros, RHP poles and robustness issues in the process. For tutorial reasons a separate derivation for stable (Chap. 4) and unstable (Chap. 5) systems was justified. Except in trivial cases (MP systems) the derivation of the IMC controllers for MIMO systems and the effect of RHP zeros and poles is quite complex. Therefore, only one derivation for both stable and unstable systems will be presented.

12.1 IMC Structure

The block diagram of the IMC structure is shown in Fig. 12.1-1A. Here P denotes the plant and P_m the measurement device transfer function. In general, neither P nor P_m are known exactly, only their approximate models $\tilde{P}$ and $\tilde{P}_m$ are available. The process transfer function P_d describes the effect of the disturbance d on the process output y. The measurement of y is corrupted by measurement noise n. The controller Q determines the value of the input (manipulated variable) u. The control objective is to keep y close to the reference (setpoint) r.

Commonly we will use the simplified block diagram in Fig. 12.1-1B. Here d denotes the effect of the disturbance on the output. Exact knowledge of the output y is assumed ($P_m = I, n = 0$). Note that the complete control system to be implemented through computer software or analog hardware is contained in the shaded box in Fig. 12.1-1B.

If the IMC controller Q and the classic controller C (Fig. 10.2-1B) are related

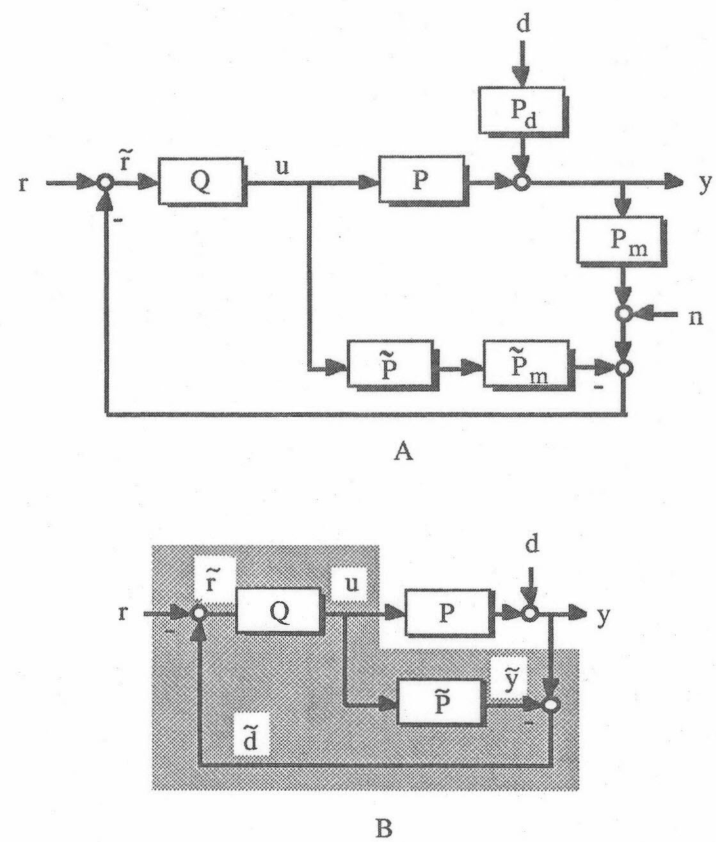

Figure 12.1-1. General (A) and simplified (B) IMC block diagram. Shaded portion indicates control system.

through

$$C = Q(I - \tilde{P}Q)^{-1} \quad (12.1-1)$$

$$Q = C(I + \tilde{P}C)^{-1} \quad (12.1-2)$$

then the IMC structure and the classic feedback structure are *equivalent* in the sense that to each input pair $\{r, d\}$ there corresponds the *same* output pair $\{y, u\}$.

For the IMC structure the sensitivity E and complementary sensitivity H are defined through

$$e = y - r = (I - \tilde{P}Q)(I + (P - \tilde{P})Q)^{-1}(d - r) \triangleq E(d - r) \quad (12.1-3)$$

$$y = PQ(I + (P - \tilde{P})Q)^{-1}r \triangleq Hr \qquad (\text{for } d = 0) \quad (12.1-4)$$

For the case that the model is perfect ($P = \tilde{P}$) these expressions yield

$$E = I - PQ \quad (12.1-5)$$

$$H = PQ \quad (12.1-6)$$

12.2 Conditions for Internal Stability

In order to test for internal stability we examine the transfer matrices between all possible system inputs and outputs. From the block diagram in Fig. 12.2-1 we note that there are three independent system inputs r, u_1, and u_2 and three independent outputs y, u, and $\tilde{y}$. If there is no model error ($P = \tilde{P}$), they are related through the following transfer matrix:

$$\begin{pmatrix} y \\ u \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} PQ & (I - PQ)P & P \\ Q & -QP & 0 \\ PQ & -PQP & P \end{pmatrix} \begin{pmatrix} r \\ u_1 \\ u_2 \end{pmatrix} \quad (12.2-1)$$

Theorem 12.2-1 follows trivially by inspection.

Theorem 12.2-1. *Assume that the model is perfect ($P = \tilde{P}$). Then the IMC system in Fig. 12.1-1B is internally stable if and only if both the plant P and the controller Q are stable.*

Thus the structure in Fig. 12.1-1B cannot be used to control plants which are open-loop unstable. Nevertheless, even for unstable plants we will exploit the features of the IMC structure for control system *design* and then *implement* the controller C (obtained from (12.1–1)) in the classic manner. For internal stability of the classic feedback structure the expression (10.2–14) has to be stable. We substitute for C from (12.1–1) to obtain for $P = \tilde{P}$

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} PQ & (I - PQ)P \\ Q & -QP \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \triangleq S \begin{pmatrix} r \\ u' \end{pmatrix} \quad (12.2-2)$$

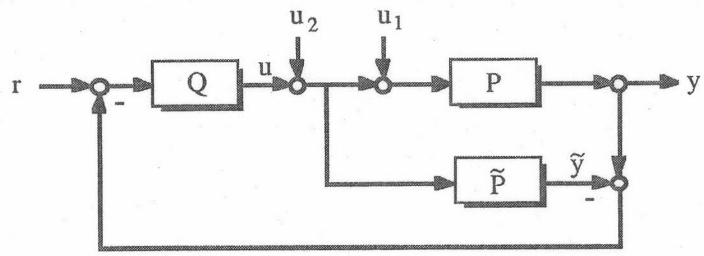

Figure 12.2-1. Block diagram for derivation of conditions for internal stability.

This implies that Q has to be stable and that in the elements of S the factors Q and $(I-PQ)$ have to cancel any unstable poles of P. Thus both Q and $(I-PQ)$ must have RHP zeros at the plant RHP poles. Special care has to be taken to cancel these common RHP zeros when the controller $C = Q(I-PQ)^{-1}$ is constructed. Minimal or balanced realization software can be used to accomplish that.

12.3 Parametrization of All Stabilizing Controllers

For SISO systems we derived a parametrization of all stabilizing controllers in terms of a stable transfer function q_1 (Thm. 5.1-2). A similar development involving coprime factorization is possible for MIMO systems. However, the resulting MIMO parameter Q cannot be interpreted as nicely as in the SISO case. Therefore, the IMC design procedure where Q is augmented by a low-pass filter for robustness and on-line adjustment would not be meaningful. We will derive an alternate parametrization which preserves the physical meaning of Q and allows us to compute the H_2-optimal controller for both stable and unstable MIMO systems. This controller will then again be a possible starting point (Step 1) for the IMC design procedure. We will make the following two assumptions.

Assumption A1. *If π is a pole of $\tilde{P}$ in the open RHP, then (a) the order of π is equal to 1 and (b) $\tilde{P}$ has no zeros at $s = \pi$.*

Assumption A1a is made solely to simplify the notation. Assumption A1b is not very restrictive because the presence of a zero at $s = \pi$ implies an exact cancellation in $\det(\tilde{P}(s))$, which is a nongeneric property — i.e., it does not happen after a slight perturbation in the coefficients of $\tilde{P}$ is introduced.

Assumption A1a is not made for poles at $s = 0$ because more than one such pole may appear in an element of $\tilde{P}$, introduced by capacitances present in the process. However, the following assumption is made which is true for all practical process control problems.

Assumption A2. *Any poles of $\tilde{P}$ or P on the imaginary axis are at $s = 0$. Also $\tilde{P}$ has no finite zeros on the imaginary axis.*

These two assumptions allow us to derive the following theorem.

Theorem 12.3-1. *Assume that Assns. A1 and A2 hold and that $Q_0(s)$ stabilizes P — i.e., that S in (12.2-2) is stable for $Q = Q_0$. Then all Q's which make S stable are given by*

$$Q(s) = Q_0(s) + Q_1(s) \tag{12.3-1}$$

where Q_1 is any stable transfer matrix such that PQ_1P is stable.

Proof. $\Rightarrow$ Assume that Q_0 makes S stable and that Q_1 and PQ_1P are stable. From substitution of (12.3-1) into (12.2-2) it follows that $S(Q)$ is stable if and only if $S' = (PQ_1 \quad Q_1P \quad PQ_1P)$ is stable. By assumption the third element of S' is stable. Stability of the other two elements follows by pre- and post-multiplication of PQ_1P by P^{-1}, since according to Assns. A1 and A2, P has no zeros at the location of its unstable poles and these are the only possible unstable poles of S'.

$\Leftarrow$ Assume that Q and Q_0 make S stable. Then the difference matrix

$$\Delta S = S(Q) - S(Q_0) = \begin{pmatrix} P(Q - Q_0) & -P(Q - Q_0)P \\ (Q - Q_0) & -(Q - Q_0)P \end{pmatrix}$$

is stable. This implies that $(Q - Q_0) = Q_1$ and PQ_1P are stable. □

Note that for SISO systems we can choose

$$Q_1 = b_p^2 s^{2\ell} Q_1', \quad Q_1' \text{ stable} \tag{12.3-2}$$

as we did for Thm. 5.1-2. Then clearly Q_1 and PQ_1P are stable. Poles of MIMO systems have a "structure" (matrix of residues) which is not reflected in (12.3–2). While even for MIMO systems any controller of the form (12.3–1 and 12.3–2) would indeed be a stabilizing controller there are many other controllers which are not of this form but are also stabilizing. Thus (12.3–1 and 12.3–2) would not constitute a *complete* parametrization of all MIMO stabilizing controllers.

12.4 Asymptotic Properties of Closed-Loop Response

"System types" were defined in Sec. 10.4.2 to classify the asymptotic closed-loop behavior.

Type m:

$$\lim_{s \to 0} s^{-k} E(s) = 0; \qquad 0 \le k < m \qquad (12.4-1)$$

Using (12.1–3) this definition becomes

Type m:

$$\lim_{s \to 0} s^{-k}(I - \tilde{P}Q)(I + (P - \tilde{P})Q)^{-1} = 0; \quad 0 \le k < m \qquad (12.4-2)$$

This is equivalent to requiring

Type m:

$$\lim_{s \to 0} \frac{d^k}{ds^k}(I - \tilde{P}Q) = 0; \qquad 0 \le k < m \qquad (12.4-3)$$

Specifically we obtain

Type 1:

$$\lim_{s \to 0} \tilde{P}Q = I \qquad (12.4-4)$$

Type 2:

$$\lim_{s \to 0} \tilde{P}Q = I \quad \text{and} \quad \lim_{s \to 0} \frac{d}{ds}(\tilde{P}Q) = 0 \qquad (12.4-5)$$

Thus, in order to track error free asymptotically constant inputs the controller gain has to be the inverse of the model steady state gain. Note that (12.4–4) and (12.4–5) are necessary and sufficient for off-set free tracking of steps and ramps, respectively, even when model error is present.

12.5 Outline of the IMC Design Procedure

In Chaps. 10 and 11 we discussed the objectives of control system design.

Nominal Performance:

Specification:

$$\|W_2 \tilde{E} W_1\|_\alpha < 1; \qquad \alpha = 2, \infty \qquad (12.5-1)$$

Optimal Design Problem:

$$\min_Q \|W_2 \tilde{E} W_1\|_\alpha = \min_Q \|W_2(I - \tilde{P}Q)W_1\|_\alpha; \qquad \alpha = 2, \infty \qquad (12.5-2)$$

In Sec. 10.4.3 we concluded that from a deterministic point of view the H_2-objective (12.5–1) is somewhat artificial. As an alternative we suggested considering the ISE for a single input or for a finite set of inputs.

Robust Stability:

Multiplicative Output Uncertainty:

$$\bar{\sigma}(\tilde{H}\bar{\ell}_O) < 1; \quad \forall\omega \quad \Leftrightarrow \quad \|\tilde{H}\bar{\ell}_O\|_\infty = \|\tilde{P}Q\bar{\ell}_O\|_\infty < 1 \tag{12.5 – 3}$$

Multiplicative Input Uncertainty:

$$\bar{\sigma}(\tilde{P}^{-1}\tilde{H}\tilde{P}\bar{\ell}_I) < 1; \quad \forall\omega \quad \Leftrightarrow \quad \|\tilde{P}^{-1}\tilde{H}\tilde{P}\bar{\ell}_I\|_\infty = \|Q\tilde{P}\bar{\ell}_I\|_\infty < 1 \tag{12.5 – 4}$$

General (block diagonal) Uncertainty (Δ_u):

$$\mu_{\Delta_u}(G^F_{11}(\tilde{P},Q)) < 1 \qquad \forall\omega \tag{12.5 – 5}$$

Robust Performance:

Specification:

$$\|W_2 E W_1\|_\infty < 1 \qquad \forall P \in \Pi \tag{12.5 – 6}$$

or

$$\mu_\Delta(G^F(\tilde{P},Q)) < 1 \qquad \forall\omega, \quad \Delta = \text{diag}\,\{\Delta_u, \Delta_p\} \tag{12.5 – 7}$$

Optimal Design Problem:

$$\min_Q \sup_\omega \mu_\Delta(G^F(\tilde{P},Q)) \tag{12.5 – 8}$$

where the superscript ^{F} in G^F is used to indicate that at the robustness stage, the design variables are the IMC filter parameters [see (12.5–11)].

For good performance a weighted norm of the sensitivity function $\tilde{E}$ should be made small (12.5–1). Usually the 2-norm, ∞-norm or the ISE for a finite set of inputs is minimized (12.5–2). Robust stability imposes constraints on the controller design: multiplicative output uncertainty constrains the maximum singular value of the complementary sensitivity function (12.5–3) but in general the constraint takes on a complex form (12.5–5). The robust performance constraint is expressed through the SSV. As an alternative to (12.5–8) one could minimize for a specific input the worst ISE that can result from any plant in the set Π.

Often what we need in practice is robust performance. However, at present, efficient and reliable techniques for the solution of (12.5–8) are not available and the specification of the performance weights which yield a practically useful controller is an art. Furthermore, sometimes it is more meaningful to consider an H_2-type performance objective [and not the H_∞-type implicit in 12.5–8] and to optimize nominal performance subject to the constraint of "robust stability" or "robust performance." Also, it is desirable to provide for convenient on-line robustness adjustment as the plant changes and the model quality degrades with time. For these reasons, we adopt the two-step IMC design approach which is capable of generating good engineering solutions to the robust control problem.

Step 1: Nominal Performance

$\tilde{Q}$ is selected to yield a "good" system response for the input(s) of interest, without regard for constraints and model uncertainty. Several H_2-type objective functions will be considered. Define

$$\Phi(v) = \|We\|_2^2 = \|W\tilde{E}v\|_2^2 = \|W(I - \tilde{P}\tilde{Q})v\|_2^2 \qquad (12.5-9)$$

Then one objective could be

Objective O1:

$$\min_{\tilde{Q}} \Phi(v)$$

for a particular input $v = (v_1, v_2, \ldots, v_n)^T$, where $\tilde{Q}$ satisfies the internal stability requirements. However, minimizing the weighted ISE for just one vector v is not very meaningful, because of the different directions in which the disturbances enter the process or the setpoints are changed. It is desirable to find a $\tilde{Q}$, that minimizes $\Phi(v)$ for every single v in a set of external inputs v of interest for the particular process. For an $n \times n$ P this set can be defined as

$$\mathcal{V} = \{v^i(s) : i = 1, \ldots, n\} \qquad (12.5-10)$$

where $v^1(s), \ldots, v^n(s)$ are vectors that describe the expected directions and frequency content of the external system inputs — e.g., steps, ramps, or other types of inputs.

The objective can then be written as

Objective O2:

$$\min_{\tilde{Q}} \Phi(v) \qquad \forall v \in \mathcal{V}$$

under the constraint that $\tilde{Q}$ satisfies the internal stability requirements. A linear time invariant $\tilde{Q}$ that meets Obj. O2 does not necessarily exist. In Sec. 12.6.3 it will be shown that it exists for a large class of $\mathcal{V}$'s. An alternative objective would be:

Objective O3:

$$\min_{\tilde{Q}}[\Phi(v^1) + \Phi(v^2) + \ldots + \Phi(v^n)]$$

In this case the objective is to minimize the sum of the ISE's that each of the inputs v^i would cause when applied to the system separately. This was the objective discussed in Sec. 10.4.3.

Step 2: Robust Stability and Performance

The aggressive controller $\tilde{Q}$ obtained in Step 1 is detuned to satisfy the robustness requirements. For that purpose $\tilde{Q}$ is augmented by a filter F of fixed structure

$$Q = Q(\tilde{Q}, F) \tag{12.5 – 11}$$

and the filter parameters are adjusted to solve (12.5–8). Generally, the weights W_i chosen for nominal performance (12.5–1, 12.5–2) and robust performance (12.5–6 to 12.5–8) are not the same. As in the SISO case the solution of the nominal performance problem suggests an ideal sensitivity operator $\bar{E}$ which cannot be realized in practice because of robustness problems. The robust performance design objective is to make the sensitivity operator as similar to $\bar{E}$ as possible while satisfying all robustness constraints. Thus, generally the performance weight for (12.5–6 to 12.5–8) is derived from the optimal $\bar{E}$ obtained in Step 1. For example, one could choose $W_2 = w_2 I$ where $w_2^{-1} > \bar{\sigma}(\bar{E})$ which implies that the magnitude (maximum singular value) of the sensitivity function is allowed to increase up to w_2^{-1}.

Sometimes F enters Q (12.5–11) in a simple manner

$$Q = \tilde{Q}F \tag{12.5 – 12}$$

Sometimes a more complicated form is required as we will discuss later. In general, it might not be possible to meet the robust performance requirement (12.5–7). The reason could be that our two-step design procedure fails to produce an acceptable Q. On the other hand, there might not exist any Q such that (12.5–7) is satisfied. Then the performance requirements have to be relaxed and/or the model uncertainty has to be reduced.

12.6 Nominal Performance

We will first make some assumptions on the input v and the set of inputs $\mathcal{V}$. Then we will find the controllers $\tilde{Q}$ which meet Objs. O1 through O3.

12.6.1 Assumptions

In Sec. 12.3 we made some assumptions on the RHP poles and zeros of $\tilde{P}$ to enable us to derive a simple controller parametrization. The following assumptions on v and $\mathcal{V}$ are of a similar nature.

Assumption A3. *Every nonzero element of v includes all open RHP poles of $\tilde{P}$, each of them with degree 1, and those are the only open RHP poles of v.*

Assumption A4. *Let ℓ_i be the maximum number of poles at $s = 0$ of any element in the i^{th} row of $\tilde{P}$. Then the i^{th} element of v, $v_i(s)$, has at least ℓ_i poles at $s = 0$. Also v has no other poles on the imaginary axis and its elements have no finite zeros on the imaginary axis.*

Assumptions A3 and A4 are not restrictive when v is an output disturbance d, generated by an input disturbance that has passed through the process. In that case d usually includes all the unstable process poles as postulated by Assns. A3 and A4. Note that the control system will also reject other disturbances with fewer unstable poles, without producing steady-state offset. Assumption A4 is different for poles at $s = 0$ because their number in each row of $\tilde{P}$ can be different (capacitancies may be associated only with certain process outputs). Also, the output disturbance may have more poles at $s = 0$ than the process.

Assumptions A3 and A4 may be restrictive for setpoints. Then, however, we can employ the two-degree-of-freedom structure (Sec. 5.2.4) which allows us to disregard the existence of any unstable poles of $\tilde{P}$ and Assns. A3 and A4 are not needed.

For Objs. O2 and O3 we are considering the set $\mathcal{V}$ (12.5–10) of n inputs v^i, $i = 1, n$. Define the square matrix

$$V(s) \triangleq (v^1(s) \quad v^2(s) \quad \ldots \quad v^n(s)) \tag{12.6 – 1}$$

Assumption A5. *The matrix V has no zeros at the location of its unstable poles and no finite zeros on the imaginary axis, and V^{-1} cancels the closed RHP poles of $\tilde{P}$ in $V^{-1}\tilde{P}$.*

Assumptions A3 and A4 for each column of V do not necessarily imply that Assn. A5 is satisfied. A matrix V which satisfies Assn. A5 can be easily constructed. One way is to obtain V as $\tilde{P}$ times a matrix with no open RHP poles and no finite zeros on the imaginary axis. This case corresponds to the physically meaningful situation, where the output disturbances v^i are generated by disturbances at the plant input. Another way is to use a diagonal V, in which case satisfaction of Assns. A3 and A4 by every column of V implies satisfaction of Assn. A5.

Note that Assns. A1-A5 are only relevant for unstable plants and impose no restrictions on stable plants.

12.6.2 H_2-Optimal Control for a Specific Input

The solution of Obj. O1 is also the first step toward the solution of Obj. O2 if such a solution exists. Let $v_0(s)$ be the scalar allpass that includes the *common*

RHP zeros of the elements of v. Factor v as follows:

$$v(s) = v_0(s)(\hat{v}_1(s)\dots\hat{v}_n(s))^T \triangleq v_0(s)\hat{v}(s)^T \qquad (12.6-2)$$

The plant P can be factored into a stable allpass portion P_A and an MP portion P_M such that

$$P = P_A P_M \qquad (12.6-3)$$

Hence P_A and P_M^{-1} are stable and $P_A^H(i\omega)P_A(i\omega) = I$. The procedure for carrying out this "inner-outer" factorization is discussed in Sec. 12.6.4.

Theorem 12.6-1. *Assume that Assns. A1-A4 hold. Then the set of controllers $\tilde{Q}$ that solves Obj. O1 satisfies*

$$\tilde{Q}\hat{v} = P_M^{-1}W^{-1}\{WP_A^{-1}\hat{v}\}_* \qquad (12.6-4)$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, all terms involving the poles of P_A^{-1} are omitted. Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (12.6–4) is infinite. Guidelines for the construction of a controller are given in the proof.

We stress that not every controller that satisfies (12.6–4) is stabilizing. Improper stabilizing controllers that satisfy (12.6–4) are accepted because in the second step of the design procedure a filter with the appropriate roll-off will produce a proper $Q(s)$. Note that any *constant* weight W cancels in (12.6–4).

The relationship (12.6–4) should be compared with the H_2-optimal controller for SISO systems stated in Thm. 5.2-1. If we assume that the plant and the disturbance have the *same* open-loop RHP poles, then the expressions (5.2–6) and (12.6–4) are equivalent. For SISO systems we did not have to make Assn. A3 because we can easily factor out the unstable portions of p and v.

Proof of Theorem 12.6-1. The proof of (12.6–4) is somewhat lengthy and can be skipped by the result-oriented reader without loss of continuity. We will assume $W = I$. The proof of the weighted case if left as an exercise. Let V_0 be a diagonal matrix where each column satisfies Assn. A3 and every element has ℓ_v poles at $s = 0$ where ℓ_v is the maximum number of such poles in any element of v. Assume that there exists Q_0 which stabilizes $\tilde{P}$ in the sense of Thm. 12.3-1 and also makes $(I - PQ_0)V_0$ stable. Its existence will be proven by construction. Substitution of (12.3–1) into (12.5–9) and use of the fact that pre- or post-multiplication of a function with an allpass does not change its L_2-norm, yields:

$$\Phi(v) = \|P_A^{-1}(I - PQ_0)\hat{v} - P_M Q_1 \hat{v}\|_2^2 \triangleq \|f - P_M Q_1 \hat{v}\|_2^2 \qquad (12.6-5)$$

L_2, the space of functions square integrable on the imaginary axis, can be decomposed into two subspaces: H_2 the subspace of functions analytic in the RHP (stable functions) and its orthogonal complement $H_2^{\perp}$ that includes all strictly unstable functions (functions with *all* their poles in the RHP). The term f has no poles on the imaginary axis because $(I-PQ_0)V_0$ has no such poles. However, f may not be an L_2 function because of the existence of a constant term in its PFE. This term is equal to $\lim_{s\to+\infty} f(s) \triangleq f(\infty)$. Then $\hat{f}(s) \triangleq f(s)-f(\infty)$ is in L_2 and it can be uniquely decomposed into two orthogonal functions $\{\hat{f}\}_+ \in H_2$ and $\{\hat{f}\}_- \in H_2^{\perp}$:

$$\hat{f}(s) \triangleq f(s) - f(\infty) = \{\hat{f}\}_+ + \{\hat{f}\}_-$$

In order for $\Phi(v)$ to be finite, the optimal Q_1 has to make $f - P_M Q_1 \hat{v}$ strictly proper. Hence Q_1 has to satisfy

$$\lim_{s\to\infty}(P_M Q_1 \hat{v}) \triangleq (P_M Q_1 \hat{v})(\infty) = f(\infty) \qquad (12.6-6)$$

Next we want to show that $P_M Q_1 \hat{v}$ has to be stable. The fact that $(I-PQ_0)V_0$ is stable and the way V_0 is constructed imply that $(I-PQ_0)v$ is stable. Then $(I-PQ_0)\hat{v}$ will also be stable because of the definition of v_0. We require that $(I-PQ)v$ have no open RHP poles and therefore that $(I-PQ)\hat{v} = (I-PQ_0)\hat{v} - PQ_1\hat{v}$ have no open RHP poles. But since $(I-PQ_0)\hat{v}$ is stable, this requirement reduces to $PQ_1\hat{v}$ having no poles in the open RHP. Also in order for $\Phi(v)$ to be finite, Q_1 must be such that $(I-PQ)v$ or $(I-PQ)\hat{v}$ has no poles on the imaginary axis. But since $(I-PQ_0)\hat{v}$ is stable this is equivalent to $PQ_1\hat{v}$ having no poles on the imaginary axis. Hence the optimal Q_1 must be such that $PQ_1\hat{v}$ is stable. Then the only possible unstable poles of $P_M Q_1 \hat{v} = P_A^{-1} P Q_1 \hat{v}$ are the poles of P_A^{-1}. But Assns. A1, A2 imply that the poles of P_A^{-1} are not among those of $P_M Q_1 \hat{v}$ and therefore $P_M Q_1 \hat{v}$ has to be stable. To proceed we will assume that Q_1 has this property. We will verify later that the solution has indeed this property.

The fact that $P_M Q_1 \hat{v}$ is stable and (12.6-6) imply that $P_M Q_1 \hat{v} - f(\infty)$ is in H_2. Thus, we can then write (12.6–5) as

$$\Phi(v) = \|\{\hat{f}\}_-\|_2^2 + \|\{\hat{f}\}_+ - (P_M Q_1 \hat{v} - f(\infty))\|_2^2$$

Because $\hat{f}$ does not depend on Q_1, the obvious solution to the minimization of $\Phi(v)$ is

$$P_M Q_1 \hat{v} - f(\infty) = \{\hat{f}\}_+$$

or

$$P_M Q_1 \hat{v} = \{\hat{f}\}_+ + f(\infty) \triangleq \{f\}_{\infty+}$$

or

$$Q_1\hat{v} = P_M^{-1}\{f\}_{\infty+} \qquad (12.6-7)$$

where the notation $\{\cdot\}_{\infty+}$ is used to indicate that after a PFE all strictly proper stable terms as well as the constant term are retained.

Clearly such a Q_1 produces a stable $P_M Q_1 \hat{v}$ as was assumed and also satisfies (12.6–6). It should now be proved that Q_1's that satisfy the internal stability requirements exist among those described by (12.6–7) so that the obvious solution is a true solution. For $n = 1$, (12.6–7) yields a unique Q_1, which we have shown to satisfy the requirements in the proof of Thm. 5.2-1. For $n \geq 2$ write

$$Q_1 \triangleq (q_1 \quad q_2) \qquad (12.6-8)$$

$$\hat{V}_2 \triangleq (\hat{v}_2 \quad \ldots \quad \hat{v}_n)^T \qquad (12.6-9)$$

where without loss of generality the first element of v, and thus $\hat{v}_1$, is assumed to be nonzero. Also q_1 is $n \times 1$ and q_2 is $n \times (n-1)$. Then it follows from (12.6–7) that

$$Q_1 = \left(\hat{v}_1^{-1}(P_M^{-1}\{f\}_{\infty+} - q_2\hat{V}_2) \qquad q_2\right) \qquad (12.6-10)$$

We now need to show that a stable q_2 exists such that Q_1 is stable and produces a stable PQ_1P. Select a q_2 of the form:

$$q_2(s) = \hat{q}_2(s)s^{3\ell_v}\prod_{i=1}^{k}(s-\pi_i)^3 \qquad (12.6-11)$$

where $\hat{q}_2$ is stable and $\{\pi_1, \ldots, \pi_k\}$ are the poles of P in the open RHP. Then if follows from (12.6–10) that in order for PQ_1P to be stable it is sufficient that $P\hat{v}_1^{-1}P_M^{-1}\{f\}_{\infty+}\{P\}_{1^{st}row}$ have no closed RHP poles. But $PP_M^{-1} = P_A$ is stable and the only possible unstable poles of $\hat{v}_1^{-1}\{P\}_{1^{st}row}$ are open RHP poles of $\hat{v}_1^{-1}$ because of Assns. A3 and A4. These are also the only possible unstable poles of Q_1. Let α be such a pole (zero of $\hat{v}_1$). Then for stability we need to find $\hat{q}_2$ such that

$$\hat{q}_2(\alpha)\hat{V}_2(\alpha) = \alpha^{-3\ell_v}\prod_{i=1}^{k}(\alpha-\pi_i)^{-3}P_M^{-1}(\alpha)\{f\}_{\infty+}\Big|_{s=\alpha} \qquad (12.6-12)$$

The above equation always has a solution because the vector $\hat{V}_2(\alpha)$ is not identically zero since any common RHP zeros in v were factored out in v_0.

We shall now proceed to obtain an expression for $Q\hat{v}$. Equations (12.3–1) and (12.6–7) yield

$$Q\hat{v} = P_M^{-1}(P_A^{-1}PQ_0\hat{v} - \{P_A^{-1}PQ_0\hat{v}\}_{\infty+} + \{P_A^{-1}\hat{v}\}_{\infty+})$$

$$= P_M^{-1}(\{P_A^{-1}PQ_0\hat{v}\}_{0-} + \{P_A^{-1}\hat{v}\}_{\infty+}) \qquad (12.6-13)$$

where $\{\cdot\}_{0-}$ indicates that in the partial fraction expansion all poles in the closed RHP are retained. For (12.6–13), these poles are the poles of $\hat{v}$ in the closed RHP; $P_A^{-1}PQ_0 = P_MQ_0$ is strictly stable because of Assn. A1 and the fact that Q_0 is a stabilizing controller. The fact that $(I - PQ_0)V_0$ has no poles at $s = 0$ and the form of V_0 imply that $(I - PQ_0)$ and its derivatives up to and including the $(\ell_v - 1)^{th}$ are equal to zero at $s = 0$. Also, the fact that $(I - PQ_0)V_0$ is stable and that the columns of this diagonal V_0 satisfy Assn. A3, imply that $(I - PQ_0) = 0$ at $\pi_1, \ldots, \pi_k$. Hence (12.6–13) simplifies to (12.6–4).

We now need to show that a stabilizing controller Q_0 exists with the property that $(I - PQ_0)V_0$ is stable. The selection of a V_0 with the properties mentioned at the beginning of this section and its use instead of V in (12.6–15) yields such a controller. □

12.6.3 H_2-Optimal Control for a Set of Inputs

We determined in Sec. 12.6.2 that the set of controllers which meets Obj. O1 for a MIMO system is infinite. Thus Obj. O1 is not a very interesting design objective. In this section we will address Objs. O2 and O3. Clearly any solution for Obj. O2 (if it exists) is also a solution for Obj. O3. Therefore, we will start by determining the controller which meets Obj. O3. If a solution for Obj. O2 exists it will be given by the solution for Obj. O3.

Factor V, defined in (12.6–1), similarly to P (see Sec. 12.6.4)

$$V = V_MV_A \qquad (12.6-14)$$

Then the following theorem holds:

Theorem 12.6-2. *Assume that Assns. A1-A5 hold. Then the controller*

$$\tilde{Q} = P_M^{-1}W^{-1}\{WP_A^{-1}V_M\}_*V_M^{-1} \qquad (12.6-15)$$

is the unique solution for Obj. O3. Here the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, all terms involving the poles of P_A^{-1} are omitted.

The formula is identical to that obtained in the SISO case when p and v have the same RHP poles.

Proof. Again we assume $W = I$ and leave the weighted case as an exercise. The L_2-norm of a matrix $G(s)$ analytic on the imaginary axis is given by

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}[G^H(i\omega)G(i\omega)] \quad d\omega\right)^{1/2} \qquad (12.6-16)$$

Then from (12.5–9), (12.6–1) and (12.6–16) it follows that

$$\Phi(v^1) + \Phi(v^2) + \ldots + \Phi(v^n) = \|(I - P\tilde{Q})V\|_2^2 \triangleq \Phi(V) \qquad (12.6-17)$$

The minimization of $\Phi(V)$ follows the steps in the proof of Thm. 12.6-1 up to (12.6–7), with V_M used instead of $\hat{v}$. In this case ℓ_v is the maximum number of poles at $s = 0$ in any element of V. From the equivalent to (12.6–7) we obtain

$$Q_1 = P_M^{-1}\{f\}_{\infty+}V_M^{-1} \qquad (12.6-18)$$

where V_M is used instead of $\hat{v}$ in f. This Q_1 makes $f - P_M Q_1 V_M$ strictly proper. We now have to establish that Q_1 is stable and produces a stable PQ_1P. In PQ_1P the unstable poles of the P on the left cancel with those of P_M^{-1}. As for the P on the right, the same follows from Assn. A5. Then in the same way that (12.6–4) follows from (12.6–13), (12.6–15) follows from (12.6–18). □

Let us now consider the more meaningful Obj. O2. Factor each of the v^i in the same way as in (12.6–2):

$$v^i(s) = v_0^i(s)\hat{v}^i(s) \qquad (12.6-19)$$

Define

$$\hat{V} \triangleq (\hat{v}^1 \quad \hat{v}^2 \quad \ldots \quad \hat{v}^n) \qquad (12.6-20)$$

Theorem 12.6-3.

i) *If $\hat{V}(s)$ is non-minimum phase, then there exists no solution to O2.*

ii) *If $\hat{V}(s)$ is minimum phase, then use of $\hat{V}$ instead of V_M in (12.6–15) yields exactly the same $\tilde{Q}$, which also solves Obj. O2. In addition $\tilde{Q}$ minimizes $\Phi(v)$ for any v that is a linear combination of v^i's that have the same v_0^i's.*

Proof. ($W = I$). A stabilizing controller that solves Obj. O2 has to solve Obj. O1 for all v^i, $i = 1, \ldots n$. Satisfying (12.6–4) for every v^i is equivalent to

$$\tilde{Q} = P_M^{-1}\{P_A^{-1}\hat{V}\}_*\hat{V}^{-1} \qquad (12.6-21)$$

Hence the above $\tilde{Q}$ is the only potential solution for Obj. O2. However, it is not necessarily a stabilizing controller since not only stabilizing $\tilde{Q}$'s satisfy (12.6–4) for some v. Indeed, if $\hat{V}$ is non-minimum phase, $\hat{V}^{-1}$ is unstable and this results in an unstable $\tilde{Q}$, which is therefore unacceptable. Hence in such a case, there exists no solution for Obj. O2, which completes the proof of part (i) of the theorem.

In the case where $\hat{V}^{-1}$ is stable ($\hat{V}$ minimum phase), the controller given by (12.6–21) is stable and therefore it is the same as the one given by (12.6–15). This fact can be explained as follows. We have

$$V = \hat{V}V_0 \tag{12.6 - 22}$$

where

$$V_0 = \text{diag}\{v_0^1, v_0^2, \ldots, v_0^n\} \tag{12.6 - 23}$$

Since $\hat{V}^{-1}$ is stable, (12.6–22) represents a factorization of V similar to that in (12.6–14). From spectral factorization theory it follows that

$$\hat{V}(s) = V_M(s)A \tag{12.6 - 24}$$

where A is a constant matrix, such that $AA^H = I$. Then (12.6–15) is not altered when $\hat{V}$ is used instead of V_M because A cancels.

Let us now assume without loss of generality that the first j v^i's have the same v_0^i's. Consider a v that is a linear combination of $v^1, \ldots, v^j$:

$$v(s) = \alpha_1 v^1(s) + \ldots + \alpha_j v^j(s) \tag{12.6 - 25}$$

Then it follows that

$$v_0(s) = v_0^1(s) = \ldots = v_0^j(s) \tag{12.6 - 26}$$

$$\hat{v}(s) = \alpha_1 \hat{v}^1(s) + \ldots + \alpha_j \hat{v}^j(s) \tag{12.6 - 27}$$

One can easily check that a $\tilde{Q}$ that satisfies (12.6–4) for $\hat{v}^1, \ldots, \hat{v}^j$, will also satisfy (12.6–4) for the $\hat{v}$ given by (12.6–27) because of the property

$$\{\alpha_1 f_1(s) + \ldots + \alpha_j f_j(s)\}_* = \alpha_1\{f_1(s)\}_* + \ldots + \alpha_j\{f_j(s)\}_* \tag{12.6 - 28}$$

But then from Thm. 12.6-1 it follows that if a stabilizing controller $\tilde{Q}$ satisfies (12.6–4) for $\hat{v}$, then it minimizes the ISE $\Phi(v)$. □

The following corollary to Thm. 12.6-3 holds for a specific choice of V.

Corollary 12.6-1. *Let*

$$V = \mathrm{diag}\{v_1, v_2, \ldots, v_n\} \qquad (12.6-29)$$

where $v_1(s), \ldots, v_n(s)$ *are scalars. Then use of* $\hat{V}$ *instead of* V_M *in (12.6–15) yields exactly the same* $\tilde{Q}$, *which minimizes* $\Phi(v)$ *for the following* n *vectors:*

$$v = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} \qquad (12.6-30)$$

and their multiples, as well as for the linear combinations of those directions that correspond to v_i*'s with the same open RHP zeros with the same degree.*

In summary it is most popular to obtain $\tilde{Q}$ by solving Obj. O3. As we pointed out in Sec. 10.4.3 the $\tilde{Q}$ which solves Obj. O3 minimizes the 2-norm of the weighted sensitivity operator. Theorem 12.6-3 can be helpful for choosing a meaningful V.

12.6.4 Algorithm for "Inner-Outer" Factorization

The following theorem is the tool for obtaining the factorizations (12.6–3) and (12.6–14).

Theorem 12.6-4 (Chu, 1985). *Let* $G(s) = C(sI - A)^{-1}B + D$ *be a minimal realization of the square transfer matrix* $G(s)$, *and let* $G(s)$ *have no zeros on the* $i\omega$*-axis including infinity. Then we have*

$$G(s) = N(s)M(s)^{-1} \qquad (12.6-31)$$

where N, M *are stable and* $N(i\omega)^H N(i\omega) = I$. $N(s)$ *and* $M(s)^{-1}$ *are given by*

$$N(s) = (C - QF)(sI - (A - BR^{-1}F))^{-1}BR^{-1} + Q \qquad (12.6-32)$$

$$M(s)^{-1} = F(sI - A)^{-1}B + R \qquad (12.6-33)$$

where

$$D = QR \qquad (12.6-34)$$

is the QR *factorization of* D *into an orthogonal matrix* Q $(Q^TQ = I)$ *and an upper triangular matrix* R, *and*

$$F = Q^TC + (BR^{-1})^TX \qquad (12.6-35)$$

with X *the stabilizing [i.e., it makes* $(A - BR^{-1}F)$ *stable] real symmetric solution of the following algebraic Riccati equation (ARE):*

$$(A - BR^{-1}Q^TC)^TX + X(A - BR^{-1}Q^TC) - X(BR^{-1})(BR^{-1})^TX = 0 \qquad (12.6-36)$$

Such a solution X exists for (12.6–36).

Methods for solving ARE's can be found in the cited literature. Many control software packages include programs for that purpose.

By comparing (12.6–31) to (12.6–3) we see that $P_A = N$ and $P_M = M^{-1}$. Note however that a $P(s)$ that represents a physical system, is usually strictly proper ($D = 0$). In order to apply Thm. 12.6-4 in such a case we need to add to $P(s)$ a small $D = \epsilon I$. Repeated applications have shown this to be a satisfactory approach. Also note that $N(s)$ in (12.6–32) is unique only up to premultiplication with a constant unitary matrix U ($U^H U = 1$). Although not necessary, one may wish for consistency with the SISO case, to choose $P_A(s)$ such that $P_A(0) = I$. This can be accomplished by premultiplying $N(s)$ (12.6–32) by $N(0)^{-1}$.

A comparison of (12.6–31) and (12.6–14) indicates that Thm. 12.6-4 cannot be directly applied to $G(s) = V(s)$. We can, however, apply Thm. 12.6-4 to $\tilde{V}(s) \triangleq V^T(s)$ to obtain

$$\tilde{V}(s) = V^T(s) = \tilde{V}_A(s)\tilde{V}_M(s) \qquad (12.6-37)$$

with $\tilde{V}_A, \tilde{V}_M^{-1}$ stable and $\tilde{V}_A(i\omega)^H \tilde{V}_A(i\omega) = I$. From (12.6–37) we find

$$V(s) = \tilde{V}_M^T(s)\tilde{V}_A^T(s) \qquad (12.6-38)$$

Since $\tilde{V}_A^T(s), (\tilde{V}_M^T(s))^{-1}$ are stable and

$$\tilde{V}_A^T(i\omega)(\tilde{V}_A^T(i\omega))^H = (\tilde{V}_A^H(i\omega)\tilde{V}_A(i\omega))^T = I$$

we can select V_A, V_M as

$$V_A(s) = \tilde{V}_A^T(s) \qquad (12.6-39)$$

$$V_M(s) = \tilde{V}_M^T(s) \qquad (12.6-40)$$

Similarly to the case for $P(s)$, we may have to introduce a small D matrix in $V(s)$. Also the factorization is only unique up to post-multiplication with a constant unitary matrix.

12.7 Robust Stability and Performance

The controller $\tilde{Q}$ is to be detuned through a lowpass filter F such that for the detuned controller $Q = Q(\tilde{Q}, F)$ both the robust stability (12.5–5) and the robust performance (12.5–7) conditions are satisfied. Because (12.5–7) implies (12.5–5) a procedure is proposed to minimize $\mu_\Delta(G^F(\tilde{P}, Q))$ as a function of the filter parameters for a filter with a fixed structure. First we will postulate reasonable filter structures. Then we will discuss an algorithm for carrying out the minimization.

12.7.1 Filter Structure

In principle the structure of F can be as complex as the designer wishes. However, in order to keep the number of variables in the optimization problem small, a simple structure like a diagonal F with first- or second-order terms is recommended. In most cases this is not restrictive because the controller $\tilde{Q}$ designed in the first step of the IMC procedure is a full matrix with high order elements. Some restrictions must be imposed on the filter in the case of an open-loop unstable plant. Also a more complex filter structure may be necessary in cases of highly ill-conditioned systems.

Open-Loop Unstable Plants.

The filter $F(s)$ is chosen to be a diagonal rational function that satisfies the following requirements:

(a) Pole-zero excess. The controller $Q = \tilde{Q}F$ must be proper. Assume that the designer has specified a pole-zero excess of ν for the filter $F(s)$.

(b) Internal stability. The transfer matrix S in (12.2–2) must be stable.

(c) Asymptotic setpoint tracking and/or disturbance rejection. $(I-\tilde{P}\tilde{Q}F)v$ must be stable.

Write

$$F(s) = \text{diag}\,\{f_1(s), \ldots, f_n(s)\} \qquad (12.7-1)$$

Under Assns. A1-A5, (b) and (c) are equivalent to the following conditions. Let $\pi_i \quad (i = 1, \ldots, k)$ be the open RHP poles of $\tilde{P}$. Let $\pi_0 = 0$ and m_{0l} be the largest multiplicity of such a pole in any element of the l^{th} row of V. From Assns. A1-A5 and the fact that $\tilde{Q}$ makes S and $(I-P\tilde{Q})V$ stable, it follows that the l^{th} element, f_l, of the filter F must satisfy:

$$f_l(\pi_i) = 1, \qquad i = 0, 1, \ldots, k \qquad (12.7-2)$$

$$\left.\frac{d^j}{ds^j} f_l(s)\right|_{s=\pi_0} = 0, \qquad j = 1, \ldots, m_{0l} - 1 \qquad (12.7-3)$$

Requirements (12.7–2) clearly show the limitation RHP poles place on the robustness properties of a control system designed for an open-loop unstable plant. Since because of (12.7–2) one cannot reduce the closed-loop bandwidth of the nominal system at frequencies corresponding to the RHP poles of the plant, only a relatively small model error can be tolerated at those frequencies.

Experience has shown that the following structure for a filter element $f_l(s)$ is reasonable:

$$f_l(s) = \frac{a_{\nu_l-1,l}s^{\nu_l-1} + \ldots + a_{1,l}s + a_{0,l}}{(\lambda s+1)^{\nu+\nu_l-1}} \qquad (12.7-4)$$

where

$$\nu_l = m_{0l} + k \qquad (12.7-5)$$

For a specific tuning parameter λ the numerator coefficients can be computed to satisfy (12.7–2) and (12.7–3). This involves solving a system of ν_l linear equations with ν_l unknowns.

Example 12.7-1. Assume that we desire a pole-zero excess of ν and that there is only one pole π. Then

$$f(s) = \frac{(\lambda\pi+1)^\nu}{(\lambda s+1)^\nu} \qquad (12.7-6)$$

If $\pi = 0$, (12.7–6) reduces to the standard filter for stable systems $f(s) = (\lambda s + 1)^{-\nu}$. □

Example 12.7-2. Assume that $\nu = 2$ and the only pole is a double pole at $s = 0$. Then

$$f(s) = \frac{3\lambda s+1}{(\lambda s+1)^3} \qquad (12.7-7)$$

□

Ill-Conditioned Plants.

Problems arise because the optimal controller $\tilde{Q}$ designed for $\tilde{P}$ tends to be an approximate inverse of $\tilde{P}$ and as a result $\tilde{Q}$ is ill-conditioned when $\tilde{P}$ is. Although robust stability can generally be achieved by significant detuning of the diagonal filter, the robust performance condition is usually not satisfied. We have shown in Sec. 11.3.3 that an ill-conditioned P and C (and therefore an inverting ill-conditioned $\tilde{Q}$ as well) can cause problems when input uncertainty is present. This problem can be addressed through a filter that acts directly on the singular values of $\tilde{Q}$, at the frequency where the condition number of $\tilde{Q}$ is highest, say ω^*. Let

$$\tilde{Q}(i\omega^*) = \mathrm{U}_Q \Sigma_Q \mathrm{V}_Q^H \qquad (12.7-8)$$

be the SVD of $\tilde{Q}$ at ω^* and let R_u, R_v be real matrices that solve the pseudo-diagonalization problems:

$$\mathrm{U}_Q^H R_u \cong I \qquad (12.7-9)$$

$$\mathrm{V}_Q^H R_v \cong I \qquad (12.7-10)$$

Then the controller Q including the filter, is chosen to be of the form

$$Q(s) = R_u F_1(s) R_u^{-1} \tilde{Q}(s) F_2(s) \qquad (12.7-11)$$

or

$$Q(s) = \tilde{Q}(s) R_v F_1(s) R_v^{-1} F_2(s) \qquad (12.7-12)$$

where $F_1(s)$, $F_2(s)$ are diagonal filters, such that $F_1(0) = F_2(0) = I$ if integral action is desired. Note that for F_1, m_0 should be used in (12.7–3), (12.7–5), for all l, instead of m_{0l}, where $m_0 = \max_l m_{0l}$. This is required for internal stability and no steady-state offset.

It should be pointed out that the success of this approach depends on the quality of either one of the pseudo-diagonalizations (12.7–9) or (12.7–10). The diagonalization will be perfect if U_Q or V_Q is real. This will happen for example if $\omega^* = 0$, that is if the problems arise because the plant is ill-conditioned at steady-state, as for example high purity distillation columns are.

12.7.2 General Interconnection Structure with Filter

Consider the block diagram in Fig. 12.1-1B. Assume that $Q = \tilde{Q}F$. In order to use the SSV effectively for designing F, some rearrangement of the structure is necessary. Figure 12.1-1B can be transformed to Fig. 12.7-1A, where $v = d - r$, $e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \qquad (12.7-13)$$

where the blocks 0 and I have the appropriate dimensions.

We will assume throughout the following that the set of plants Π can be modeled in a way that allows Fig. 12.7-1A to be transformed into Fig. 12.7-1B where Δ is a block diagonal matrix with the additional property that $\bar{\sigma}(\Delta) \leq 1$, $\forall\omega$ (see Sec. 11.2.1). The superscript u in G^u denotes the dependence of G^u not only on G but also on the specific uncertainty description available for the model $\tilde{P}$. Next G^u is derived for some typical examples.

1. *Additive Uncertainty* (11.1–1), (11.1–5)

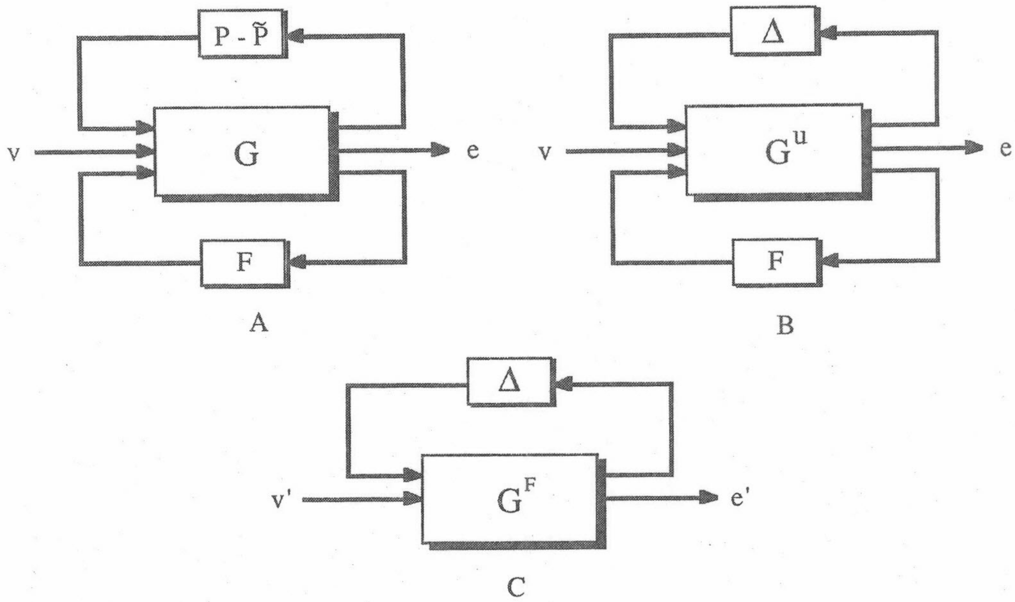

Figure 12.7-1. Model uncertainty block diagrams.

$$G^u = G^A = \begin{pmatrix} \bar{\ell}_A I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \qquad (12.7-14)$$

where G is defined in (12.7–13)

2. *Multiplicative Output Uncertainty* (11.1–2), (11.1–5)

$$G^u = G^O = \begin{pmatrix} \bar{\ell}_O \tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \qquad (12.7-15)$$

3. *Multiplicative Input Uncertainty* (11.1–3), (11.1–5)

$$G^u = G^I = G \begin{pmatrix} \bar{\ell}_I \tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \qquad (12.7-16)$$

4. *Independent Uncertainty in the Transfer Matrix Elements* (11.2–32), (11.2–33)

$$G^u = G^e = \begin{pmatrix} W_{e1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \begin{pmatrix} W_{e2} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \qquad (12.7-17)$$

Note that all the above relations yield a G^u already partitioned as

$$G^u = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} \\ G^u_{21} & G^u_{22} & G^u_{23} \\ G^u_{31} & G^u_{32} & G^u_{33} \end{pmatrix} \qquad (12.7-18)$$

Then Fig. 12.7-1B can be written as Fig. 12.7-1C with $e' = W_2 e$ and $v = W_1 v'$

$$G^F = \begin{pmatrix} G^u_{11} & G^u_{12}W_1 \\ W_2G^u_{21} & W_2G^u_{22}W_1 \end{pmatrix} + \begin{pmatrix} G^u_{13} \\ W_2G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} F (G^u_{31} \quad G^u_{32}W_1)$$

$$\triangleq \begin{pmatrix} G^F_{11} & G^F_{12} \\ G^F_{21} & G^F_{22} \end{pmatrix} \qquad (12.7-19)$$

If the special filter structure for ill-conditioned systems is used, the procedure for deriving G^u is slightly modified. Define

$$F(s) = \text{diag}\{F_1(s), F_2(s)\} \qquad (12.7-20)$$

and

$$\tilde{Q}_A(s) = R_u; \qquad A(s) = R_u^{-1}\tilde{Q}(s) \qquad (12.7-21)$$

or

$$\tilde{Q}_A(s) = \tilde{Q}(s)R_v; \qquad A(s) = R_v^{-1} \qquad (12.7-22)$$

depending on whether (12.7–11) or (12.7–12) is chosen. Then in Fig. 12.7-1B use $G^{u,ill}$ instead of G^u, where

$$G^{u,ill} = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} & 0 \\ G^u_{21} & G^u_{22} & G^u_{23} & 0 \\ 0 & 0 & 0 & A \\ G^u_{31} & G^u_{32} & G^u_{33} & 0 \end{pmatrix} \qquad (12.7-23)$$

and G^u is obtained from G (12.7–13) with $\tilde{Q}$ replaced by $\tilde{Q}_A$.

12.7.3 Robust Control: H_∞ Performance Objective

We can write

$$F \triangleq F(s; \Lambda) \qquad (12.7-24)$$

where Λ is an array of filter parameters. Then the design problem (12.5-8) has been converted into a nonlinear program with the constraint that the filter F must be stable. This is easily accomplished by expressing the filter denominator as a product of polynomials of degree 2 whose coefficients must be positive. If an odd degree is desired, an additional first-order term with positive coefficients can be included. The positivity constraint can be handled by writing the coefficients as squares of the independent variables Λ. The design objective (12.5–8) becomes

Objective O4:

$$\min_{\Lambda} \sup_{\omega} \mu_\Delta(G^F(P, \tilde{Q})) \qquad (12.7-25)$$

One should note that the objective function is not convex. Good initial guesses for the filter parameters (elements of Λ) can usually be obtained by matching them with the frequencies where the peaks of $\mu_\Delta(G^F)$ appear for $F = I$.

The SSV μ in Obj. O4 is computed by minimizing its upper bound (11.2–15). In the computation of the supremum in Obj. O4 only a finite number of frequencies is considered. Hence Obj. O4 is transformed into

Objective O4':

$$\min_{\Lambda} \max_{\omega \in \Omega} \inf_{D \in \mathcal{D}} \bar{\sigma}(DG^F D^{-1})$$

where Ω is a finite set of frequencies. Define

$$\Phi_\infty(\Lambda) \triangleq \max_{\omega \in \Omega} \inf_{D \in \mathcal{D}} \bar{\sigma}(DG^F D^{-1}) \qquad (12.7-26)$$

The gradient of Φ_∞ with respect to Λ can be computed analytically except when two or more of the largest singular values of $DG^F D^{-1}$ coincide. This is quite uncommon, however, and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega \in \Omega}$ is attained at more than one frequency, but again the use of a mean direction seems to be sufficient. We shall now proceed to obtain the expression for the gradient of $\Phi_\infty(\Lambda)$ in the general case.

Assume that for the value of Λ where the gradient of $\Phi_\infty(\Lambda)$ is computed, the $\max_{\omega \in \Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D \in \mathcal{D}} \bar{\sigma}(DG^F(i\omega_0)D^{-1})$ is obtained at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition be

$$D_0 G^F(i\omega_0) D_0^{-1} = (\mathrm{u}_1 \quad \mathrm{U}) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} \mathrm{v}_1^H \\ \mathrm{V}^H \end{pmatrix} \qquad (12.7-27)$$

Then for the element of the gradient vector corresponding to the filter parameter λ_k we have under the above assumptions:

$$\frac{\partial}{\partial \lambda_k} \Phi_\infty = \frac{\partial}{\partial \lambda_k} \sigma_1(D_0 G^F(i\omega_0) D_0^{-1}) \qquad (12.7-28)$$

because $\nabla_{D_0}(\sigma_1) = 0$ since we are at an optimum with respect to the D's. To simplify the notation use

$$A \triangleq D_0 G^F(i\omega_0) D_0^{-1} = \mathrm{U}_A \Sigma_A \mathrm{V}_A^H \qquad (12.7-29)$$

Then the gradient can be computed as follows:

$$AA^H = \mathrm{U}_A \Sigma_A^2 \mathrm{U}_A^H$$

$$\Rightarrow \frac{\partial}{\partial \lambda_k}(AA^H) = \frac{\partial}{\partial \lambda_k}(\mathrm{U}_A)\Sigma_A^2 \mathrm{U}_A^H + \mathrm{U}_A \frac{\partial}{\partial \lambda_k}(\Sigma_A^2)\mathrm{U}_A^H + \mathrm{U}_A \Sigma_A^2 \frac{\partial}{\partial \lambda_k}(\mathrm{U}_A^H)$$

Multiply this expression by u_1^H on the left and u_1 on the right. Consider the first term on the RHS

$$\mathrm{u}_1^H \frac{\partial}{\partial \lambda_k}(\mathrm{U}_A)\Sigma_A^2 \mathrm{U}_A^H \mathrm{u}_1 = \mathrm{u}_1^H \frac{\partial}{\partial \lambda_k}(\mathrm{U}_A)(\sigma_1^2 \quad 0 \quad \ldots \quad 0)^T = \sigma_1^2 \mathrm{u}_1^H \frac{\partial}{\partial \lambda_k}\mathrm{u}_1 = 0$$

because u_1 is a vector on the unit sphere whose gradient is orthogonal to u_1. Thus both the first and the last term on the RHS vanish and we find

$$\mathrm{u}_1^H \frac{\partial}{\partial \lambda_k}(AA^H)\mathrm{u}_1 = \mathrm{u}_1^H\Big(\frac{\partial}{\partial \lambda_k}(A)A^H + A\frac{\partial}{\partial \lambda_k}(A^H)\Big)\mathrm{u}_1 = \mathrm{u}_1^H \mathrm{U}_A\Big(2\Sigma_A \frac{\partial}{\partial \lambda_k}(\Sigma_A)\Big)\mathrm{U}_A^H \mathrm{u}_1$$

$$\Rightarrow \mathrm{u}_1^H \frac{\partial}{\partial \lambda_k}(A)\mathrm{v}_1\sigma_1 + \sigma_1 \mathrm{v}_1^H \frac{\partial}{\partial \lambda_k}(A^H)\mathrm{u}_1 = 2\sigma_1 \frac{\partial}{\partial \lambda_k}(\sigma_1)$$

$$\Rightarrow \frac{\partial}{\partial \lambda_k}(\sigma_1) = \mathrm{Re}\Big[\mathrm{u}_1^H\Big(\frac{\partial}{\partial \lambda_k}(D_0 G^F(i\omega_0)D_0^{-1})\Big)\mathrm{v}_1\Big] \qquad (12.7-30)$$

From (12.7–19) we get

$$\frac{\partial}{\partial \lambda_k}(D_0 G^F(i\omega_0)D_0^{-1}) = D_0 \begin{pmatrix} G_{13}^u \\ W_2 G_{23}^u \end{pmatrix} \frac{\partial}{\partial \lambda_k}\left[(I - FG_{33}^u)^{-1}F\right](G_{31}^u \qquad G_{32}^u W_1)D_0^{-1} \qquad (12.7-31)$$

We use the identity

$$\frac{d}{dz}(M(z)^{-1}) = -M(z)^{-1}\frac{d}{dz}(M(z))M(z)^{-1} \qquad (12.7-32)$$

in

$$\frac{\partial}{\partial \lambda_k}((I - FG^u_{33})^{-1}F) = \Big(\frac{\partial}{\partial \lambda_k}(I - FG^u_{33})^{-1}\Big)F + (I - FG^u_{33})^{-1}\frac{\partial F}{\partial \lambda_k} \quad (12.7-33)$$

to obtain

$$\frac{\partial}{\partial \lambda_k}((I - FG^u_{33})^{-1}F) = -(I - FG^u_{33})^{-1}\Big(\frac{\partial}{\partial \lambda_k}(I - FG^u_{33})\Big)(I - FG^u_{33})^{-1}F + (I - FG^u_{33})^{-1}\frac{\partial F}{\partial \lambda_k}$$

$$= (I - FG^u_{33})^{-1} \cdot \frac{\partial F}{\partial \lambda_k} \cdot (G^u_{33}(I - FG^u_{33})^{-1}F + I)$$

$$= (I - FG^u_{33})^{-1} \cdot \frac{\partial F}{\partial \lambda_k} \cdot (I - G^u_{33}F)^{-1} \quad (12.7-34)$$

Equation (12.7–28) can now be expressed in terms of (12.7–30), (12.7–31) and (12.7–34):

$$\frac{\partial}{\partial \lambda_k}\Phi_\infty = \mathrm{Re}\Big[\mathrm{u}_1^H D_0 \begin{pmatrix} G^u_{13} \\ W_2 G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1}\frac{\partial}{\partial \lambda_k}(F(i\omega_0))$$

$$(I - G^u_{33}F)^{-1}(G^u_{31} \quad G^u_{32}W_1)D_0^{-1}\mathrm{v}_1\Big] \quad (12.7-35)$$

where F, G^u_{ij}, W_1, and W_2 are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of Λ) depend on the particular filter form selected by the designer and can be computed easily.

12.7.4 Robust Control: H_2-Type Performance Objective

In the previous section we outlined how to design the filter such that in the presence of model uncertainty the sensitivity operator remains "close" to its nominal value. If the performance for a *particular* external input v is of primary interest then it will be more appropriate to minimize the ISE for this particular input in the presence of model uncertainty — i.e., to minimize

$$\max_{P \in \Pi} \|W_2 E v\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \beta_0^2 \quad d\omega \quad (11.3-32)$$

Hence the filter parameters are obtained by solving

Objective O5:

$$\min_{\Lambda} \|\beta_0\|_2$$

Note that in Sec. 12.5 we defined H_2-type objective functions not only for a single input v but also for finite sets of inputs. For the robust performance case here we have to restrict our problem definition to a single input.

Objective O5 defines a nonconvex nonlinear program. The solution is simplified by the fact that the gradient $\partial\beta_0/\partial\lambda_k$ can be computed explicitly as we will show next.

Define G^β from G^F (12.7-19) as

$$G^\beta = \begin{pmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{pmatrix} G^F \qquad (12.7-36)$$

The function β_0 is defined through [see (11.3–28), (11.3–29)]

$$\mu(G^\beta(i\omega)) = 1 \quad \Leftrightarrow \quad \beta(\omega) = \beta_0(\omega) \qquad (12.7-37)$$

First, β_0 has to be computed at a finite set Ω of frequencies. Theorem 11.3-2 implies that any basic descent method should be sufficient. To obtain the gradient of $\|\beta_0\|_2$ with respect to the filter parameters, we need to compute the gradient of $\beta_0(\omega)$ with respect to these parameters for every frequency $\omega \in \Omega$. From the definition of β_0 in (12.7-37) we see that as some filter parameter λ_k changes, $\beta_0(\omega)$ must also change so that $\mu(G^{\beta_0}(i\omega))$ remains constantly equal to 1. Hence we find

$$\frac{\partial\mu}{\partial\beta_0}\frac{\partial\beta_0}{\partial\lambda_k} + \frac{\partial\mu}{\partial\lambda_k} = 0 \quad \Rightarrow \quad \frac{\partial\beta_0}{\partial\lambda_k} = -\frac{\partial\mu}{\partial\lambda_k} / \frac{\partial\mu}{\partial\beta_0} \qquad (12.7-38)$$

where μ is computed by minimizing its upper bound $\inf_{D\in\mathcal{D}} \bar{\sigma}(DG^\beta D^{-1})$ [see (11.2–15)]. We assume again that the two largest singular values of $DG^\beta D^{-1}$ for the optimal D's at the value of β where the gradient is computed, are not equal to each other. If this is not the case a mean direction can be used as mentioned above.

The numerator $\partial\mu/\partial\lambda_k$ of (12.7–38) can be evaluated in the same way as (12.7–28) and (12.7–35) but with G^β instead of G^F:

$$\frac{\partial}{\partial\lambda_k}(\mu(G^\beta(i\omega)) = \mathrm{Re}\Big[\mathrm{u}_1^H D_0 \begin{pmatrix} G_{13}^u \\ \beta^{-1}W_2 G_{23}^u \end{pmatrix} (I - FG_{33}^u)^{-1}$$

$$\frac{\partial F}{\partial\lambda_k}(I - G_{33}^u F)^{-1}(G_{31}^u \quad G_{32}^u W_1) D_0^{-1}\mathrm{v}_1\Big] \qquad (12.7-39)$$

where

$$W_1 = (v \qquad 0) \qquad (12.7-40)$$

For the computation of the denominator $\partial\mu/\partial\beta_0$ let the $\inf_{D\in\mathcal{D}}\bar{\sigma}(DG^{\beta}(i\omega)D^{-1})$ be attained for $D_0 = D_0(\omega;\beta)$ and let σ_1 be the maximum singular value and u_1, v_1 the corresponding singular vectors of $D_0G^{\beta}D_0^{-1}$. Then the same steps as for obtaining (12.7–30) are valid and we find after some algebra

$$\frac{\partial}{\partial\beta}(\mu(G^{\beta}(i\omega)) = -\frac{1}{\beta^2}\mathrm{Re}\left[u_1^H D_0 \begin{pmatrix} 0 & 0 \\ G_{21}^{\beta} & G_{22}^{\beta} \end{pmatrix} D_0^{-1} v_1\right] \qquad (12.7-41)$$

Substitution of (12.7–39) and (12.7–41) into (12.7–38) yields the desired gradient.

12.8 Application: High-Purity Distillation

Consider again the distillation column described in the Appendix where the overhead composition is to be controlled at $y_D = 0.99$ and the bottom composition at $x_B = 0.01$ using the reflux L and boilup V as manipulated inputs. Approximating the dynamics by a first-order system we find the linear model

$$\tilde{P}(s) = \frac{1}{75s+1}\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \qquad (12.8-1)$$

Problems arise from the fact that high purity distillation columns tend to be ill-conditioned. For (12.8–1) the condition number is 142. Hence any controller Q based on the inverse of $\tilde{P}$ will also be ill conditioned and this might result in a control system which is not robust. For simplicity we will use the (conservative) uncertainty description (11.1–22)

$$\bar{\ell}_I(s) = 0.2\frac{5s+1}{0.5s+1} \qquad (12.8-2)$$

The performance weights

$$W_2^{-1}(s) = \frac{20s}{10s+1}I \qquad (12.8-3)$$

$$W_1 = I \qquad (12.8-4)$$

require a closed loop time constant of about 20 and restrict the maximum peak of the sensitivity operator to less than two.

The plant $\tilde{P}$ is MP and therefore

$$\tilde{Q}(s) = \tilde{P}(s)^{-1} \qquad (12.8-5)$$

First a diagonal filter structure is chosen. A one-parameter search based on the analytic gradient expression derived in Sec. 12.7.3 yields

$$F(s) = \frac{1}{7.28s + 1} I \qquad (12.8-6)$$

Plots of μ for robust stability and robust performance are shown in Fig. 12.8-1A. Clearly, although the system is guaranteed to remain stable in the presence of modeling error within the bound defined by (12.8–2), the performance is expected to deteriorate. This is confirmed by the simulations shown in Fig. 12.8-2. For the nominal case ($P = \tilde{P}$), the outputs are decoupled and the performance is acceptable (Fig. 12.8-2A). However in the case where

$$P(s) = \tilde{P}(s) \begin{pmatrix} 1.2 & 0 \\ 0 & 0.8 \end{pmatrix} \qquad (12.8-7)$$

the performance deteriorates to the point where it is totally unacceptable (Fig. 12.8-2). Note that the plant in (12.8-7) includes a 20% error in each plant input and is within the bound (12.8–2). The same plant is used in all other simulations in this section when model-plant mismatch is assumed. Higher order filters with different elements were also tried but were found not to improve the performance substantially. The reason is that, in general, a diagonal filter cannot affect the condition number of $\tilde{Q}$ significantly.

We shall proceed with the filter structure suggested in Sec. 12.7.1 for ill-conditioned systems. For our example $\omega^* = 0$ and therefore the diagonalizations (12.7–9) and (12.7–10) are exact. Hence (12.7–11) and (12.7–12) yield the same Q. Objective O4' was solved with a gradient search method using the analytic gradient expression of Sec. 12.7.3. Different filter orders were tested and a few different initial guesses were tried to avoid local minima. The final result for filters with two parameters in each element was:

$$F_1(s) = \begin{pmatrix} \frac{0.244s+1}{(0.184s+1)^2} & 0 \\ 0 & \frac{0.00284s+1}{(8.72s+1)^2} \end{pmatrix} \qquad (12.8-8)$$

$$F_2(s) = \begin{pmatrix} \frac{0.164s+1}{(0.446s+1)^2} & 0 \\ 0 & \frac{0.213s+1}{(0.476s+1)^2} \end{pmatrix} \qquad (12.8-9)$$

The values of μ for robust stability and performance are shown in Fig. 12.8-1B. The clear improvement over the diagonal filter is verified by the simulations in Fig. 12.8-3. It is interesting to note that settling times for the nominal case are similar as with the diagonal filter (Fig. 12.8-2 and 3). Hence, as expected, the robustness improvement was not the result of additional detuning, but of the two-filter structure which allowed us to affect the singular values of $\tilde{Q}$ directly and reduce its condition number in the critical frequency range.

Finally, a comparison will be made between the performance obtained by the two-filter IMC controller and the "true" μ-optimum controller, defined as the

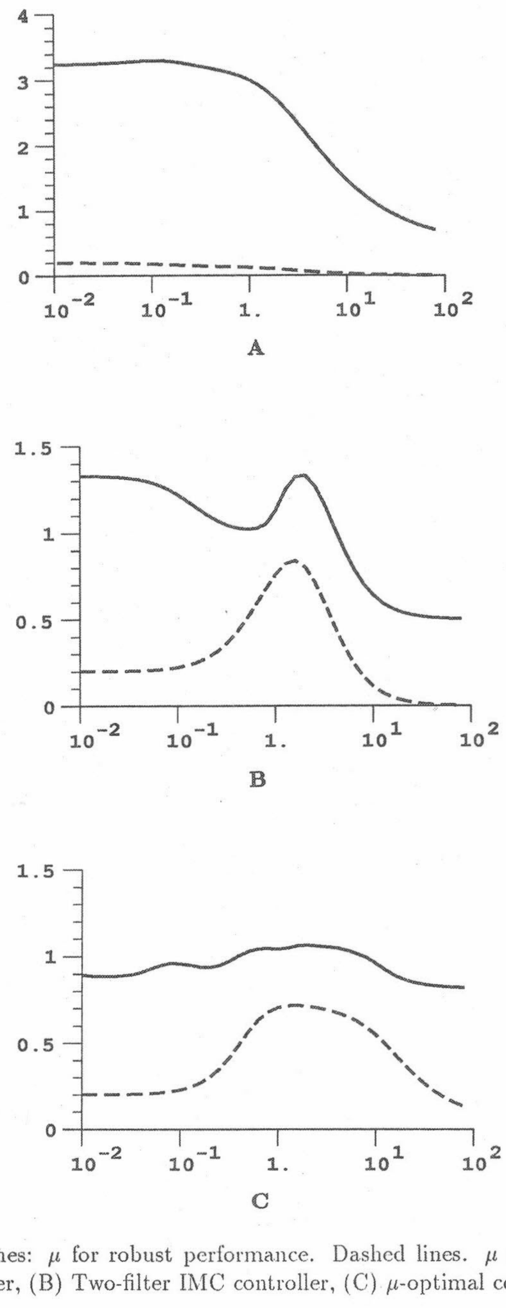

Figure 12.8-1. Solid lines: μ for robust performance. Dashed lines. μ for robust stability. (A) One-filter IMC controller, (B) Two-filter IMC controller, (C) μ-optimal controller.

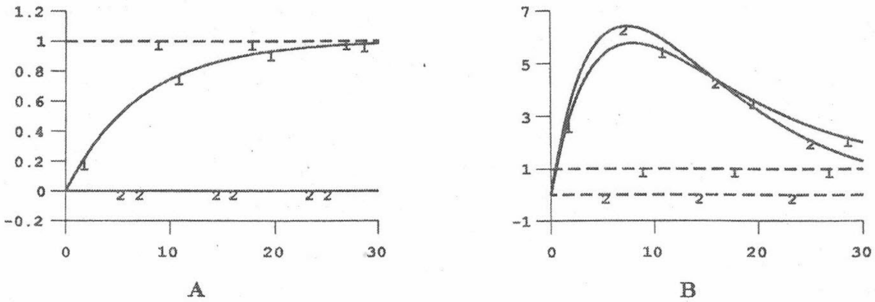

Figure 12.8-2. Time response for the one-filter IMC controller for unit step setpoint change for distillate (output 1). Dashed line: Setpoint; Solid lines: Outputs. (A) $P = \tilde{P}$, (B) $P \neq \tilde{P}$.

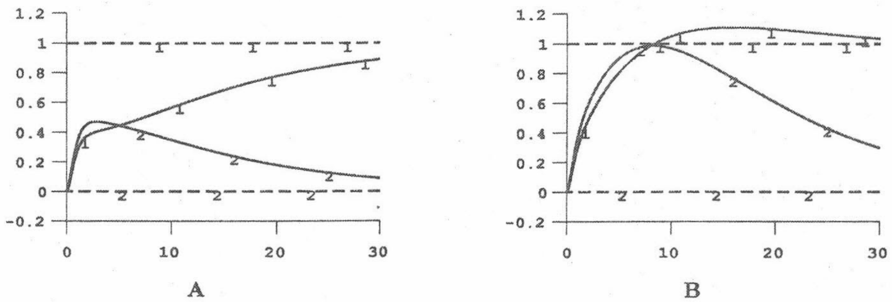

Figure 12.8-3. Time response for the two-filter IMC controller for unit step setpoint change for distillate (output 1). Dashed line: Setpoint; Solid lines: Outputs. (A) $P = \tilde{P}$, (B) $P \neq \tilde{P}$.

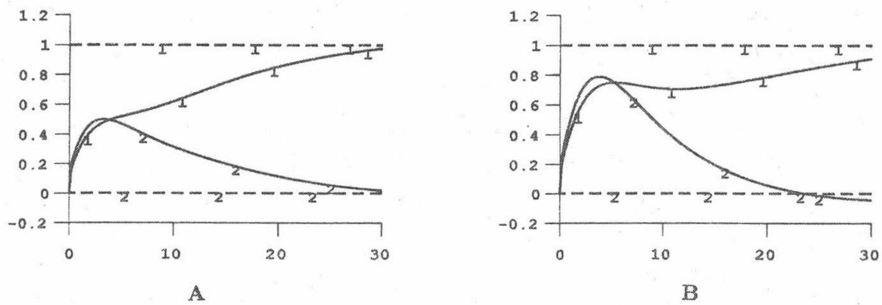

Figure 12.8-4. Time response for the μ-optimal controller for unit step setpoint change for distillate (output 1). Dashed line: Setpoint; Solid lines: Outputs. (A) $P = \tilde{P}$, (B) $P \neq \tilde{P}$.

result of minimizing $\sup_\omega \mu_\Delta(G^F)$ over *all* stabilizing controllers Q (or C). However, the iterative approach presently available for solving this problem is not guaranteed to converge and indeed, in our experience it has often failed to converge. For this particular example though, we were able to obtain a μ-optimal controller. The values of μ for robust performance and stability are shown in Fig. 12.8-1C. Clearly the difference is not significant and this is verified by the simulations shown in Fig. 12.8-4.

12.9 Summary

For internal stability of the IMC structure (12.1–1) both the plant P and the IMC controller Q have to be stable. For open-loop unstable plants it is convenient to *design* the IMC controller Q but for *implementation* the classic feedback structure must be chosen. Under some mild assumptions about pole-zero cancellation (Sec. 12.3) *all* stabilizing controllers Q for the plant P are parametrized by

$$Q_1(s) = Q_0(s) + Q_1(s) \qquad (12.3-1)$$

where $Q_0(s)$ is an arbitrary stabilizing controller for P and Q_1 is any stable transfer matrix such that PQ_1P is stable.

The IMC design procedure consists of two steps. In the *first step*, $\tilde{Q}$ is selected to yield a good system response for the inputs of interest, without regard for constraints and model uncertainty. As one option we propose to define a set of n inputs

$$\mathcal{V} = \left\{v^i(s) : \quad i = 1, \dots n\right\} \qquad (12.5-10)$$

and to minimize the *sum* of the ISE's that each of the inputs v^i would cause when applied to the system separately:

Objective O3:

$$\min_{\tilde{Q}} \left[\Phi(v^1) + \Phi(v^2) + \dots + \Phi(v^n)\right]$$

where

$$\Phi(v^i) = \|We^i\|_2^2 = \|W\tilde{E}v^i\|_2^2 = \|W(I - \tilde{P}\tilde{Q})v^i\|_2^2 \qquad (12.5-9)$$

The unique controller $\tilde{Q}$ which meets Obj. O3 is given by

$$\tilde{Q} = P_M^{-1} W^{-1} \left\{W P_A^{-1} V_M\right\}_* V_M^{-1} \qquad (12.6-15)$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, all terms involving the poles of P_A^{-1} are omitted. The factorization of the plant

$$P = P_A P_M \qquad (12.6-3)$$

into an allpass portion P_A and an MP portion P_M can be accomplished via inner-outer factorization (Sec. 12.6.4). The input matrix $V = (v^1 \quad v^2 \quad \ldots \quad v^n)$ can be factored similarly

$$V = V_M V_A \qquad (12.6-14)$$

In some special cases, $\tilde{Q}$ defined by (12.6–15) is also H_2-optimal for *each* one of the inputs v^i separately and their linear combinations.

In the *second step* of the design procedure, the controller $\tilde{Q}$ is detuned through a lowpass filter F such that for the detuned controller $Q = Q(\tilde{Q}, F)$ the robust performance condition

$$\mu_\Delta(G^F(\tilde{P}, Q)) < 1 \qquad \forall\omega, \qquad \Delta = \text{diag}\{\Delta_u, \Delta_p\} \qquad (12.5-7)$$

is satisfied. A nonlinear program was formulated (Sec. 12.7.3) to minimize $\mu_\Delta(G^F(\tilde{P}, Q))$ as a function of the filter parameters for a filter with a fixed diagonal structure. For unstable plants the filter has to be identity at all the unstable plant poles. For ill conditioned plants two filters can be necessary to meet the requirement (12.5–7).

12.10 Discussion and References

12.2. The idea of a "balanced realization" as a framework for model reduction is described by Moore (1981).

12.3. A general stable parametrization of all stabilizing controllers for a particular plant was developed by Youla, Jabr & Bongiorno (1976). It involves matrix coprime factors.

12.6, 12.7. The MIMO procedure for obtaining the controller for optimal nominal performance is patterned after the SISO case. For SISO systems the tradeoff between robustness and performance is straightforward because there are very few degrees of freedom. It makes sense to start from (almost) "perfect control" and to detune for robustness. Usually, the robust performance obtained in this manner is only marginally inferior to what can be accomplished via an integrated procedure which optimizes robust performance directly.

For MIMO systems this is not true anymore. There are tradeoffs between the different outputs and between the different inputs. One or even two diagonal filters might not offer enough degrees of freedom to reach the desirable performance-robustness compromise when the starting point is the (improper) H_2-optimal controller as proposed in Sec. 2.6. Short of going to the integrated procedure of optimizing robust performance directly, there are two options: a more complex filter or a different design for nominal performance. Because of the nonconvex nonlinear programming problem (Sec. 12.7.3 and 4) necessary to determine the filter parameters, the first option is likely to cause many difficulties. It appears more promising to investigate alternate nominal-performance designs. One possibility is to include a penalty term for the control action in the H_2-objective. The linear quadratic optimal controller minimizing the modified objective can be found by solving two Riccati equations (Kwakernaak & Sivan, 1972). The controller is always proper and should be better conditioned because of the control action penalty. The other possibility is to postulate $\tilde{H}(= \tilde{P}\tilde{Q})$ and to determine $\tilde{Q}$ from $\tilde{H}$. The advantage is that the ***structure*** of $\tilde{H}$ (decoupled, one-way decoupled, etc.) can be postulated directly by the designer. The limitations on $\tilde{H}$ imposed by NMP elements in $\tilde{P}$ have been explored by Holt & Morari (1985a,b), Zafiriou & Morari (1987) and Morari, Zafiriou & Holt (1987).

12.6-1-12.6-3. These sections follow the development in Chap. V of Zafiriou (1987).

12.6.4. Theorem 12.6-4 is Thm. 4 of Chu (1985, p. 40), applied to square systems and with some notational changes. Methods for solving ARE's can be found in Laub (1979) and Molinari (1973). Details on the QR factorization can be found in many books — e.g., Sec. 5.7.2 of Dahlquist and Björck (1974).

12.7.1. An algorithm for pseudo-diagonalization is described by Rosenbrock (1974).

12.7.3, 12.7.4. These sections follow the discussion in Zafiriou and Morari (1986b).

12.8. For computing the μ-optimal controller the procedure proposed by Doyle (1985) was used.

Chapter 13

PERFORMANCE LIMITATIONS FOR MIMO SYSTEMS

We know from practical experience that for a SISO system with a small gain, a long deadtime or severe "inverse response" it is impossible to achieve good closed-loop performance. Our thereotical analysis in Sec. 3.3.4 corroborated these observations. NMP characteristics are detrimental to closed-loop performance regardless of the employed controllers, in particular long delays and RHP zeros close to the origin should be avoided. Also, when the plant gain is small, only small disturbances can be controlled effectively without saturation of the manipulated variable.

Unlike SISO systems, MIMO systems display "directionality" which makes the assessment of their inherent performance limitations much more difficult. There is no single "gain" but the MIMO "gain" depends on the *direction* of the input vector. This is most apparent from the singular value decomposition which was discussed in Sec. 10.1.5. Whether a MIMO RHP zero leads to poor closed-loop performance or not, depends not only on its location in the RHP but also on its "structure." If it affects primarily a process output which is of minor importance, its presence might be irrelevant, even when it is located close to the origin. Finally, in Chap. 11 we pointed out that model uncertainty is a very critical component in the design of MIMO controllers. SISO systems do not display a similar sensitivity to model uncertainty.

The objective of this chapter is to develop a quantitative understanding of the factors which limit the achievable closed loop performance and to demonstrate how they can be used to screen alternate choices of "control structures" — i.e., manipulated variables.

13.1 Effect of Plant Gain

We will use the process description

$$y = Pu + P_d d' \tag{13.1-1}$$

as suggested by Fig. 10.2-1. P_d is the disturbance model expressing the relationship between the physical disturbances d_i' and their effects d_i on the output. For a distillation column the components of $d' = (d_1', \ldots, d_i', \ldots, d_n')^T$ may correspond to disturbances in feed rate, feed composition, boilup rate, etc. The column vector p_{di} of P_d represents the disturbance model for the disturbance d_i'. The effect of a particular disturbance d_i' on the process output is d_i,

$$d_i = p_{di} d_i' \tag{13.1-2}$$

The direction of the vector d_i will be referred to as the *direction of the disturbance* i. The overall effect of all disturbances d_i' on the output is d,

$$d = \sum_i d_i = \sum_i p_{di} d_i' = P_d d' \tag{13.1-3}$$

In most cases we will consider the effect of one particular disturbance d_i'. To simplify notation we will usually drop the subscript i, and $d = p_d d'$ will denote the effect of this *single* disturbance $d_i' = d'$ on the outputs. We will be referring to d as a "disturbance," although, in general, it represents the *effect* of the physical disturbance. The "disturbance" d can also represent a setpoint change $(-r)$ as indicated in Fig. 10.2-1.

We assume that the disturbance model and the process model have been scaled such that at steady state $-1 \leq d_i' \leq 1$ corresponds to the expected range of each disturbance and $-1 \leq u_j \leq 1$ corresponds to the acceptable range for each manipulated variable. For process control $u_j = -1$ may correspond to a closed valve and $u_j = 1$ to a fully open valve.

13.1.1 Constraints on Manipulated Variable

In this section we investigate the magnitude of the manipulated variables necessary to cancel the influence of a disturbance on the process output at steady state. It should be obvious that this magnitude is independent of the controller. This analysis will allow us to identify problems with *actuator* (e.g., valve) *constraints at steady state.* However, we should point out that the issue of constraints at steady state is not really a *control* problem, but rather a plant *design* problem. Any well designed plant should be able to reject disturbances at steady state.

For complete disturbance rejection ($y = 0$) at steady state

$$u = -P(0)^{-1}p_d(0)d' \qquad (13.1-4)$$

To avoid actuator saturation we must require

$$|u|_\infty \leq 1 \qquad \forall d' \ni |d'|_\infty \leq 1$$

or equivalently

$$|P(0)^{-1}p_d(0)|_\infty \leq 1 \qquad (13.1-5)$$

where $|\cdot|_\infty$ denotes the ∞-vector norm which was defined in Sec. 10.1. Whether (13.1–5) is violated and saturation causes problems depends both on the process P and the disturbance p_d. Even if $\|P(0)^{-1}\|_\infty$ is large (implying a small plant "gain"), $|P(0)^{-1}p_d(0)|_\infty$ can still be small if p_d is "aligned" with P^{-1} in a certain manner. This is the topic of the next section.

13.1.2 Disturbance Condition Number

When saturation is not an issue it is more reasonable to use the Euclidean (2-) norm as a measure of magnitude because it "sums up" the deviations of all manipulated variables rather than accounting for the maximum deviation only (like the ∞-norm). Consider a particular disturbance $d = p_d d'$. For complete rejection of this disturbance

$$u = -P^{-1}d \qquad (13.1-6)$$

The quantity

$$\frac{|u|_2}{|d|_2} = \frac{|P^{-1}d|_2}{|d|_2} \qquad (13.1-7)$$

depends only on the direction of the disturbance d but not on its magnitude. It measures the magnitude of u needed to reject a disturbance d of unit magnitude which enters in a particular direction expressed by $d/|d|_2$.

From the discussion of the SVD in Sec. 10.1.5 we note that the RHS of (13.1–7) is minimized for

$$d = \underline{v}(P^{-1}) = \bar{u}(P) \qquad (13.1-8)$$

For d defined by (13.1-8), (13.1-7) becomes

$$\frac{|u|_2}{|d|_2} = |P^{-1}\underline{v}(P^{-1})|_2 = \underline{\sigma}(P^{-1}) = \frac{1}{\bar{\sigma}(P)} \qquad (13.1-9)$$

Thus, the best disturbance direction requiring the *least* action by the manipulated variables, is that of the singular vector $\bar{u}(P)$ associated with the largest singular value of P.

By normalizing (13.1–7) with $|u|_2/|d|_2$ for the "best" disturbance defined by (13.1–8) we obtain the *disturbance condition number of the plant P*

$$\kappa_d(P) = \frac{|P^{-1}d|_2}{|d|_2}\bar{\sigma}(P)$$

$$= \frac{|P^{-1}p_d|_2}{|p_d|_2}\bar{\sigma}(P) \qquad (13.1-10)$$

It expresses the magnitude of the *manipulated variable* needed to reject a disturbance in the direction d relative to rejecting a disturbance with the same magnitude, but in the "best" direction $[\bar{u}(P)]$.

The "worst" disturbance direction is

$$d = \bar{v}(P^{-1}) = \underline{u}(P)$$

In this case we get

$$\kappa_d(P)_{max} = \bar{\sigma}(P^{-1})\bar{\sigma}(P) = \kappa(P)$$

and therefore for all disturbance directions

$$1 \leq \kappa_d(P) \leq \kappa(P) \qquad (13.1-11)$$

Thus, $\kappa_d(P)$ may be viewed as a generalization of the condition number $\kappa(P)$ of the plant, which also takes into account the direction of the disturbances. It measures how well the disturbance direction d is aligned with the direction of maximum effectiveness of the manipulated variables. A large value of $\kappa(P)$ indicates a large degree of directionality in the plant P. If this directionality is not compensated by the controller the closed-loop performance for different disturbance directions is vastly different as we will show next.

13.1.3 Implications of κ_d for Closed-Loop Performance

The objective of the control system is to minimize the effect of the disturbances on the outputs y. Consider a particular disturbance $d(s) = p_d(s)d'(s)$. The closed-loop relationship between this disturbance and the outputs is

$$y(s) = (I + P(s)C(s))^{-1}d(s) = E(s)d(s) \qquad (13.1-12)$$

Let $|y(i\omega)|_2$ denote the Euclidean norm of y evaluated at each frequency. The quantity

$$\alpha(\omega) = \frac{|Ed(i\omega)|_2}{|d(i\omega)|_2} \qquad (13.1-13)$$

depends only on the disturbance direction but not on its magnitude. $\alpha(\omega)$ measures the magnitude of the output vector $y(i\omega)$ resulting from a sinusoidal disturbance $d(i\omega)$ of unit magnitude and frequency ω.

The "best" disturbance direction causing the smallest output deviation is that of the right singular vector $\underline{v}(E)$ associated with the smallest singular value $\underline{\sigma}(E)$. By normalizing $\alpha(\omega)$ with this best disturbance we obtain the ***disturbance condition number of*** E^{-1}

$$\kappa_d(E^{-1}) = \frac{|Ed|_2}{|d|_2}\frac{1}{\underline{\sigma}(E)} = \frac{|Ed|_2}{|d|_2}\bar{\sigma}(E^{-1}) \qquad (13.1-14)$$

Again

$$1 \leq \kappa_d(E^{-1}) \leq \kappa(E^{-1}) = \kappa(E) \qquad (13.1-15)$$

At low frequencies, where the controller gain is high we have $E(i\omega) \cong (PC(i\omega))^{-1}$. In particular, this expression is exact at steady state ($\omega = 0$) when the controller includes integral action. Based on this approximation we derive the ***disturbance condition number of PC.***

$$\kappa_d(PC) = \frac{|(PC)^{-1}d|_2}{|d|_2}\bar{\sigma}(PC) \qquad (13.1-16)$$

As stated above this quantity has physical significance only when $\underline{\sigma}(PC) >> 1$. To avoid problems when this measure is evaluated at $\omega = 0$ write

$$C(s) = c(s)D(s) \qquad (13.1-17)$$

where $c(s)$ is a scalar transfer function which includes any integral action present in C. $D(s)$ may be viewed as a "decoupler." Then we have

$$\kappa_d(PC) = \frac{|(PD)^{-1}d|_2}{|d|_2}\bar{\sigma}(PD) \qquad (13.1-18)$$

For $D = I$ (i.e., the controller $c(s)I$) we find from (13.1–18) the disturbance condition number of P (13.1–10) derived previously. Thus $\kappa_d(P)$ can be interpreted in terms of closed loop performance as follows: If a scalar controller $C = c(s)I$ is chosen (which keeps the directionality of the plant unchanged), then $\kappa_d(P)$ measures the magnitude of the *output* y for a particular disturbance d, compared to the magnitude of the output if the disturbance were in the "best" direction (corresponding to the large plant gain). If $\kappa_d(P) = \kappa(P)$, the disturbance has all its components in the "bad" direction corresponding to low plant gain and low bandwidth. If $\kappa_d(P) = 1$, the disturbance has all its components in the "good" direction corresponding to high plant gain and high bandwidth.

Though a large value of $\kappa_d(P)$ does not necessarily imply bad performance, it usually does. In principle we could choose a compensator C which makes $\kappa_d(PC) = 1$ for all disturbances. However, such a controller can lead to serious robustness problems as we will show in Sec. 13.3.

13.1.4 Decomposition of d along Singular Vectors

The objective here is to gain insight into the type of dynamic response which is to be expected when disturbances along a particular direction affect a system with a high degree of directionality ($\kappa(E^{-1})$ is "large"). The singular vectors $v_j(E)$ form an orthonormal basis. The disturbance vector d can be represented in terms of this basis

$$d = \sum_{j=1}^{n} (v_j(E)^T \cdot d) v_j(E) \qquad (13.1-19)$$

Then the output y is described by

$$y(i\omega) = Ed(i\omega) = \sum_{j=1}^{n} E v_j(E)(v_j(E)^T \cdot d)(i\omega)$$

$$= \sum_{j=1}^{n} \sigma_j(E) u_j(E)(v_j(E)^T \cdot d) = \sum_{j=1}^{n} \sigma_j(E) d^j(i\omega) \qquad (13.1-20)$$

where we have defined the new "disturbance components"

$$d^j = (v_j(E)^T \cdot d) u_j(E) \qquad (13.1-21)$$

(13.1–20) shows that the response to a particular disturbance can be viewed as the sum of responses to the disturbances d^j passing through the scalar transfer function $\sigma_j(E)$. The magnitude of d^j depends on the alignment of the disturbance d with the singular vector $v_j(E)$. The characteristics of the (speed of) response to d^j depend on $\sigma_j(E)$.

For the controller $C = c(s)D(s)$ with integral action in $c(s)$ the approximation $E(i\omega) \cong c^{-1}(PD)^{-1}(i\omega)$ is valid for small ω. Then with $\ell = n - j + 1$ (13.1–20) becomes

$$y(i\omega) = \sum_{\ell=1}^{n} \frac{1}{c\sigma_\ell(PD)} \tilde{d}^\ell \qquad (13.1-22)$$

where

$$\tilde{d}^\ell = d^{n-\ell+1} = (u_\ell^T(PD) \cdot d) v_\ell(PD) \qquad (13.1-23)$$

The magnitude of $\tilde{d}^\ell$ is given by the component of d in the direction of the singular vector $u_\ell(PD)$ and $\tilde{d}^\ell$ affects the output along the direction of the singular vector $v_\ell(PD)$. If the loop transfer matrix PD has a high gain in this direction [i.e., $\sigma_\ell(PD)$ is large] then the control will be quick and good. If the gain is low the response will be slow and poor. If PD is ill-conditioned [$\kappa(PD)$ is large], the widely different response characteristics for different disturbance components will result in unusual overall system responses. This will be observed in the following example.

Example 13.1-1. LV-Distillation Column. Consider the distillation column described in the Appendix with L and (-V) as manipulated variables and the product compositions y_d and x_B as controlled outputs. The model is given by

$$P(s) = \frac{1}{75s+1}\begin{pmatrix} 0.878 & 0.864 \\ 1.082 & 1.096 \end{pmatrix} \tag{13.1-24}$$

We assume there are no problems with constraints. We want to study how well the system rejects various disturbances using a diagonal controller $C(s) = c(s) \cdot I$. Since we are only concerned about the outputs (y_D and x_B), the scaling does not matter provided the outputs are scaled such that an output of magnitude one is equally "bad" for both y_D and x_B. We have

$$\bar{\sigma}(P) = 1.972, \quad \underline{\sigma}(P) = 0.0139, \quad \kappa(P) = 141.7$$

Consider disturbances d' of unit magnitude in feed composition, x_F, feed flowrate, F, feed liquid fraction, q_F, and boilup rate, $-V_d$. The linearized steady state disturbance models are

$$d = P_d d' = \begin{pmatrix} 0.881 \\ 1.119 \end{pmatrix} x_F + \begin{pmatrix} 0.394 \\ 0.586 \end{pmatrix} F + \begin{pmatrix} 0.868 \\ 1.092 \end{pmatrix} q_F + \begin{pmatrix} 0.864 \\ 1.096 \end{pmatrix} (-V_d) \tag{13.1-25a}$$

Also consider setpoint changes in y_D and x_B of magnitude one. These are mathematically equivalent to disturbances with

$$d = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{13.1-25b}$$

The steady state values of the disturbance condition number, $\kappa_d(P)$, are given for these disturbances in Table 13.1-1. The disturbance condition number of E^{-1}, with the controller described below, is shown as a function of frequency in Fig. 13.1-1. From these data we see that disturbances in x_F, q_F and V are very well aligned with the plant, and there is little need for using a decoupler to change the direction of P. The feed flow disturbance is the "worst" disturbance, but even it has its dominant effect in the "good" direction.

A "decoupler" is desirable if we want to follow setpoint changes which have a large component in the "bad" direction corresponding to low plant gains. However, a decoupler is *not* recommended for this distillation column because of severe

Table 13.1.1. Disturbance condition numbers for distillation example ($\kappa(P) = 141.7$).

	Disturbance/Setpoint Change d'					
	x_F	F	q_F	$-V_d$	y_{Ds}	x_{Bs}
d	$\begin{pmatrix} 0.881 \\ 1.119 \end{pmatrix}$	$\begin{pmatrix} 0.394 \\ 0.586 \end{pmatrix}$	$\begin{pmatrix} 0.868 \\ 1.092 \end{pmatrix}$	$\begin{pmatrix} 0.864 \\ 1.096 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\kappa_d(P)$	1.48	11.75	1.09	1.41	110.7	88.5

robustness problems caused by uncertainty (see Sec. 13.3). Therefore we cannot expect good setpoint tracking for this LV-configuration. If setpoint changes are of little or no interest, the LV-configuration with a diagonal controller may be a good choice. The response to a feed-rate disturbance is then expected to be somewhat sluggish because of the high value of $\kappa_d(P)$.

We will now confirm the predictions based on the data in Table 13.1-1 by studying some time responses. We use a diagonal controller of the form $C(s) = c(s)I$ where $c(s)$ is the PI controller $c(s) = 0.1(75s + 1)s^{-1}$.

Time domain simulations are shown for "disturbances" in x_F and F and for setpoint changes in y_D in Figs. 13.1-2 to 13.1-4. We have simulated all responses as step setpoint changes of size d (13.1–25) to make comparisons easier. Dynamics have not been included in the "disturbances" for x_F and F (which is clearly unrealistic) to make the example simpler. The time responses confirm the predictions based on Table 13.1-1. The rather odd-looking response can be explained easily by decomposing the disturbances along the singular vector directions of the closed-loop system, as shown before. For each disturbance, the closed-loop frequency response at low frequencies can be approximated by $y(i\omega) \cong c^{-1}P^{-1}d(i\omega)$. By decomposing d along the "directions" of P as in (13.1-22) and (13.1-23) we may write this response as the sum of two SISO responses

$$y(i\omega) \cong \left[\frac{1}{c\bar{\sigma}(P)}\tilde{d}^1 + \frac{1}{c\underline{\sigma}(P)}\tilde{d}^2\right] \qquad (13.1-26)$$

where

$$\tilde{d}^1 = (\bar{u}^T \cdot d)\bar{v}(P) \qquad (13.1-27a)$$

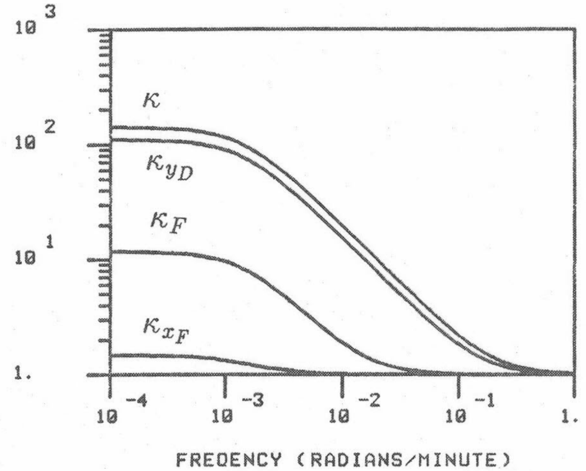

Figure 13.1-1. Disturbance condition number of E^{-1} for disturbances in feed rate F, feed composition x_F, and setpoint change in y_D. $C(s) = 0.1(75s + 1)s^{-1} \cdot I$. (Reprinted with permission from *Ind. Eng. Chem. Res.*, **26**, 2033 (1987), American Chemical Society.)

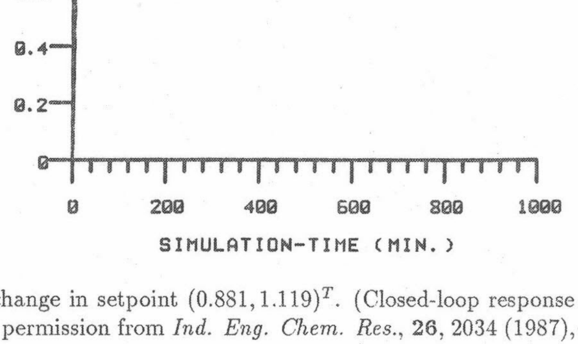

Figure 13.1-2. Step change in setpoint $(0.881, 1.119)^T$. (Closed-loop response to "disturbance" in x_F.) (Reprinted with permission from *Ind. Eng. Chem. Res.*, **26**, 2034 (1987), American Chemical Society.)

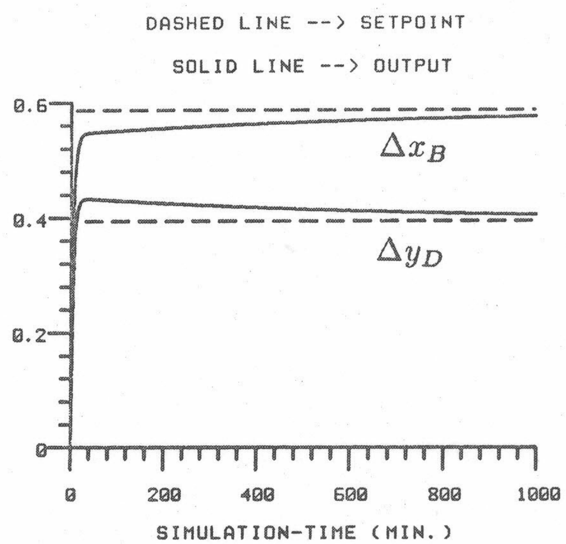

Figure 13.1-3. Step change in setpoint $(0.394, 0.586)^T$. (Closed-loop response to "disturbance" in F.) (Reprinted with permission from *Ind. Eng. Chem. Res.*, **26**, 2034 (1987), American Chemical Society.)

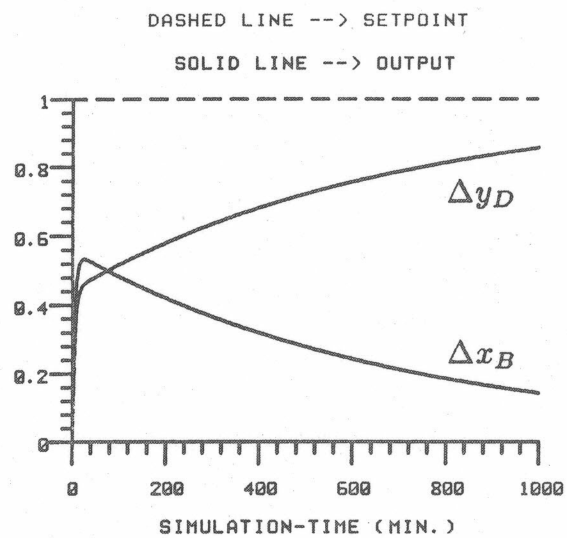

Figure 13.1-4. Step change in setpoint $(1,0)^T$. (Closed-loop response to setpoint change for y_D.) (Reprinted with permission from *Ind. Eng. Chem. Res.*, **26**, 2034 (1987), American Chemical Society.)

Table 13.1-2. $\tilde{d}^1$ and $\tilde{d}^2$ (13.1-27) for distillation example.

	Disturbance/Setpoint Change		
	x_F	F	y_{Ds}
d	$\begin{pmatrix} 0.881 \\ 1.119 \end{pmatrix}$	$\begin{pmatrix} 0.394 \\ 0.586 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\tilde{d}^1$	$\begin{pmatrix} 1.00 \\ 1.00 \end{pmatrix}$	$\begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix}$	$\begin{pmatrix} 0.44 \\ 0.44 \end{pmatrix}$
$\tilde{d}^2$	$\begin{pmatrix} -0.008 \\ 0.008 \end{pmatrix}$	$\begin{pmatrix} -0.04 \\ 0.04 \end{pmatrix}$	$\begin{pmatrix} 0.55 \\ -0.55 \end{pmatrix}$

$$\tilde{d}^2 = (\underline{u}^T \cdot d)\underline{v}(P) \qquad (13.1-27b)$$

Thus, each disturbance response is the sum of two responses: one fast in the direction $\tilde{d}^1$ and one slow in the direction $\tilde{d}^2$ (see Table 13.1-2). The singular value decomposition $P = U\Sigma V^H$ gives

$$\Sigma = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} = \begin{pmatrix} 1.972 & 0 \\ 0 & 0.01391 \end{pmatrix}$$

$$U = (\bar{u} \quad \underline{u}) = \begin{pmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{pmatrix}$$

$$V = (\bar{v} \quad \underline{v}) = \begin{pmatrix} 0.707 & -0.708 \\ 0.708 & 0.707 \end{pmatrix}$$

The decomposition expressed through (13.1–26) and (13.1–27) which holds for low frequencies, explains the actual responses very well: Initially there is a very fast response in the direction of $\bar{v}^T = (0.707, \quad -0.708)$. This response arises from the overall open-loop transfer function $c\bar{\sigma}(P) = 0.197/s$ corresponding to a first-order response with time constant $(0.1\bar{\sigma}(P))^{-1} = 5.1$ min. Added to this is a slow first-order response with time constant $(0.1\underline{\sigma}(P))^{-1} = 720$ min in the direction of $\underline{v}^T = (-0.708, 0.707)$.

Note that the slow disturbance component $\tilde{d}^2$ is the "error" at $t \approx 40$ min, because the fast response has almost settled at this time. As an example consider the disturbance in feed rate F (Fig. 13.1-3). At $t \approx 40$ *min* the deviation from the desired setpoint, $(0.394, 0.586)^T$, is approximately equal to $\tilde{d}^2 = (-0.04, 0.04)^T$. Similarly, for the setpoint change in y_D (Fig. 13.1-4) the deviation from the desired setpoint, $(1,0)^T$, at $t \approx 40$ min is approximately equal to $\tilde{d}^2 = (0.55, -0.55)^T$. □

13.1.5 Summary

The disturbance condition number of a matrix A with respect to a disturbance with direction d is defined as follows:

$$\kappa_d(A) = \frac{|A^{-1}d|_2}{|d|_2}\bar{\sigma}(A)$$

For non-square A (e.g., plants with more inputs than outputs) one should replace $|A^{-1}d|_2$ by $\{\min|m|_2 \text{ s.t. } Am = d\}$, that is — one should replace A^{-1} by $A^{\#} \triangleq A^T(AA^T)^{-1}$ (the pseudo-inverse of A).

We have introduced the disturbance condition number of the plant P (13.1–10), of E^{-1} (13.1–14) and of PC (13.1–16). The disturbance condition number $\kappa_d(E^{-1})$ measures the performance (error) for a disturbance in direction $d/|d|$ relative to the performance for a disturbance which enters in the "optimal" direction. For high gain feedback ($\underline{\sigma}(PC) >> 1$) $E^{-1} \cong PC$, which justifies the use of $\kappa_d(PC)$ and also $\kappa_d(P)$ (when the controller C is scalar) instead of $\kappa_d(E^{-1})$. The disturbance condition number is a measure of control *performance* and therefore *must* be scaling *dependent*: We define performance in terms of a weighted average of output deviations. Any measure of performance must depend on the relative importance of the outputs which is expressed through scaling factors. (On the other hand, the issue of *stability* is *independent* of scaling, and any measure used as a tool for evaluating a system's stability should be independent of scaling.)

If $\kappa_d(P)$ is large then for good performance in all directions a controller is needed which makes $\kappa_d(PC)$ small. As we will discuss in Sec. 13.3 robustness problems might prevent us from choosing such a controller. On the other hand, if $\kappa_d(P)$ is small then a *scalar* controller C can be expected to yield good performance. Thus we can use $\kappa_d(P)$ in two different ways:

1. Discriminating between process alternatives and selecting manipulated inputs. Plants with large values of $\kappa_d(P)$ are not necessarily bad, but if other factors are equal [e.g., RGA-values (Sec. 13.3), RHP-zeros (Sec. 13.2)], then we should prefer a design with a low value of $\kappa_d(P)$. This measure may therefore be used as *one* criterion for selecting manipulated variables and discriminating among alternatives.

2. Selecting controller structure (e.g., diagonal versus multivariable inverse-based controller). Guidance can be obtained from $\kappa_d(P)$ when it is used together with the RGA, as discussed in Sec. 13.3.

13.2 NMP Characteristics

In our discussions we stressed repeatedly that NMP characteristics limit the achievable closed loop performance. For example, the complementary sensitivity resulting from an H_2-optimal controller (Sec. 12.6.3) includes the factor P_A, the allpass incorporating all NMP characteristics of P. If $P_A = I$, perfect control is theoretically possible. Also, Thm. 4.1-4 shows that RHP zeros close to the origin are more detrimental to the performance than zeros far out in the RHP. In this section we want to capture quantitatively the "directional" effect of RHP zeros — i.e., which outputs are most and which are least affected by a particular zero. We will make the following three assumptions:

Assumption 13.2-1: *$P(s)$ is square ($n \times n$) and nonsingular except for isolated values of s.*

Assumption 13.2-2. *All RHP zeros $\zeta_1, \ldots, \zeta_m$ are of degree one — i.e., the rank of $P(\zeta_i)$ is $n-1$.*

Assumption 13.2-3. *No RHP poles are located at $\zeta_1, \ldots, \zeta_m$.*

Usually, these assumptions are not restrictive. They can be easily relaxed at the expense of a more involved notation.

13.2.1 Zero Direction

Definition 13.2-1: *Let ζ_i be a zero of $P(s)$. The vector z_i ($z_i \neq 0$) satisfying $z_i^T P(\zeta_i) = 0$ is called the direction of the zero ζ_i.*

Note that $P(\zeta_i)$ is of rank $n-1$ because the zero was assumed to be of degree one. The vector z_i is the eigenvector of $P(\zeta_i)^T$ associated with the eigenvalue zero. It is called zero direction because for any system input of the form $ke^{\zeta_i t}$, where k is an arbitrary complex vector, the output in the direction of z_i is identically equal to zero (given appropriate initial conditions).

Because the IMC controller Q has to be stable it cannot cancel the RHP zeros of P and they will appear unchanged in the complementary sensitivity $H = PQ$. Furthermore from Def. 13.2-1 we find the following relation for the sensitivity

$$z_i^T E(\zeta_i) = z_i^T (I - PQ(\zeta_i)) = z_i^T \qquad (13.2-1)$$

which implies that the magnitude of any disturbance component $z_i e^{\zeta_i t}$ entering along the zero direction z_i is passing through to the output y unaffected by feedback (given appropriate initial conditions). Thus RHP zeros affect both the achievable H and E.

RHP poles also impose limitations on the choice of Q as discussed in Sec. 12.2. With a two-degree-of-freedom structure, however, the achievable H is not constrained by the presence of RHP plant poles (Sec. 5.2.4).

We would like to characterize in a convenient manner all complementary sensitivity operators which can be achieved for a plant P. The following theorem is self-evident from the preceding discussion.

Theorem 13.2-1. *For a plant P the complementary sensitivity H is achievable by a two-degree-of-freedom controller (such that the closed-loop system is internally stable) if and only if there exists a stable Q such that $H = PQ$.*

A direct test for the existence of a stable Q is provided by the following theorem.

Theorem 13.2-2. *There exists a stable Q such that the complementary sensitivity is equal to a desired transfer matrix H if and only if H satisfies*

$$z_i^T H(\zeta_i) = 0 \qquad (13.2-2)$$

for all RHP zeros ζ_i of the plant $P(s)$ where z_i is the direction of the zero ζ_i ($z_i^T P(\zeta_i) = 0$).

Proof.

$\Rightarrow$ Assume there exists a stable Q such that $H = PQ$. Then $z_i^T H(\zeta_i) = z_i^T PQ(\zeta_i) = 0$.

$\Leftarrow$ Find the partial fraction expansion of P^{-1}.

$$P(s)^{-1} = \frac{1}{s - \zeta_i} R_i + P_\zeta(s) \qquad (13.2-3)$$

where R_i is the matrix of residues and $P_\zeta(s)$ is a remainder term with no poles at $s = \zeta_i$. Postmultiply both sides of (13.2–3) by $P(s)$

$$I = \frac{1}{s - \zeta_i} R_i P(s) + P_\zeta(s) P(s) \qquad (13.2-4)$$

Because the LHS of (13.2–4) is the identity the RHS cannot have a pole at $s = \zeta_i$. $P(s)P_\zeta(s)$ does not have a pole at $s = \zeta_i$ and therefore it must be that

$$R_i P(\zeta_i) = 0 \qquad (13.2-5)$$

Because ζ_i is of degree one, $P(\zeta_i)$ is of rank $(n-1)$. Hence R_i is of rank 1 and as a result of Def. 13.2-1, the rows of R_i are multiples of z_i^T. Therefore $z_i^T H(\zeta_i) = 0$ implies

$$R_i H(\zeta_i) = 0 \qquad (13.2-6)$$

Now postmultiply both sides of (13.2–3) by H.

$$Q = P^{-1}H = \frac{1}{s-\zeta_i} R_i H(s) + P_\zeta(s)H(s) \qquad (13.2-7)$$

$P_\zeta(s)H(s)$ does not have a pole at $s = \zeta_i$. Hence (13.2–6) implies that Q does not have a pole at $s = \zeta_i$. □

Example 13.2-1. Consider the plant P_1

$$P_1 = \frac{1}{s+1}\begin{pmatrix} 1 & 1 \\ 1+2s & 2 \end{pmatrix}$$

which has a zero at $s = \zeta = \frac{1}{2}$. The zero direction $z = (2, -1)$ satisfies

$$z^T P_1(\zeta) = z^T \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0$$

We will use condition (13.2–2) to construct decoupled and one-way decoupled H's for which stable Q's exist. Trivially for the decoupled plant

$$H = \begin{pmatrix} \frac{-2s+1}{2s+1} & 0 \\ 0 & \frac{-2s+1}{2s+1} \end{pmatrix}$$

$H(\zeta) = 0$ and therefore (13.2–2) is satisfied. Let us postulate

$$H^{LT} = \begin{pmatrix} 1 & 0 \\ x_1 & \frac{-2s+1}{2s+1} \end{pmatrix}$$

and

$$H^{UT} = \begin{pmatrix} \frac{-2s+1}{2s+1} & x_2 \\ 0 & 1 \end{pmatrix}$$

where x_1 and x_2 are to be determined. We find from (13.2–2)

$$2 - x_1(\zeta) = 0 \quad \text{or} \quad x_1(\zeta) = 2$$

$$2x_2(\zeta) - 1 = 0 \quad \text{or} \quad x_2(\zeta) = \frac{1}{2}$$

If we postulate x_1 and x_2 to be of the form $2\beta s/(2s+1)$ (with the constant β to be determined) so that there are no steady state interactions ($H(0) = I$) then we find

$$x_1(\zeta) = \frac{2\beta}{4} = 2$$

and

$$x_1(s) = \frac{8s}{2s+1}$$

Similarly

$$x_2(s) = \frac{2s}{2s+1}$$

□

13.2.2 Implications of Zero Direction for Achievable Performance

The vector z_i is constant and possibly complex. Condition (13.2–2) requires that each column of H, evaluated at the plant zero ζ_i, is orthogonal to z_i. For a particular element of the input vector v the elements of the output vector y cannot be selected independently but have to satisfy the linear "interpolation conditions" (13.2–2). The presence of RHP plant zeros requires some relationship between the elements of a column of H but the columns themselves can be selected independently of each other.

By assumption, ζ_i is a zero of $P(s)$ of degree one. Assume that H is selected to be diagonal – i.e., completely decoupled. Then according to the theorem the degree of ζ_i in H has to be at least equal to the number of nonzero entries in z_i. Usually this number is larger than one, generally equal to n. Thus, requiring a decoupled response generally leads to the introduction of RHP zeros not originally present in the plant $P(s)$. This is the price to be paid for decoupling.

The zero ζ_i is "pinned" to the outputs corresponding to nonzero entries in z_i: the zero has to affect at least one of these outputs and it cannot affect any of the outputs corresponding to zero entries in z_i. This is illustrated in the following example.

Example 13.2-2. The system

$$P_2(s) = \frac{1}{s+2}\begin{pmatrix} -s+1 & -s+1 \\ 1 & 2 \end{pmatrix}$$

has a zero at $\zeta = 1$ with the direction $z^T = (1, 0)$. Because $z_2 = 0$, it can only affect the first output of H — i.e., it is "pinned" to the first output. Pinned zeros are nongeneric and therefore somewhat of a mathematical artifact. In the preceding example an infinitesimal change in the polynomial coefficients of p_{11} or p_{12} removes the RHP zero altogether.

Corollary 13.2-1. *Assume that the k^{th} element z_{ik} of the zero direction z_i is nonzero. Then it is possible to obtain "perfect" control on all outputs $j \neq k$ with the remaining output exhibiting no steady state offset.*

This and the next result follow trivially from Thm. 13.2-2.

Corollary 13.2-2. *Assume that $P(s)$ has a single zero ζ and that the k^{th} element z_k of the zero direction z is nonzero. Then H can be chosen of the form*

$$H = \begin{pmatrix} 1 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 & 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \frac{\beta_1 s}{s+\zeta} & \frac{\beta_2 s}{s+\zeta} & \cdot & \frac{\beta_{k-1} s}{s+\zeta} & \frac{-s+\zeta}{s+\zeta} & \frac{\beta_{k+1} s}{s+\zeta} & \cdot & \frac{\beta_n s}{s+\zeta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 1 \end{pmatrix} \quad (13.2-8)$$

where

$$\beta_j = -\frac{2z_j}{z_k} \text{ for } j \neq k \quad (13.2-9)$$

The interaction terms will be insignificant if $z_k >> z_j$ $(\forall j \neq k)$ – i.e., when the zero is "aligned" predominantly with output k. If for some j, $z_j >> z_k$ then the zero is aligned predominantly with output j. It can be pushed to output k only at the cost of generating significant interactions (large β's).

As a demonstration of the alignment effect recall Ex. 13.2-1 with the zero aligned with the first output $(z^T = (2, -1))$. Pushing the effect of the zero to the second output (H^{LT}) leads to strong interactions $(\beta = 4)$, while aligning the zero with the first output (H^{UT}) is much more favorable $(\beta = 1)$. Thus, if one-way decoupling is contemplated the zero direction should be used as a guideline.

13.2.3 Summary

The concept of zero direction is a convenient tool to judge the feasibility of alternate forms of decouplers. In the generic case when all elements of the zero direction vector are nonzero all kinds of decouplers are feasible, in principal. However, if the zero is very close to the origin complete decoupling is not advisable because it introduces more RHP zeros at this same location into the complementary sensitivity. Also, though generically the effect of a zero can be "pushed" to any arbitrary output, this can cause large interactions and violent moves in the manipulated variables unless the zero direction is aligned with this output — i.e., it has a large component in the direction of this output.

13.3 Sensitivity to Model Uncertainty

We have learned that model uncertainty limits the achievable closed-loop performance. The SSV introduced in Chap. 11 can be used to measure the performance deterioration caused by uncertainty. The extent of the deterioration depends on the (nominal) system, the controller and the type (structure) of the uncertainty. It is desirable that the system be of such a kind that even with a simplistic controller the sensitivity of closed loop performance to model uncertainty is small. Then the modelling and controller design effort necessary to achieve good performance will be small.

We know from Sec. 11.3.2 that in the presence of multiplicative output uncertainty alone simple SISO-type design techniques suffice to achieve good performance. On the other hand multiplicative input uncertainty (Sec. 11.3.3) can cause serious performance problems if the plant condition number is high: with a low condition number controller poor nominal performance is expected; when the controller condition number is high, robust performance is generally bad. Thus, high condition number plants should be avoided by appropriate *process design* and in particular by the appropriate choice of actuators.

In Secs. 11.2.5 and 11.2.6 we studied "element-by-element" uncertainty. A large *minimized* condition number $\kappa^*(P)$ is an indication that closed-loop performance is sensitive to this type of uncertainty. We were able to correlate $\kappa^*(P)$ with the RGA $\Lambda(P)$ and concluded that the stability and performance of plants with large RGA elements is strongly affected by independent element uncertainty. Example 11.2-2 illustrated, however, that this type of uncertainty description is usually inappropriate because the transfer matrix elements do not vary independently. In the following we will show that large RGA elements are also an indication of sensitivity to *diagonal* multiplicative input uncertainty, which is a good model of actuator uncertainty. Actuator uncertainty is clearly always present to some extent and therefore plants with large RGA elements should generally be avoided.

13.3.1 Sensitivity to Diagonal Input Uncertainty

Let u denote the actual plant input and u_c the input computed by the controller. Let Δ_i represent the relative uncertainty on the i^{th} manipulated input. Then $u_i = u_{ci}(1+\Delta_i)$ or in vector form $u = u_c(I+\Delta_I)$ where $\Delta_I = \text{diag}\{\Delta_i\}$. Alternatively we may define the perturbed plant

$$P = \tilde{P}(I + \Delta_I), \qquad \Delta_I = \text{diag}\{\Delta_i\} \tag{13.3-1}$$

The loop gain matrix PC which is closely related to performance, may be written in terms of the nominal $\tilde{P}C$

$$PC = \tilde{P}C(I + C^{-1}\Delta_I C) \tag{13.3-2a}$$

$$= (I + \tilde{P}\Delta_I\tilde{P}^{-1})\tilde{P}C \tag{13.3-2b}$$

For SISO plants, a relative input error of magnitude Δ results in the same relative change in $PC = \tilde{P}C(1 + \Delta)$, but for multivariable plants the effect of the input uncertainty on PC may be amplified significantly as we will show.

For 2×2 plants the error term in (13.3–2a) may be expressed in terms of the RGA of the *controller* C as follows

$$C^{-1}\Delta_I C = \begin{pmatrix} \lambda_{11}(C)\Delta_1 + \lambda_{21}(C)\Delta_2 & \lambda_{11}(C)\frac{c_{12}}{c_{11}}(\Delta_1 - \Delta_2) \\ -\lambda_{11}(C)\frac{c_{21}}{c_{22}}(\Delta_1 - \Delta_2) & \lambda_{12}(C)\Delta_1 + \lambda_{22}(C)\Delta_2 \end{pmatrix} \tag{13.3-3}$$

where

$$\Lambda(C) = C \times (C^{-1})^T \tag{13.3-4}$$

For $n \times n$ plants the *diagonal* elements of the error matrix $C^{-1}\Delta_I C$ may be written as a straightforward generalization of the 2×2 case

$$(C^{-1}\Delta_I C)_{ii} = \sum_{j=1}^{n} \lambda_{ji}(C)\Delta_j \tag{13.3-5}$$

That is, the diagonal elements of $C^{-1}\Delta_I C$ depend on the elements of $\Lambda(C)$ in the same column and the magnitude of the uncertainty. Similarly, for 2×2 plants the error term in (13.3–2b) may be expressed in terms of the RGA of the *plant*

$$\tilde{P}\Delta_I\tilde{P}^{-1} = \begin{pmatrix} \lambda_{11}\Delta_1 + \lambda_{12}\Delta_2 & -\lambda_{11}\frac{\tilde{p}_{12}}{\tilde{p}_{22}}(\Delta_1 - \Delta_2) \\ \lambda_{11}\frac{\tilde{p}_{21}}{\tilde{p}_{11}}(\Delta_1 - \Delta_2) & \lambda_{21}\Delta_1 + \lambda_{22}\Delta_2 \end{pmatrix} \tag{13.3-6}$$

For $n \times n$ plants the diagonal elements in $(P\Delta_I P^{-1})$ depend on the elements of $\Lambda(P)$ in the same row:

$$(\tilde{P}\Delta_I\tilde{P}^{-1})_{ii} = \sum_{j=1}^{n} \lambda_{ij}(\tilde{P})\Delta_j \tag{13.3-7}$$

Controllers with large RGA-elements will lead to large elements in the matrix $C^{-1}\Delta_I C$, and plants with large RGA-elements will lead to large elements in the matrix $\tilde{P}\Delta_I\tilde{P}^{-1}$. Equations (13.3–2) seem to imply that either of these cases will lead to large elements in PC and therefore poor performance when there is input uncertainty ($\Delta_I \neq 0$). However, this interpretation is generally not correct since the "directionality" of PC may be such that the elements in PC remain small even though $C^{-1}\Delta_I C$ *or* $\tilde{P}\Delta_I\tilde{P}^{-1}$ have large elements. This should be clear from the following two extreme cases

1. Assume the controller has a small RGA. In this case the elements in the error term $C^{-1}\Delta_I C$ are similar to Δ_I in magnitude (13.3–2a and 13.3–5). Consequently, PC is not particularly influenced by input uncertainty, even though the plant itself may be strongly ill-conditioned with a large RGA.

2. Assume the plant has a small RGA. In this case PC is not particularly influenced by input uncertainty (13.3–2b and 13.3–6), even though the controller itself may have a large RGA. (From a practical point of view, one can argue that it is unlikely that anyone would design a controller with large RGA-elements for a plant with small RGA-elements.)

From items 1 and 2 we conclude that for a system to be sensitive to diagonal input uncertainty, *both* the controller and the plant must have large RGA-elements. This is consistent with Sec. 11.3.3 where we found that for block input uncertainty to cause robust performance problems the condition numbers of *both* the plant and the controller have to be large. Thus the RGA plays a similar role for *diagonal* input uncertainty as the condition number does for *block* input uncertainty.

13.3.2 Sensitivity with Different Controller Structures

Inverse-based controller. For "tight" control it is desirable to use an inverse-based controller $C(s) = P^{-1}(s)K(s)$ (where $K(s)$ is *diagonal*). A special case of such an inverse-based controller is a decoupler. With $C(s) = P^{-1}(s)K(s)$ we find $PC = K(s)$ and $\Lambda(C) = \Lambda(P^{-1}K) = \Lambda(P^{-1}) = \Lambda(P)^T$. Thus, if the elements of $\Lambda(P)$ are large, so will be the elements of $\Lambda(C)$ and from the discussion above we expect high sensitivity to input uncertainty. We also see directly from

$$PC = K(s)(I + \tilde{P}\Delta_I\tilde{P}^{-1}) = K(s)(I + C^{-1}\Delta_I C) \qquad (13.3-8)$$

that large elements in $\tilde{P}\Delta_I\tilde{P}^{-1}$ (or equivalently large elements in $C^{-1}\Delta_I C$) imply that the loop transfer matrix PC is very different from nominal ($\tilde{P}C$) and poor response or even instability is expected when $\Delta_I \neq 0$.

Decouplers have been discussed extensively in the chemical engineering literature, in particular in the context of distillation columns. The purpose of the decoupler (D) is to take care of the multivariable aspects and to reduce the tuning of the control system to a series of single-loop problems. Let $K(s)$ denote these "single-loop" controllers. Then the overall controller C including the decoupler is $C(s) = DK(s)$. Typical decoupler forms are:

Ideal Decoupling:

$$D_I = \tilde{P}^{-1}\tilde{P}_{diag}$$

Simplified Decoupling:

$$D_S = \tilde{P}^{-1}((\tilde{P}^{-1})_{diag})^{-1}$$

Steady State Decoupling:

$$D_0 = D_I(0) \text{ or } D_0 = D_S(0)$$

where we used the notation $A_{diag} = \text{diag}\{a_{11}, \ldots a_{nn}\}$. One-way decouplers are triangular and chosen such that $\tilde{P}C = \tilde{P}DK$ is triangular.

For systems with large RGA the closed loop performance has been reported to be sensitive to errors in the decoupler. This can be easily explained from the results in Sec. 11.2.5 and 11.2.6. However, the most important reason for the robustness problems encountered with decouplers is probably diagonal (actuator) and full-block *input* uncertainty. Because all decouplers (except one-way decouplers) lead to inverse-based controllers the closed-loop performance is highly sensitive to input uncertainty when the plant is ill-conditioned or has a large RGA.

Diagonal Controllers. Closed-loop systems with diagonal controllers are insensitive to diagonal input uncertainty because $\Lambda(C) = I$ when C is diagonal. However, when P is ill-conditioned or has a large RGA, then the performance with a diagonal controller is generally poor, except when for all inputs of interest the disturbance condition number $\kappa_d(PC)$ is close to unity.

In one special case a diagonal controller can lead to good performance for *arbitrary* input directions even for an ill-conditioned plant: when the plant is naturally decoupled at the input. Let the SVD of P be $P = U\Sigma V^H$ and assume $V = I$ (or more generally that V has only one nonzero element in each row and column, so that the inputs can be rearranged to give $V = I$). Then we can choose the diagonal controller $C(s) = c(s)\Sigma^{-1}$ to get $PC = c(s)U$ which has $\kappa_d(PC) = 1$ for *all* inputs d — i.e., good disturbance rejection independent of direction. Note, however that the response is not decoupled (unless U is diagonal).

Also note that in this case

$$\kappa^*(P) = \min_{D_1, D_2} \kappa(D_1 P D_2)$$

$$= \min_{D_1,D_2} \kappa(D_1 U \Sigma D_2)$$
$$= \min_{D_1,D_2} \kappa(D_1 U D_2) = 1$$

because Σ is diagonal and U is unitary. Furthermore, because of (11.2–43), (11.2–45) and (11.2–46) all elements of Λ have to be positive and smaller than one.

General Controller Structure. We have learned that an inverse-based controller is generally bad for plants with a large RGA. (In this case $\Lambda(C)$ is also large.) We also know that for diagonal controllers ($\Lambda(C) = I$) the closed-loop system is always insensitive to diagonal input uncertainty. This suggests that a plot of the magnitude of the elements of $\Lambda(C)$ as a function of frequency may be useful for evaluating a system's sensitivity to input uncertainty. In particular, a large $\Lambda(C)$ around the crossover frequency is undesirable.

We stress that a large $\Lambda(C)$ does not necessarily imply sensitivity to input uncertainty. For example, if $\Lambda(P)$ is small there are no sensitivity problems as we discussed. It is unlikely, however, that a controller with large $\Lambda(C)$ would lead to good *nominal* performance for a plant with small $\Lambda(P)$. Thus, if the controller is designed for good nominal performance, then $\Lambda(C)$ should be a good indicator of sensitivity to input uncertainty.

13.3.3 "Worst-Case" Uncertainty

It is of interest to know the "worst" possible input uncertainty — i.e., the "worst-case" combination of Δ_j's. Consider (13.3–7). If all Δ_j have the same magnitude ($|\Delta_j| < r_I$) then the largest possible magnitude (worst case) of any diagonal element in $P\Delta_I P^{-1}$ is given by $r_I \|\Lambda(P)\|_\infty$. To obtain this value the signs of the Δ_j's should be the same as those in the row of $\Lambda(P)$ with the largest elements.

Example 13.1-3. Consider a plant with steady-state gain matrix

$$P(0) = \begin{pmatrix} 1 & 0.1 & -2 \\ 1 & 2 & -3 \\ -0.1 & -1 & 1 \end{pmatrix}$$

The RGA is

$$\Lambda(P(0)) = \begin{pmatrix} -1.89 & -0.13 & 3.02 \\ 3.59 & 3.02 & -5.61 \\ -0.7 & -1.89 & 3.59 \end{pmatrix}$$

Assume the relative uncertainties Δ_1, Δ_2 and Δ_3 on each manipulated input have the same magnitude $|\Delta|$. Then the second row of $\Lambda(P)$ has the largest row sum ($\|\Lambda(P)\|_\infty = 12.21$) and the worst combination of input uncertainty for an inverse-based controller is

$$\Delta_1 = \Delta_2 = -\Delta_3 = \Delta$$

We find

$$\operatorname{diag}(P\Delta_I P^{-1}) = \operatorname{diag}\{-5.0\Delta, 12.2\Delta, -6.2\Delta\}$$

Note that in this specific example we would arrive at the same worst case by considering the first or third row. Therefore the worst case will always be obtained with Δ_1 and Δ_2 with the same sign and Δ_3 with a different sign even if their magnitudes are different. □

In some cases we may arrive at a different conclusion by considering other frequencies. Also note that, unless an inverse-based controller is used, it is not guaranteed that the worst case uncertainties are deduced using this approach.

13.3.4 Example

The high-purity distillation column described by the simple one-time constant model (see Appendix) will be used to demonstrate the effects of input (actuator) uncertainty. Two control configurations will be compared. In one case reflux (L) and boilup (V) are used for composition control, in the other case the distillate flow (D) and the boilup (V). We will study the responses for two setpoint changes: $r_1 = (1,0)^T$ and $r_2 = (0.4, 0.6)^T$, where r_2 is roughly equivalent to the effect of a change in feed flow to the column. The two configurations are compared in Table 13.3-1. Two controllers were designed for each one of these configurations, an inverse-based and a diagonal controller. The controller gains were adjusted to guarantee robust stability for diagonal input uncertainty with a magnitude bound $w_I(s) = 0.2(5s+1)/(0.5s+1)$. Robust stability is guaranteed for this kind of uncertainty if and only if (Ex. 11.2-1)

$$\mu(CP(I+CP)^{-1}) \leq 1/|w_I|, \qquad \forall\omega$$

where the structured singular value μ is computed with respect to a diagonal matrix. The condition is shown graphically in Fig 13.3-1 for the controllers used in this example.

Responses are obtained both for the nominal case ($\Delta_I = 0$) and with 20% actuator uncertainty: $\Delta_I = \operatorname{diag}\{0.2, -0.2\}$ which yields the following error terms (13.3–6) for PC when an inverse-based controller is used:

$$(\tilde{P}\Delta_I\tilde{P}^{-1})_{LV} = \begin{pmatrix} 35.1\Delta_1 - 34.1\Delta_2 & -27.7(\Delta_1 - \Delta_2) \\ 43.2(\Delta_1 - \Delta_2) & -34.1\Delta_1 + 35.1\Delta_2 \end{pmatrix} = \begin{pmatrix} 13.8 & -11.1 \\ 17.2 & -13.8 \end{pmatrix}$$

$$(\tilde{P}\Delta_I\tilde{P}^{-1})_{DV} = \begin{pmatrix} 0.45\Delta_1 + 0.55\Delta_2 & 0.45(\Delta_1 - \Delta_2 \\ -0.55(\Delta_1 - \Delta_2) & 0.55\Delta_1 + 0.45\Delta_2 \end{pmatrix} = \begin{pmatrix} -0.02 & 0.18 \\ -0.22 & 0.02 \end{pmatrix}$$

The simulations shown in Figs. 13.3-2 to 13.3-5 illustrate the following points:

Table 13.3-1. Comparison of LV and DV configurations.

	LV	DV
$\begin{pmatrix} dy_d \\ dx_B \end{pmatrix} = \frac{1}{75s+1} \times$	$\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \begin{pmatrix} dL \\ dV \end{pmatrix}$	$\begin{pmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{pmatrix} \begin{pmatrix} dD \\ dV \end{pmatrix}$
RGA $:\lambda_{11}$	35.1	0.45
RGA $: \sum_{i,j} \lvert\lambda_{ij}\rvert$	138.3	2
$\kappa(P)$	141.7	70.8
$\kappa_{r_1}, r_1 = (1,0)^T$	110.7	54.9
$\kappa_{r_2}, r_2 = (0.4, 0.6)^T$	11.8	4.3

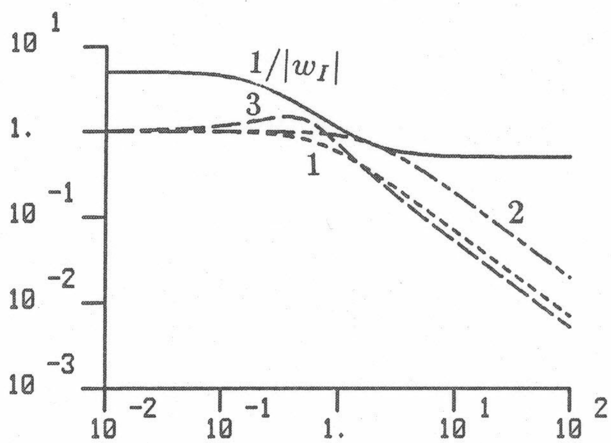

Figure 13.3-1. Robust stability test: $\mu(CP((I + CP)^{-1})$ is shown for 1: Inverse-controller for LV- and DV-configurations; 2: Diagonal LV-controller; 3: Diagonal DV-controller. (Reprinted with permission from *Ind. Eng. Chem. Res.*, **26**, 2329 (1987), American Chemical Society.)

- An inverse-based controller gives poor response when the plant has large RGA-elements (λ_{11} is large) and there is input uncertainty (Fig. 13.3-2).

- A diagonal controller cannot correct for the strong directionality of a plant with large RGA-elements. This results in responses which are strongly dependent on the disturbance (or setpoint) directions (Fig. 13.3-3). The response to r_2 (disturbance in F) is acceptable, but the response to the setpoint change r_1 is extremely sluggish. This system may be acceptable despite the large value of λ_{11}, provided setpoint changes are not important.

- An inverse-based controller may give very good response for an ill conditioned plant even with diagonal input uncertainty, provided λ_{11} is small (Fig. 13.3-4).

- A diagonal controller may remove most of the directionality in the plant if $V \approx I$. However, "interactions" are still present because U is not diagonal (Fig. 13.3-5).

13.3.5 Summary

To some extent actuator (diagonal input) uncertainty is present in every physical system. From the RGA information can be obtained on the sensitivity of the closed loop performance to input uncertainty for different controller structures.

1. Closed loop systems with diagonal controllers are insensitive to diagonal input uncertainty. However when P is ill-conditioned or has a large RGA, then the performance is poor, except when for all inputs of interest the disturbance condition number $\kappa_d(PC)$ is close to one.

2. An inverse-based (e.g., decoupling) controller should *never* be used for a plant with large RGA because it leads to extreme sensitivity to input uncertainty.

3. Inverse-based controllers may give poor response even if the RGA is small. This may happen in the case of a 2×2 system if p_{12}/p_{22} or p_{21}/p_{11} is large (see 13.3–6). One example is a triangular plant which always has $\lambda_{11} = 1$, but where the response obtained with an inverse-based controller may display large "interactions" in the presence of uncertainty.

4. One-way decouplers are generally much less sensitive to input uncertainty.

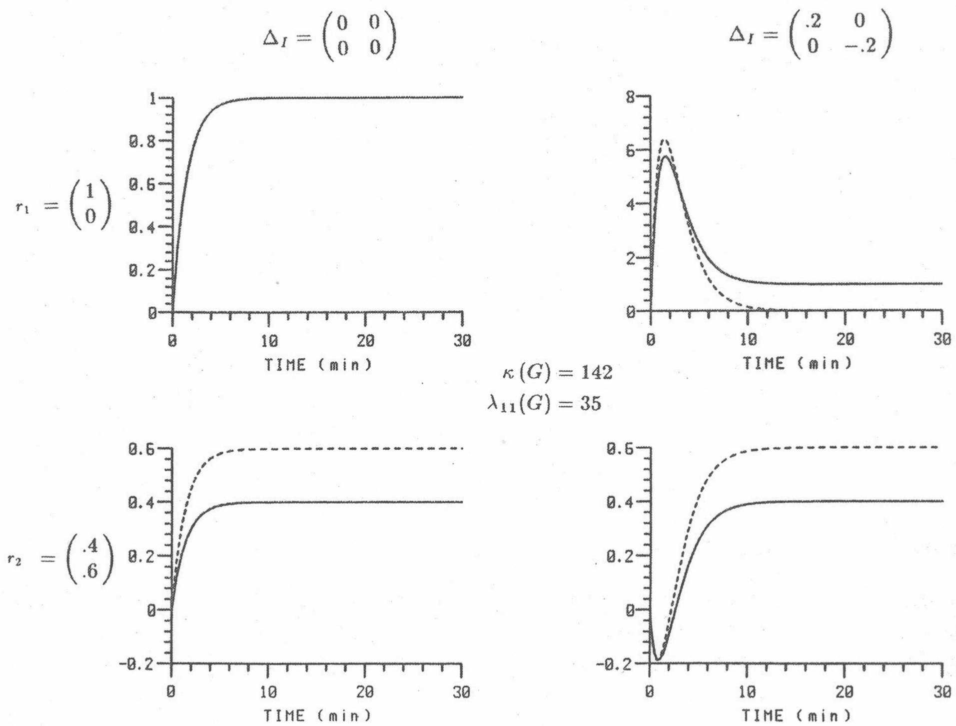

Figure 13.3-2. LV-configuration. Closed-loop responses y_1 and y_2 for inverse-based controller:

$$C(s) = \frac{0.7(75s+1)}{s} G_{LV}^{-1} = \frac{0.7(75s+1)}{s} \begin{pmatrix} 39.94 & -31.49 \\ 39.43 & -32.00 \end{pmatrix}$$

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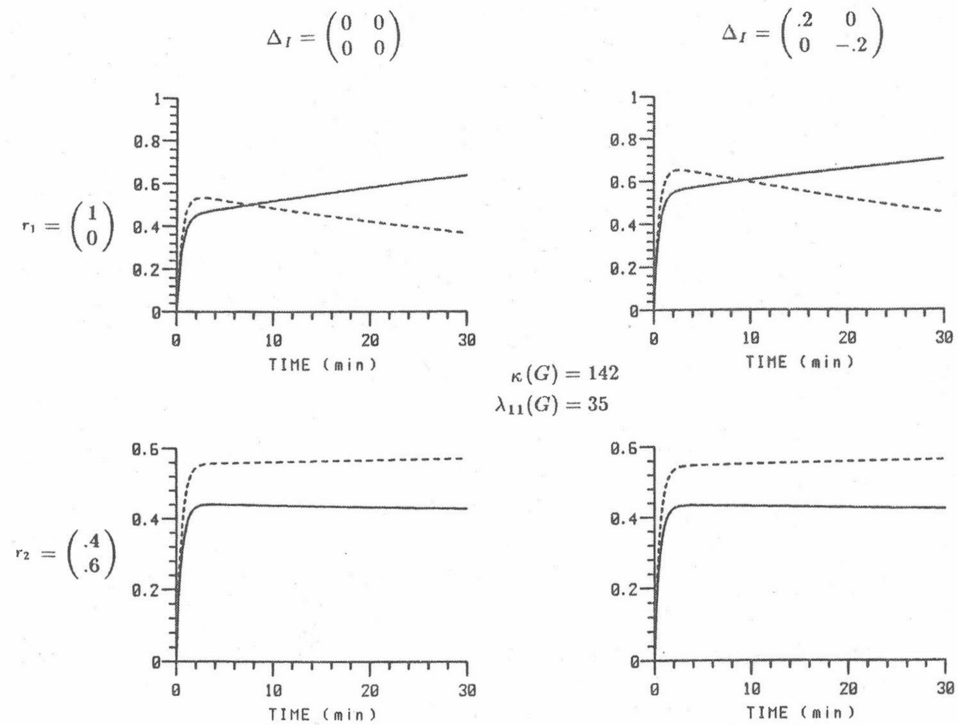

Figure 13.3-3. LV-configuration. Closed-loop responses y_1 and y_2 for diagonal controller:

$$C(s) = \frac{(75s+1)}{s}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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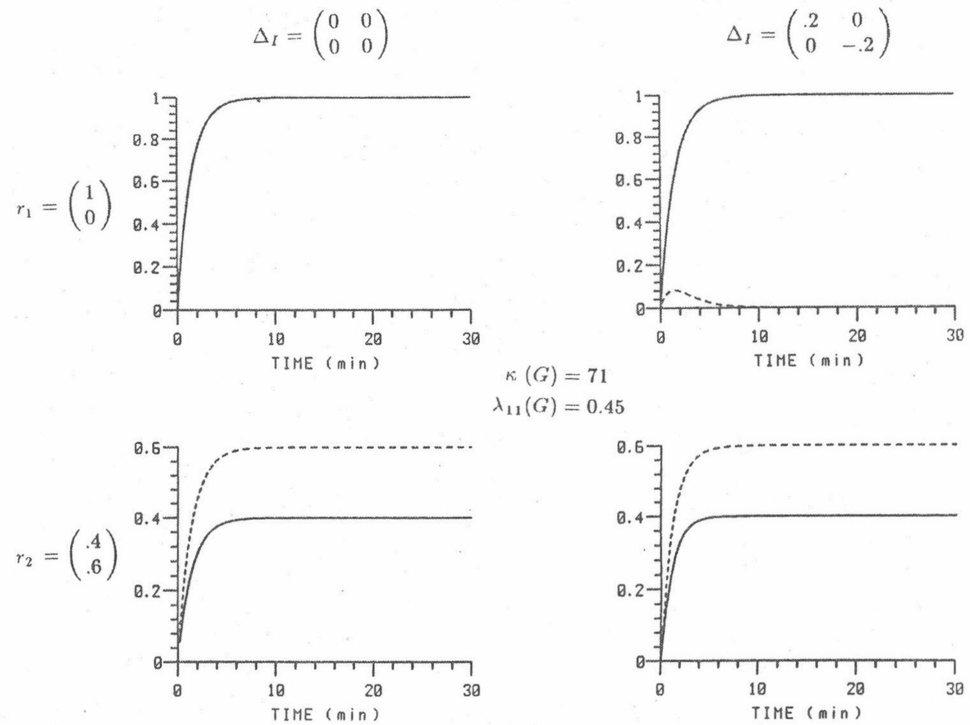

Figure 13.3-4. DV-configuration. Closed-loop responses y_1 and y_2 for inverse-based controller:

$$C(s) = \frac{0.7(75s+1)}{s} G_{DV}^{-1} = \frac{0.7(75s+1)}{s} \begin{pmatrix} -0.5102 & -0.5102 \\ 39.43 & -32.00 \end{pmatrix}$$

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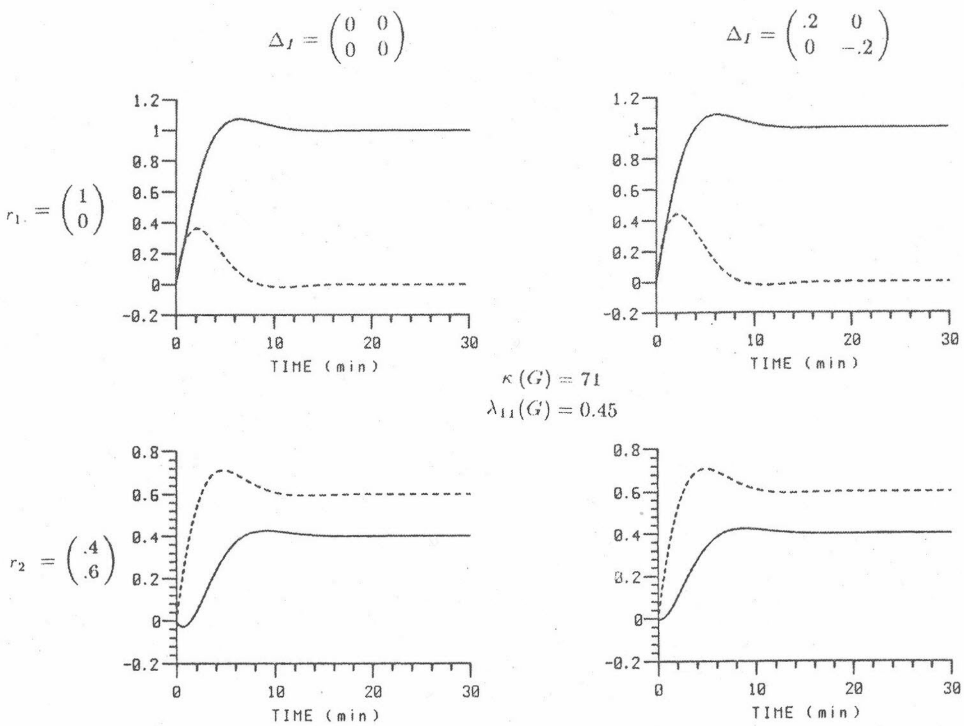

Figure 13.3-5. DV-configuration. Closed-loop responses y_1 and y_2 for diagonal controller:

$$C(s) = \frac{-0.2(75s+1)}{s}\Sigma^{-1} = \frac{0.2(75s+1)}{s}\begin{pmatrix} -0.718 & 0 \\ 0 & -50.8 \end{pmatrix}$$

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Table 13.3-2. Guidelines for choice of preferred multivariable controller structure ("large" implies a comparison with one, typically > 10), $\Vert\Lambda\Vert_A = \sum_{i,j} |\lambda_{ij}|$.

		$max_d \kappa_d(P)$	
		Large	Small
$\Vert\Lambda(P)\Vert_A$	Large	–	Diagonal
	Small	Inverse-based ($V = I$: diagonal)	Inverse-based (diagonal)

5. For a general multivariable controller designed for good performance a plot of the magnitude of the elements of $\Lambda(C)$ vs. frequency is a good indicator of sensitivity to input uncertainty. In particular, a large $\Lambda(C)$ in the crossover frequency range is undesirable.

6. Table 13.3-2 is helpful for choosing between inverse-based (decoupling) and diagonal controllers in extreme situations.

13.4 References

13.1. The ideas in this section were first presented by Skogestad & Morari (1987d). Stanley, Marino-Galarraga & McAvoy (1985) also observed the dependence of the performance on the disturbance direction and developed the Relative Disturbance Gain as an indicator. Their observations regarding the behavior of high purity distillation columns parallel those made for Ex. 13.1-1.

13.2. This section follows the paper by Morari, Zafiriou & Holt (1987). Zafiriou & Morari (1987) presented a more general treatment removing some of the assumptions. A discrete framework was used in this paper, which allows one to handle multiple delays in exactly the same way as NMP zeros in one single step. The transmission blocking interpretation of zeros is due to Desoer & Schulman (1974).

13.2.2. Holt & Morari (1985b) give a general overview of the effect of RHP zeros on closed-loop performance. They review the concept of "pinned zeros" introduced by Bristol (1980). Desoer & Gündes (1986) point out in a very general context that additional RHP zeros can be introduced by decouplers.

13.3. Skogestad & Morari (1987c) discovered the RGA as an indicator of sensitivity to diagonal input (i.e., actuator) uncertainty.

13.3.2. The use of decouplers for distillation control is discussed by Luyben (1970) and Arkun, Manousiouthakis & Palozoglu (1984). Toijala & Fagervik (1982) report high sensitivity to decoupler errors for systems with large RGA.

13.3.3. The method developed by Fan & Tits (1986) for computing the SSV yields also the "worst-case" uncertainty.

Chapter 14

DECENTRALIZED CONTROL

14.1 Motivation

Let $P(s)$ be an $n \times n$ rational transfer function matrix relating the vector of system inputs y to the vector of system outputs u. Let r be the vector of reference signals or setpoints for the closed loop system. Assume that u, y, and r have been partitioned in the same manner: $u = (u_1, u_2, \ldots u_m)^T, y = (y_1, y_2, \ldots y_m)^T, r = (r_1, r_2, \ldots, r_m)^T$. In this book decentralized control means that the controller C is block diagonal (Fig. 14.1-1)

$$u_i = C_i(y_i - r_i) \tag{14.1 - 1}$$

(It should be obvious that any control system where every input u_i and every output y_j are processed by a single controller block C_k, can be rearranged such that C is diagonal.) The constraints on the controller structure invariably lead to performance deterioration when compared to the system with a full controller matrix. This sacrifice has to be weighed against the following two factors:

1. *Hardware simplicity.* If u_i, y_i are physically close but u_i, y_j $(i \neq j)$ are far apart, a full controller could require expensive communication links. Also, the controller hardware costs could be high if an implementation through analog circuitry is required. These considerations are relevant, for example, for large networks of power stations where the distances between the stations can be significant. Hardware issues are generally irrelevant in the context of process control; in all modern plants all measurement signals are sent into a central control room from where all the actuator signals originate.

2. *Design simplicity.* If all the blocks $P_{ij} = 0$ $(i \neq j)$ then each controller C_i can be designed for the isolated subsystem P_{ii} without any loss of performance. If P_{ij} $(i \neq j)$ is "small" then it should still be possible to design the controller for the essentially independent subsystem P_{ii}. The advantage is that fewer controller parameters need to be chosen than for the full system. This is particularly relevant in process control where often thousands of variables have to be controlled, which

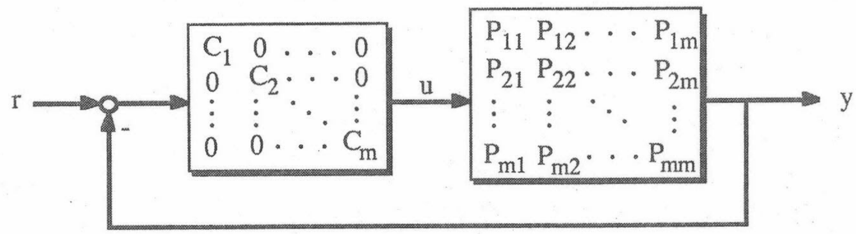

Figure 14.1-1. General decentralized control structure.

could lead to an enormously complex controller. It is also important that at least stability but hopefully also performance is preserved to some degree when individual sensors or actuators fail. This failure tolerance is generally easier to achieve with decentralized control systems, where parts can be turned off without significantly affecting the rest of the system, provided that the controller blocks are designed appropriately.

The designer of decentralized controllers is faced with two issues: (1) The control structure or "pairing" problem — i.e., which set of measurements should be used to affect which set of inputs. (2) The controller design problem — i.e., how to tune the individual controller blocks. The pairing problem can be formidable. Even for relatively small systems, there are many distinct decentralized control system structures to choose from: For a 4×4 system there are 130 possibilities, for a 5×5 system 1495, and so on. Thus efficient screening techniques are needed which are capable of eliminating quickly all the control structures which are definitely inappropriate according to certain criteria like failure tolerance.

All controller design techniques discussed in this book so far yield controllers whose structure and order are determined by the structure and order of the system to be controlled. In this section we will develop guidelines on how to design controllers with a fixed (decentralized) structure which is usually different from the structure of the system to be controlled. We will also develop a series of so-called "Interaction Measures" (IM's). Their purpose is, in general, to help in the screening of different control structures and to guide the design of the controller blocks. Some IM's indicate, for example, if it is possible to design a fault tolerant control system for a specific control structure. Others express the performance degradation caused by the decentralized control structure.

After defining the decentralized control problem more rigorously, we will derive a number of *necessary* conditions the plant and the associated control structure

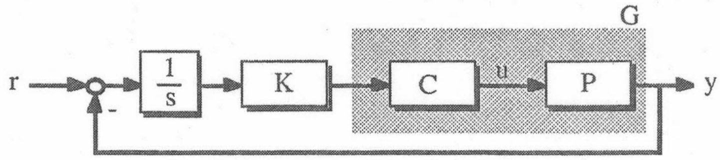

Figure 14.2-1. System with integrator and diagonal compensator K.

have to satisfy for the design of a fault tolerant decentralized control system to be possible. These conditions can be used to screen alternate control structures. Then we will present a decentralized design procedure which guarantees that the overall closed-loop system satisfies certain robust performance conditions. To simplify the notation we will often assume that the controller is fully decentralized — i.e., that the controller matrix C is diagonal rather than block diagonal. Most results can be extended in a obvious manner to the block diagonal case. Usually we will also assume that the square $n \times n$ plant P is stable. If we do so, then the results do *not* carry over easily to systems which are open-loop unstable.

14.2 Definitions

Consider the block diagram in Fig. 14.2-1 where the diagonal controller $C(s)$ is stable and where K is a diagonal gain matrix with positive entries:

$$K = \text{diag}\{k_i\} \qquad k_i > 0, \quad i = 1, n \qquad (14.2-1)$$

The factor s^{-1} implies that the control system features integral control in all channels. The following property is very desirable from a practical point of view.

Definition 14.2-1. *A plant P is Decentralized Integral Controllable (DIC) if there exists a diagonal controller CKs^{-1} with integral action such that (a) the closed loop system shown in Fig. 14.2-1 is stable and (b) the gains of any subset of loops can be reduced to $K_\epsilon = \text{diag}\{k_i\epsilon_i\}$, $0 \le \epsilon_i \le 1$ without affecting the closed loop stability.*

Condition (b) implies that any subset of loops can be detuned or taken out of service (put on "manual") while maintaining the stability of the rest of the system. If a system is DIC then it is possible to achieve stable closed-loop performance of the overall system by tuning every loop separately. DIC is a property of the system and in particular of the selected control structure. It is desirable to select a control structure such that the system is DIC. Unfortunately, no necessary *and* sufficient conditions for DIC are available which can be used to screen alternate control structures. There are some sufficient conditions but they should be used for screening only with great caution because they might eliminate attractive alternatives erroneously. Next we will define some properties which are weaker than DIC. They will lead in turn to *necessary* conditions for DIC. These necessary conditions are usually powerful enough to reduce the number of alternatives drastically.

Definition 14.2-2. *The system $G = PC$ is Integral Controllable (IC) if there exists a $k > 0$ such that (a) the closed-loop system shown in Fig. 14.2-1 is stable for $K = kI$ and (b) the gains of the loops can be reduced to $K_\epsilon = \epsilon k I$, $0 < \epsilon \leq 1$ without affecting the closed-loop stability.*

For a system to be IC, stability must be preserved when the gains k are reduced *simultaneously*, while for it to be DIC one must be able to change the gains *independently*. We will show later than when P is not IC with a certain class of compensators C then it cannot be DIC. Thus, control structures which fail the IC test can be eliminated when searching for a system which is DIC.

Definition 14.2-3. *The system $G = PC$ is Integral Stabilizable (IS) if there exists a $k > 0$ such that the closed-loop system shown in Fig. 14.2-1 is stable for $K = kI$.*

Clearly IS is necessary for IC. Easy tests for IS are available which can help to eliminate systems which cannot be DIC.

14.3 Necessary Conditions for Controllability

14.3.1 Results

We will start with some necessary conditions for IS and IC which will then lead to necessary conditions for DIC. In order not to interrupt the flow of the presentation all proofs are collected in Sec. 14.3.2.

Theorem 14.3-1. *Assume G is a proper rational transfer matrix. G is IS only if* $\det G(0) > 0$.

Theorem 14.3-2. *Assume G is a rational transfer matrix. G is IC if all the eigenvalues of $G(0)$ lie in the open RHP. G is not IC if any of the eigenvalues of $G(0)$ lie in the open LHP (the test is inconclusive if any of the eigenvalues are purely imaginary).*

Corollary 14.3-1. *Assume $g(s)$ is a proper rational transfer function. The SISO system described by $g(s)$ is IS and IC if and only if $g(0) > 0$.*

Corollary 14.3-1 simply states the well known fact that positive feedback leads to instability. An SISO system with positive steady state gain can only be controlled with a negative feedback loop. Theorems 14.3-1 and 14.3-2 are generalizations of the negative feedback condition to multivariable systems. These theorems involve steady state information only, the dynamic components of $P(s)$ and $C(s)$ are not important. Because $P(s)$ and $C(s)$ are assumed to be stable it is always possible to choose the dynamic elements of $C(s)$ conservatively enough for the overall system to be stable and to remain stable with controller gain changes — as long as the steady state requirements — expressed, for example, through Thm. 14.3-2 – are satisfied.

For a system to be DIC all the individual loops have to be stable. Therefore, according to Cor. 14.3-1 the controller gains $c_i(0)$ have to be chosen such that $p_{ii}(0)c_i(0) > 0$. Alternatively we can define the matrix $P^+(0)$ which is derived from $P(0)$ by multiplying each column with $+1$ or -1 such that all diagonal elements are positive. Then for the individual loops to be stable we require $c_i(0) > 0$ so that $p_{ii}^+(0)c_i(0) > 0$. Now we can state necessary conditions for DIC in terms of $P^+(0)$ as corollaries of Th. 14.3-1 and 14.3-2.

Corollary 14.3-2. *Let $P(s)C(s)$ be proper and rational. The system $P(s)$ is DIC only if* $\det(P^+(0)) > 0$.

Corollary 14.3-3. *The rational system $P(s)$ is DIC only if all eigenvalues of the matrix product $P^+(0)C(0)$ are in the closed RHP for all nonnegative diagonal gain matrices $C(0)$.*

These corollaries follow from the fact that for DIC it must be possible to adjust the controller gains $k_i > 0$ or equivalently $c_i(0) > 0$ arbitrarily without affecting closed loop stability: if for some gain matrix $C(0)$ the system is not IS or not IC then it cannot be DIC.

Finally it can be shown that the system is not DIC if any of the diagonal elements of the RGA are negative.

Corollary 14.3-4. *If the RGA element $\lambda_{jj}(P) < 0$ then for any diagonal compensator $C(s)$ chosen so that $P(s)C(s)$ is proper and any K the closed loop system shown in Fig. 14.2-1 has at least one of the following properties. (a) The closed-loop system is unstable. (b) Loop j is unstable by itself — i.e., with all the other loops opened. (c) The closed loop system is unstable as loop j is removed.*

Conditions for DIC. In summary, a system $P(s)$ is DIC with a controller $C(s)$ selected such that PC is proper *only if all* the following conditions are met:

(a) $\det(P^+(0)) > 0$

(b) $\mathrm{Re}\{\lambda_i(P^+(0)C(0))\} \geq 0 \quad \forall i$, for all diagonal $C(0) \geq 0$

(c) $\mathrm{Re}\{\lambda_i(P^+(0))\} \geq 0 \quad \forall i$

(d) $\mathrm{Re}\{\lambda_i(L(0))\} \geq -1 \quad \forall i$, where $L = (P - \bar{P})\bar{P}^{-1}$, $\bar{P} = \mathrm{diag}\{p_{11}, \dots, p_{nn}\}$

(e) RGA: $\lambda_{ii}(P(0)) > 0 \qquad \forall i$

Conditions (a), (b) and (e) follow from Cor. 14.3-2, 14.3-3 and 14.3-4 respectively. Condition (c) and (d) are special cases of (b) for $C(0) = I$ and $P^+(0)C(0) = P(0)\bar{P}(0)^{-1}$. Condition (a) is implied by (c) and is therefore redundant. Condition (b) is difficult to test. Condition (c), (d), and (e) are popular tools for screening control structures in terms of DIC. They are independent as the following examples illustrate.

Example 14.3-1.

$$P(0) = \begin{pmatrix} 10 & 0 & 20 \\ 0.2 & 1 & -1 \\ 11 & 12 & 10 \end{pmatrix}$$

$$\lambda_i(P^+(0)) = \{24.7, -3.0, -0.65\}$$

$$\lambda_i(L(0)) = \{1.19, -0.59 \pm 0.232i\}$$

$$\mathrm{RGA}: \quad \lambda_{ii} = \{4.6, -2.5, 2.1\}$$

Here $\lambda_i(L(0))$ is inconclusive, the other two tests indicate that the system is not DIC. □

Example 14.3-2.

$$P(0) = \begin{pmatrix} 8.72 & 2.81 & 2.98 & -15.80 \\ 6.54 & -2.92 & 2.50 & -20.79 \\ -5.82 & 0.99 & -1.48 & -7.51 \\ -7.23 & 2.92 & 3.11 & 7.86 \end{pmatrix}$$

$$\lambda_i(P^+(0)) = \{-9.7, 4.7, 6.1, 19.9\}$$

$$\lambda_i(L(0)) = \{-3.3, 1.9, 0.7 \pm i\}$$

$$\mathrm{RGA} \quad \lambda_{ii} = \{0.41, 0.45, 0.17, 0.04\}$$

Here the RGA is inconclusive, the other two tests indicate that the system is not DIC. □

Example 14.3-3.

$$P(0) = \begin{pmatrix} 0.5 & 0.5 & -0.005 \\ 1 & 2 & -0.01 \\ -30 & -250 & 1 \end{pmatrix}$$

$$\lambda_i(P^+(0)) = \{3.43, 0.036 \pm 0.32i\}$$

$$\lambda_i(L(0)) = \{1.7, -0.85 \pm 0.38i\}$$

$$\text{RGA} \quad \lambda_{ii} = \{-0.71, 2, 1.43\}$$

Here only the RGA allows one to conclude that the system is not DIC. □

Note that sometimes it is not possible to find any control structure for which a system is DIC. This is illustrated in the next example.

Example 14.3-4.

$$P(0) = \begin{pmatrix} 1 & 1 & -0.1 \\ 0.1 & 2 & -1 \\ -2 & -3 & 1 \end{pmatrix}$$

$$\text{RGA} = \begin{pmatrix} -1.89 & 3.59 & -0.7 \\ -0.13 & 3.02 & -1.89 \\ 3.02 & -5.61 & 3.59 \end{pmatrix}$$

Clearly it is not possible to permute the rows and columns of $P(0)$ and equivalently the RGA such that all elements on the diagonal are positive. Thus the system is not DIC for any pairing of variables. □

For 2×2 systems much stronger results are available.

Corollary 14.3-4. *For* 2×2 *systems all tests (a)-(e) are equivalent and necessary and sufficient for DIC. Moreover, for every* 2×2 *system, there is always a pairing such that the system is DIC.*

14.3.2 Proofs

Proof of Theorem 14.3-1. The proof is based on the Routh test. The characteristic equation (CE) for the closed-loop system in Fig. 14.2-1 is given by

$$\phi(s) \cdot \det(I + G(s)\frac{k}{s}) = 0 \qquad (14.3-1)$$

where $\phi(s)$ is the open-loop characteristic polynomial of $G(s) = P(s)C(s)$. Express $G(s)$ as $G(s) = N(s)d^{-1}(s)$ where $d(s)$ is the common denominator of the elements of $G(s)$ and $N(s)$ is a polynomial matrix. Equation (14.3-1) can then be expressed as

$$\frac{\phi(s)}{sd(s)} \cdot \det(sd(s)I + kN(s)) = 0 \qquad (14.3-2)$$

Upon expansion of the determinant, this expression becomes

$$\frac{\phi(s)}{sd(s)} \cdot (s^n d^n(s) + \ldots + k^n \det N(0)) = 0 \qquad (14.3-3)$$

If $G(s)$ is proper, the coefficient of the highest power of s in (14.3–3) will be the coefficient of the highest power of s in $d(s)$. This coefficient will be positive because of the stability assumption. The closed-loop system will be stable only if all the coefficients in $\det(sd(s)I + kN(s))$ are positive. The constant coefficient is $\det(kN(0))$ and therefore for closed-loop stability it is required that $\det(N(0)) > 0$ and $\det(G(0)) > 0$. □

Proof of Theorem 14.3-2. Let the Nyquist D-contour be indented at the origin to the right to exclude the pole of $s^{-1}G(s)$ at the origin. The system will be closed-loop stable if none of the characteristic loci (CL) encircles the point $(-1/k, 0)$. For IC it is necessary and sufficient that the CL intersect the negative real axis only at *finite* values. An intersection at $(-\infty, 0)$ could only occur because of the pole of $1/s\ G(s)$ at the origin. Along the indentation, the small semicircle with radius ϵ around the origin, the CL can be described by

$$\lambda_j(G(0))\frac{1}{\epsilon}e^{i\phi} \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}; \quad j = 1, n$$

for small ϵ. Let $\lambda_j(G(0)) = r_j e^{i\theta_j}$; then the expression for the CL can be rewritten as

$$\frac{r_j}{\epsilon}e^{i(\theta_j+\phi)} \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}; \qquad j = 1, n$$

The CL do not cross the negative real axis if $-\pi < \theta_j + \phi < \pi$ or $-\pi/2 < \theta_j < \pi/2$, which means $\lambda_j(G(0))$, $j = 1, n$ has to be in the open RHP. The characteristic locus j crosses the negative real axis if $\pi/2 < \theta_j < 3\pi/2$, which means $\lambda_j(G(0))$ is in the open LHP. Nothing can be said from this proof about the systems for which the spectrum of $G(0)$ is constrained to the closed right-half plane and includes eigenvalues on the imaginary axis. □

Proof of Corollary 14.3-4. Because the RGA element λ_{ij} is invariant under input and output scaling we have for any diagonal $C(0)$

$$\lambda_{ij} = (-1)^{i+j} p_{ij} \frac{\det(P^{ji}(0))}{\det(P(0))} \qquad (14.3-4)$$

$$\lambda_{ij} = (-1)^{i+j} g_{ij} \frac{\det(G^{ji}(0))}{\det(G(0))} \qquad (14.3-5)$$

where the notation A^{ij} indicates that row i and column j of matrix A have been deleted.

If $\lambda_{jj} < 0$ then one or three of the terms in (14.3-5) is negative. For property (a) $\det(G(0)) < 0$; for property (b) $g_{jj} < 0$; for property (c) $\det(G^{jj}(0)) < 0$. □

The proofs of the other corollaries are straightforward and are left as an exercise.

14.4 Stability Conditions – Interaction Measures

Let us define the term Interaction Measure (IM) more precisely with reference to Fig. 14.4-1.

A controller

$$C = \text{diag}\{C_1, C_2, \ldots, C_m\} \tag{14.4-1}$$

is to be designed for the system

$$\bar{P} \triangleq \text{diag}\{P_{11}, P_{22}, \ldots, P_{mm}\} \tag{14.4-2}$$

such that the block diagonal closed-loop system with the transfer matrices

$$\bar{H} \triangleq \bar{P}C(I + \bar{P}C)^{-1} \tag{14.4-3}$$

$$\bar{E} \triangleq (I + \bar{P}C)^{-1} \tag{14.4-4}$$

is stable ($\delta = 0$ in Fig. 14.4-1). An IM expresses the constraints imposed on the choice of the closed-loop transfer matrix $\bar{H}$ or $\bar{E}$ for the block diagonal system, which guarantee that the full closed-loop system

$$H = PC(I + PC)^{-1} \tag{14.4-5}$$

$$E = (I + PC)^{-1}$$

is stable (i.e., $\delta = 1$) in Fig. 14.4-1.

This definition of an IM has its limitations and therefore the results should be interpreted with caution. The reason is that the IM is based on the block diagonal $\bar{H}$ or $\bar{E}$ which might or might not be indicative of the actual full closed-loop transfer matrix H or E. Though this definition of the IM guarantees the system to be stable it can be very badly behaved. Even if the IM indicates "small" interactions, the performance can be arbitrarily poor. The following matrices play central roles in interaction analysis:

$$L_H = (P - \bar{P})\bar{P}^{-1} \tag{14.4-6}$$

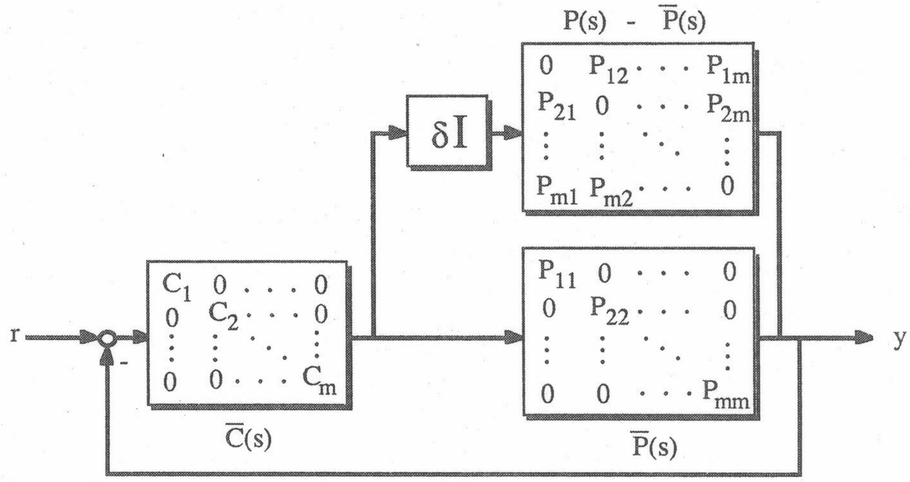

Figure 14.4-1. Block diagram representation of interactions as additive uncertainty.

$$L'_H = \bar{P}^{-1}(P - \bar{P}) \qquad (14.4-7)$$

$$L_E = (P - \bar{P})P^{-1} \qquad (14.4-8)$$

They can be viewed as "relative errors" arising from the "approximation" of the full system P by $\bar{P}$.

14.4.1 Necessary and Sufficient Stability Conditions

The following two stability conditions follow from the multivariable Nyquist criterion. Let us denote by $N(k, g(s))$ the net number of clockwise encirclements of the point $(k, 0)$ by the image of the Nyquist D contour under $g(s)$. Then we can state

Theorem 14.4-1. *Assume that P and $\bar{P}$ have the same number of RHP poles and that $\bar{H}$ is stable. Then the closed-loop system H is stable if and only if*

$$N(0,\ \det(I + L_H\bar{H})) = 0 \qquad (14.4-9)$$

Proof. The return difference operator for the full system H can be factored as

$$(I + PC) = (I + L_H\bar{H})(I + \bar{P}C) \qquad (14.4-10)$$

Let the number of open loop unstable poles of P and $\bar{P}$ be p_0. H is stable if and only if

$$N(0, \det(I+PC)) = N(0, \det(I+\bar{P}C)) + N(0, \det(I+L_H\bar{H})) = -p_0 \quad (14.4-11)$$

Because $\bar{H}$ is stable by assumption

$$N(0, \det(I+\bar{P}C)) = -p_0 \quad (14.4-12)$$

Substituting (14.4–12) into (14.4–11), (14.4–9) follows immediately. □

Theorem 14.4-2. *Assume that P and $\bar{P}$ have the same number of RHP zeros and that $\bar{H}$ is stable. Then the closed-loop system H is stable if and only if*

$$N(0, \det(I-\bar{E}L_E)) = 0 \quad (14.4-13)$$

Proof. Consider the following identity

$$P^{-1}(I+PC) = \bar{P}^{-1}(I+\bar{P}C)(I-\bar{E}L_E) \quad (14.4-14)$$

Let the number of RHP zeros of P and $\bar{P}$ be z_0. H is stable if and only if

$$N(0, \det(P^{-1}(I+PC))) = N(0, \det(\bar{P}^{-1}(I+\bar{P}C))) + N(0, \det(I-\bar{E}L_E)) = -z_0 \quad (14.4-15)$$

Because $\bar{H}$ is stable by assumption and the number of RHP zeros of $\bar{P}$ is z_0

$$N(0, \det(\bar{P}^{-1}(I+\bar{P}C))) = -z_0 \quad (14.4-16)$$

Substituting (14.4–16) into (14.4–15), (14.4–13) follows immediately. □

Theorems 14.4-1 and 14.4-2 form the cornerstone for much of the further development and deserve some discussion. First note that the poles of P are always a subset of the poles of $\bar{P}$. Generically the subset is proper and therefore generically the number of unstable poles of P and $\bar{P}$ is not the same except trivially for stable systems. Therefore for all practical purposes Thm. 14.4-1 is limited to open-loop stable systems. By the same arguments Thm. 14.4-2 is limited to minimum phase systems.

Requirement (14.4–9) can be interpreted as a "robustness" condition for $\bar{H}$ to remain stable under the multiplicative perturbation L_H. Ideally one would want to select $\bar{H} = I$ — i.e., perfect control. If $P = \bar{P}$ — i.e. P itself is block diagonal, then (14.4–9) is trivially satisfied. Thus the closed loop system is stable regardless of how $\bar{H}$ is chosen. If $P \neq \bar{P}$, $\bar{H}$ has to be chosen such that (14.4–9) remains satisfied. Qualitatively, at least, it is clear that when L_H is "large" $\bar{H}$ has to be made "small" to avoid encirclements. A small $\bar{H}$ implies poor performance. IMs derived from (14.4–9) provide a quantitative indication of the constraints on $\bar{H}$ imposed by L_H.

Similarly (14.4–13) can be interpreted as a "robustness" condition for $\bar{H}$ to remain stable under the inverse multiplicative perturbation L_E. Because $\bar{E}$ should be small for good performance (14.4–13) seems easier to satisfy than (14.4–9). However, for strictly proper systems $\lim_{s \to +\infty} E = I$ and therefore encirclements can occur if L_E is too large.

14.4.2 Sufficient Stability Conditions

By invoking the Small Gain Theorem (Thm. 10.2-2) sufficient conditions for the stability of the decentralized control system can be easily derived from Thm. 14.4-1 and 14.4-2.

Theorem 14.4-3. *Assume that P and $\bar{P}$ have the same RHP poles and that $\bar{H}$ is stable. Then the closed loop system H is stable if*

$$\rho(L_H(i\omega)\bar{H}(i\omega)) < 1 \qquad \forall\omega \qquad (14.4-17)$$

or if

$$\|L_H(i\omega)\bar{H}(i\omega)\| < 1 \qquad \forall\omega \qquad (14.4-18)$$

where $\|\cdot\|$ denotes any compatible matrix norm.

Theorem 14.4-4. *Assume that P and $\bar{P}$ have the same RHP zeros and that $\bar{H}$ is stable. Then the closed loop system is stable if*

$$\rho(\bar{E}(i\omega)L_E(i\omega)) < 1 \qquad \forall\omega \qquad (14.4-19)$$

or if

$$\|\bar{E}(i\omega)L_E(i\omega)\| < 1 \qquad \forall\omega \qquad (14.4-20)$$

Several approaches can be taken to derive IMs — i.e., explicit bounds on $\bar{H}$ from (14.4–17) and (14.4–18). There is a trade-off between the assumptions made about the structure of $\bar{H}$ and the restrictiveness of the derived bounds. Less restrictive bounds on the magnitude of $\bar{H}$ are obtained as more restrictive assumptions are made about the structure of $\bar{H}$. Assuming a highly structured form for $\bar{H}$ — i.e., $\bar{H}(s) = \bar{h}(s)I$, leads to the following bound.

Corollary 14.4-1. *Assume $\bar{H}(s) = \bar{h}(s)I$. Under the assumptions of Thm. 14.4-3 the closed-loop system H is stable if*

$$|\bar{h}(i\omega)| < \rho^{-1}(L_H(i\omega)) \qquad \forall\omega \qquad (14.4-21)$$

The form of $\bar{H}(s)$ assumed for Cor. 14.4-1 is very restrictive but $\rho^{-1}(L_H(i\omega))$ is the least restrictive magnitude bound that can be derived for $\bar{H}(s)$ from (14.4–17). (A similar corollary can be derived from Thm. 14.4-4.)

If integral action is employed in all channels then $\bar{h}(0) = 1$ and the requirement (14.4–21) becomes for $\omega = 0$, $\rho(L_H(0)) < 1$. If the system is open-loop stable then it is always possible to satisfy (14.4–21) at all other frequencies ($\omega \neq 0$) by rolling off $\bar{h}(s)$ sufficiently fast. Thus we have the following corollary.

Corollary 14.4-2. *The stable system P is DIC if*

$$\rho(L_H(0)) < 1 \qquad (14.4-22)$$

The small gain condition (14.4–18) does not take into account the special structure of the matrix $\bar{H}$. Therefore it tends to be conservative though the conservativeness depends on the type of norm used. It is natural to look for an expression which takes into account the structure of $\bar{H}$ and represents an optimal bound in the following sense. Let $\bar{H}$ be block diagonal and norm bounded

$$\bar{H}(s) = \text{diag}\{\bar{H}_1(s), \ldots, \bar{H}_m(s)\} \qquad (14.4-23)$$

$$\bar{\sigma}(\bar{H}_i(i\omega)) < \delta(\omega) \qquad \forall i, \omega \qquad (14.4-24a)$$

or equivalently

$$\bar{\sigma}(\bar{H}(i\omega)) < \delta(\omega) \qquad \forall\omega \qquad (14.4-24b)$$

A real positive function $\mu(L_H(i\omega))$ is desired with the property that (14.4–9) is satisfied for *all* matrices $\bar{H}(i\omega)$ satisfying (14.4–23) and (14.4–24) if and only if

$$\bar{\sigma}(\bar{H}(i\omega)) < \mu^{-1}(L_H(i\omega)) \qquad \forall\omega \qquad (14.4-25)$$

It follows directly from Thm. 11.2-1 that μ is the structured singular value (SSV).

Theorem 14.4-5. *Assume that P and $\bar{P}$ have the same RHP poles and that $\bar{H}$ is stable. Then the closed loop system H is stable if*

$$\bar{\sigma}(\bar{H}(i\omega)) < \mu^{-1}(L_H(i\omega)) \qquad \forall\omega \qquad (14.4-26)$$

This is the tightest norm bound, in the sense that if there is a system $\bar{H}_1$ which violates (14.4–26)

$$\bar{\sigma}(\bar{H}_1(i\omega)) > \mu^{-1}(L_H(i\omega)) \qquad (14.4-27)$$

then there exists another system $\bar{H}_2$ such that

$$\bar{\sigma}(\bar{H}_1(i\omega)) = \bar{\sigma}(\bar{H}_2(i\omega)) \qquad (14.4-28)$$

for which (14.4–9) is violated and H is unstable.

A similar result can be derived from Thm. 14.4-2.

Theorem 14.4-6. *Assume that P and $\bar{P}$ have the same RHP zeros and that $\bar{H}$ is stable. Then the closed loop system H is stable if*

$$\bar{\sigma}(\bar{E}(i\omega)) < \mu^{-1}(L_E(i\omega)) \qquad \forall\omega \qquad (14.4-29)$$

Different IM's can be derived readily from Thms. 14.4-1 through 14.4-6.

14.4.3 Diagonal Dominance Interaction Measures

Assume that $\bar{P}$ and $\bar{H}$ are diagonal and that the 1-norm is used in (14.4–18). Then (14.4–18) becomes

$$\sum_{i\neq j} |(p_{ij}(i\omega)/p_{jj}(i\omega))\bar{h}_j(i\omega)| < 1 \qquad \forall j,\omega \qquad (14.4-30)$$

or

$$|\bar{h}_j(i\omega)| < \left[\sum_{i\neq j} |p_{ij}(i\omega)/p_{jj}(i\omega)|\right]^{-1} \qquad \forall j,\omega \qquad (14.4-31)$$

This is the "IMC interaction measure" for column dominance. It expresses the constraints the individual loops, $\bar{h}_j(s)$ must satisfy for the overall system to be stable. A plot of the RHS of (14.4–31) as a function of frequency is a good indicator of the bandwidth over which good control can be achieved.

Definition 14.4-1. *Let $\bar{P}(i\omega)$ be diagonal. The complex matrix $P(i\omega)$ is column dominant if*

$$\|L_H(i\omega)\|_1 < 1 \qquad (14.4-32a)$$

or

$$\sum_{i\neq j} |p_{ij}(i\omega)| < |p_{jj}(i\omega)| \qquad \forall j \qquad (14.4-32b)$$

When $P(i\omega)$ is column dominant for all ω, a fortunate and rare situation, then the constraint (14.4–31) on $|\bar{h}_j(i\omega)|$ is very mild

$$|\bar{h}_j(i\omega)| < \alpha \quad \text{where} \quad \alpha > 1 \qquad \forall j,\omega$$

and there are no limitations on the bandwidth imposed by the interactions.

Note that

$$\det(I + (P - \bar{P})\bar{P}^{-1}\bar{H}) = \det(I + \bar{P}^{-1}\bar{H}(P - \bar{P})) \qquad (14.4-33)$$

Therefore another sufficient stability condition which is similar to (14.4–18) is

$$\|\bar{P}^{-1}(i\omega)\bar{H}(i\omega)(P(i\omega) - \bar{P}(i\omega))\| < 1 \qquad \forall\omega \qquad (14.4-34)$$

The IMC interaction measure for row dominance can be derived by employing the ∞-norm in (14.4–34).

14.4.4 Generalized Diagonal Dominance Interaction Measures

Assume again that $\bar{P}$ and $\bar{H}$ are diagonal. If the inputs of P are scaled by a diagonal nonsingular matrix D_2 and the outputs by a diagonal nonsingular matrix D_1 and if the controller C is scaled accordingly, the stability of the system should be unaffected. This can be seen easily from (14.4–9)

$$\det(I + (D_1PD_2 - D_1\bar{P}D_2)D_2^{-1}\bar{P}^{-1}D_1^{-1} \cdot D_1\bar{H}D_1^{-1}) \qquad (14.4-35a)$$

$$= \det(I + D_1(P - \bar{P})\bar{P}^{-1}\bar{H}D_1^{-1}) \qquad (14.4-35b)$$

$$= \det(I + D_1(P - \bar{P})\bar{P}^{-1}D_1^{-1}\bar{H}) \qquad (14.4-35c)$$

$$= \det(I + L_H\bar{H}) \qquad (14.4-35d)$$

A similar development holds for the right-hand side of (14.4–33). Though stability is independent of scaling the sufficient stability condition

$$\|\bar{H}(i\omega)\| < \|L_H(i\omega)\|^{-1} \qquad \forall\omega \qquad (14.4-36)$$

derived from (14.4–18) is not. Therefore it is natural to seek the scaling which makes (14.4–36) least conservative. Equation (14.4–35c) shows that for the error matrix $L_H(s)$ only one scaling matrix (D_1) is necessary, the other one cancels. Thus the minimization problems requiring solutions are

$$\min_{D_1} \|D_1L_H(i\omega)D_1^{-1}\|_1 \qquad (14.4-37a)$$

and

$$\min_{D_2} \|D_2^{-1}L'_H(i\omega)D_2\|_\infty \qquad (14.4-37b)$$

The solutions of (14.4–37) are provided by the Perron-Frobenius Theorem.

Theorem 14.4-7.

$$\min_{D_1} \|D_1 L_H(i\omega) D_1^{-1}\|_1 = \min_{D_2} \|D_2^{-1} L_H'(i\omega) D_2\|_\infty = \rho(|L_H(i\omega)|) \qquad (14.4-38)$$

where $|A|$ denotes the matrix A with all its elements replaced by their magnitudes.

Corollary 14.4-3. *Assume that $P(s)$ and $\bar{P}(s)$ have the same RHP poles and that $\bar{H}(s)$ is stable. Then the closed loop system $H(s)$ is stable if*

$$|\bar{h}_i(i\omega)| < \rho^{-1}(|L_H(i\omega)|) \qquad \forall i, \omega \qquad (14.4-39)$$

Note the similarity between Cor. 14.4-1 and 14.4-3. Equation (14.4–21) is a tighter bound than (14.4–39) because the spectral radius bounds any norm — even when it is minimized — from below. However, for obtaining this tighter bound $\bar{H}(s)$ had to be restricted to $\bar{H}(s) = \bar{h}(s)I$.

Definition 14.4-2. *Assume $\bar{P}(i\omega)$ to be diagonal. The complex matrix $P(i\omega)$ is generalized diagonal dominant if*

$$\rho(|L_H(i\omega)|) < 1 \qquad (14.4-40)$$

When $G(i\omega)$ is generalized diagonal dominant for all ω then the constraint (14.4–39) is very mild and there are no limitations on the bandwidth imposed by the interactions.

It would be incorrect to view (14.4–39) as less conservative than (14.4–31). Inequality (14.4–31) provides individual bounds for each of the single loop transfer functions $\bar{h}_j(s)$ and thus allows trade-offs between the different loops. The optimization giving rise to (14.4–39) minimizes the worst bound and in the process makes all the bounds even. However (14.4–39) has the advantage that independent of the number of system inputs and outputs the design engineer can determine from a single curve whether or not the selected control structure leads to significant performance deterioration.

14.4.5 The μ-Interaction Measure

Theorem 14.4-5 states that for stability the magnitude of the diagonal blocks $\bar{H}_i(s)$ has to be constrained by the reciprocal of the SSV μ of the relative error matrix

$$\bar{\sigma}(\bar{H}_j(i\omega)) < \mu^{-1}(L_H(i\omega)) \qquad \forall j, \omega \qquad (14.4-41)$$

The value of μ depends on the structure assumed for $\bar{H}(s)$. In Sec. 11.2.2 we found the bounds

$$\rho(L_H(i\omega)) \leq \mu(L_H(i\omega)) \leq \bar{\sigma}(L_H(i\omega)) \qquad (14.4-42)$$

This confirms the finding of Cor. 14.4-1 that $\rho^{-1}(L_H(j\omega))$ constitutes the loosest bound but that it is only correct for the rather restricted structure $\bar{H}(s) = \bar{h}(s)I$. The upper bound on μ is consistent with the conservative result (14.4–36) when the spectral norm is used and when no structural constraints are put on $\bar{H}(s)$. Not surprisingly $\mu(L_H(j\omega))$ lies between the extremes when $\bar{H}(s)$ has no specific structure at all and when $\bar{H}(s) = \bar{h}(s)I$.

From Thm. 14.4-6 we obtain the interaction constraint

$$\bar{\sigma}(\bar{E}_i(i\omega)) < \mu^{-1}(L_E(i\omega)) \qquad \forall i, \omega \qquad (14.4-43)$$

for minimum phase systems. For strictly proper open-loop systems with integral control $H \longrightarrow 0$, $\bar{E}_i \longrightarrow I$ as $\omega \longrightarrow \infty$ and $H_i \longrightarrow I$, $\bar{E}_i \longrightarrow 0$ as $\omega \longrightarrow 0$. Therefore (14.4–41) can be easily satisfied at high frequencies and (14.4–43) at low frequencies. Unfortunately it is not possible to combine these two bounds over different frequency ranges.

If $\mu(L_H(0)) < 1$ then the system is DIC. If the individual loops are designed in accordance with (14.4–41) then they can be detuned or turned off without affecting the stability of the rest of the system. Thus, design constraint (14.4–41) leads to controllers which have attractive practical properties.

Note that μ treats both diagonal and block diagonal $\bar{H}(s)$ in a unified optimal manner. Just as in the case of generalized diagonal dominance all loops are given equal preference.

14.4.6 Interaction Measures for 2 × 2 Systems

The interaction measures for 2×2 systems which are most widely used industrially are the RGA and the Rijnsdorp interaction measure

$$\kappa_R(s) = \frac{p_{12}(s)p_{21}(s)}{p_{11}(s)p_{22}(s)} \qquad (14.4-44)$$

It is related to the RGA through

$$\lambda_{11}(s) = \frac{1}{1-\kappa_R(s)} \qquad (14.4-45)$$

It can be easily shown that these empirical IMs are closely related to those we derived earlier on theoretical grounds.

Corollary 14.4-4. *Assume that $P(s)$ and $\bar{P}(s)$ have the same RHP poles and that $\bar{H}(s)$ is stable. Then the closed-loop system $H(s)$ is stable if*

$$|\bar{h}_i(i\omega)| < |\kappa_R(i\omega)|^{-1/2} = \rho^{-1}(L_H(i\omega)) = \mu^{-1}(L_H(i\omega)) \qquad \forall i, \omega \qquad (14.4-46)$$

Corollary 14.4-5. *The stable 2×2 system P is DIC if any one of the following equivalent conditions is satisfied:*

(a) $P(0)$ *is generalized diagonal dominant.*

(b) $\rho(L_H(0)) < 1$

(c) $\rho(|L_H(0)|) < 1$

(d) $\mu(L_H(0)) < 1$

(e) $|\kappa_R(0)| < 1$

(f) RGA: $\lambda_{11}(0) > \frac{1}{2}$

Compare these conditions with the necessary *and* sufficient conditions for DIC derived in Sec. 14.3.

$$\mathrm{Re}\{\lambda_i(L_H(0))\} \geq -1 \qquad (14.4-47)$$

$$\mathrm{RGA}: \lambda_{ii}(0) > 0 \qquad (14.4-48)$$

The conservativeness of (b) and (f), and therefore of all IMs derived in this section is immediately apparent.

14.4.7 Examples

Example 14.4-1. Consider the distillation column of Doukas and Luyben (1978). The transfer function matrix for a 3×3 subsystem is shown in Table 14.4-1.

Line 1 in Fig. 14.4-2 shows a plot of $\mu^{-1}(L_H(i\omega))$ for the fully decentralized control system with (diagonal) pairings ((1,1), (2,2), (3,3)). This curve shows that a fully decentralized controller with integral action cannot be designed on the basis of Thm. 14.4-5 since $\mu^{-1}(L_H(0)) < 1$. This constraint can be relaxed by considering a more complex controller structure. Line 2 in the same figure shows a plot of $\mu^{-1}(L_H(i\omega))$ for the block decentralized control system with pairings ((1-2,1-2), (3,3)). In this case, a controller with integral action is possible since $\mu^{-1}(L_H(0)) > 1$. However, the interactions limit the achievable closed loop bandwidth to about 0.2 rad min^{-1}.

Table 14.4-1. Distillation Column Transfer Matrix.

	Reflux Ratio	Side Draw	Reboil Duty
Toluene in bottom	$\frac{0.37e^{-7.75s}}{(22.2s+1)^2}$	$\frac{-11.3e^{-3.70s}}{(21.74s+1)^2}$	$\frac{-9.811e^{-1.59s}}{(11.36s+1)}$
Toluene in tops	$\frac{-1.986e^{-0.71s}}{(66.67s+1)^2}$	$\frac{5.24e^{-60s}}{(400s+1)}$	$\frac{5.94e^{-2.24s}}{(14.29s+1)}$
Benzene in side draw	$\frac{0.204e^{-0.59s}}{(7.14s+1)^2}$	$\frac{0.33e^{-0.68s}}{(2.38s+1)^2}$	$\frac{2.38e^{-0.42s}}{(1.43s+1)^2}$

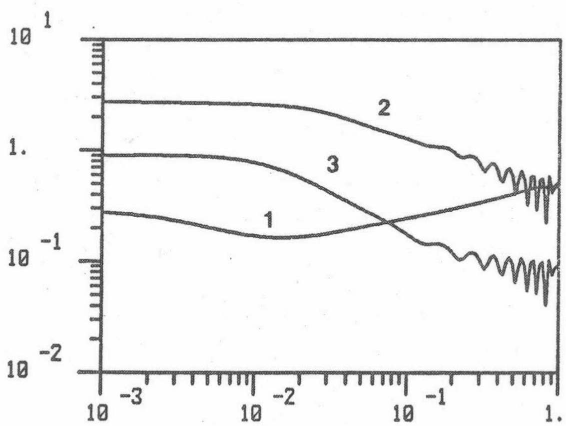

Figure 14.4-2. Interaction Measures. Line 1 = $\mu^{-1}(E(j\omega))$ for fully decentralized controller; line 2 = $\mu^{-1}(E(j\omega))$; line 3 = $\rho^{-1}(|E(j\omega)|)$ for block decentralized controller. (Reprinted with permission from *Automatica*, **22**, 316 (1986), Pergamon Press, plc.)

The generalized diagonal dominance bound $\rho^{-1}(|L_H(i\omega)|)$ for the block decentralized controller is shown as line 3 in Fig. 14.4-2. A comparison of lines 2 and 3 demonstrates the conservativeness associated with $\rho^{-1}(|L_H(j\omega)|)$ as IM. □

Example 14.4-2. Assume now a fully decentralized control structure for the system in Table 14.4-1 implemented on the variable pairs ((1,2),(2,1),(3,3)). The objective is to demonstrate the response of the closed-loop system when the three controllers are designed on the basis of Thm. 14.4-5. Figure 14.4-3 shows a plot of $\mu^{-1}(L_H(i\omega))$ (Line 1). The interactions limit the closed loop bandwidth to about 0.08 rad min^{-1}. Figure 14.4-3 also shows plots of $|\bar{h}_i(i\omega)|, (i = 1,3)$, for three different sets of controllers. In the first case, the three controllers are chosen such that

$$\bar{h}_i(s) = e^{-\tau_i s}(4s+1)^{-3} \qquad i = 1,3 \qquad (14.4-49)$$

while in the second and third cases the controllers are chosen such that

$$\bar{h}_i(s) = e^{-\tau_i s}(10s+1)^{-3} \qquad i = 1,3 \qquad (14.4-50)$$

and

$$\bar{h}_i(s) = e^{-\tau_i s}(25s+1)^{-3} \qquad i = 1,3 \qquad (14.4-51)$$

respectively. Here, τ_i, $(i = 1,3)$, are the time delays in p_{12}, p_{21}, and p_{33}, respectively. The responses of the closed-loop systems for these three sets of controllers were tested for step changes in r_2. The responses corresponding to (14.4–49 to 51) are shown in Fig. 14.4-4. It is apparent that as the $\bar{h}_i(s)$ are moved away from the stability bound (Line 1) and thus made more conservative, the closed-loop responses become more sluggish but less oscillatory. □

14.5 Robust Performance Conditions

In the spirit of the previous section we will derive bounds on the magnitude of the sensitivity $\tilde{E}$ and the complementary sensitivity $\tilde{H}$. These bounds will have the following property: If each block i of the decentralized controller is designed such that $\tilde{E}_i$ or $\tilde{H}_i$ satisfies the corresponding bound, then *robust performance* (in the H_∞ sense) of the overall coupled system is assured and not just nominal stability as before. Because robust stability is just a special case of robust performance, it is not discussed separately.

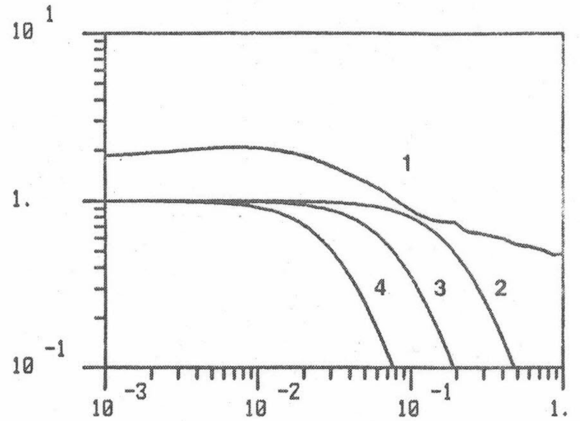

Figure 14.4-3. Line 1 = $\mu^{-1}(E(j\omega))$; lines 2-4 = amplitude ratio for (14.4–49), (14.4–50) and (14.4–51), respectively. (Reprinted with permission from *Automatica*, **22**, 317 (1986), Pergamon Press, plc.)

14.5.1 Sufficient Conditions for Robust Performance

Let us assume that the robust performance requirement is expressed as

$$\mu(M) < 1, \quad \forall\omega \tag{14.5-1}$$

We will proceed in three steps:

1. Express M as an LFT of $\tilde{H}$
2. Express $\tilde{H}$ as an LFT of $\bar{H}$
3. Express M as an LFT of $\bar{H}$. Based on this LFT state a bound on $\bar{\sigma}(\bar{H})$ which guarantees RP of the overall system — i.e., (14.5–1).

The same procedure can be defined for E instead of H. It is based on the results in Sec. 11.4 where we show how robustness conditions can be expressed in terms of bounds on specific transfer matrices related to M by LFT's.

1. *Express M as an LFT of $\tilde{H}$ or $\tilde{E}$.* In Sec. 11.4 we showed in detail how such an LFT can be constructed if it is not obvious from inspection. In general it has the form

$$M = N_{11}^T + N_{12}^T T N_{21}^T \tag{14.5-2}$$

where $T = \tilde{H}$ or $T = \tilde{E}$.

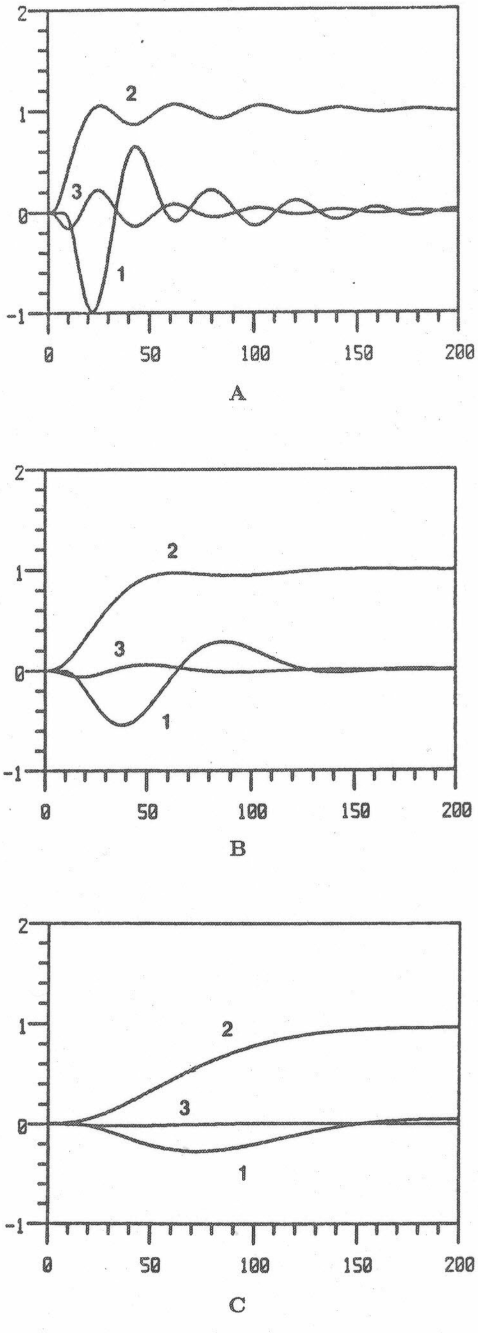

Figure 14.4-4. Closed-loop response to unit step change in $r_2(s)$ for different loop designs $\bar{h}_i$; A: (14.4–49), B: (14.4–50), C: (14.4–51). (Reprinted with permission from *Automatica*, **22**, 317 (1986), Pergamon Press, plc.)

2. *Express $\tilde{H}(\tilde{E})$ as an LFT of $\bar{H}(\bar{E})$.* It can be easily verified that

$$\tilde{H} = \tilde{P}\bar{P}^{-1}\bar{H}(I + L_H\bar{H})^{-1} \qquad (14.5-3)$$

$$\tilde{E} = \bar{E}(I - L_E\bar{E})^{-1}\bar{P}\tilde{P}^{-1} \qquad (14.5-4)$$

3. *Express M as an LFT of $\bar{H}(\bar{E})$ and compute a bound on $\bar{\sigma}(\bar{H})(\bar{\sigma}(\bar{E}))$ which guarantees robust performance.* Substitute (14.5–3) into (14.5–2) to obtain M as an LFT of $\bar{H}$:

$$M = N_{11}^H + N_{12}^H\tilde{P}\bar{P}^{-1}\bar{H}(I + L_H\bar{H})^{-1}N_{21}^H \qquad (14.5-5)$$

Similarly from (14.5–4)

$$M = N_{11}^E + N_{12}^E\bar{E}(I - L_E\bar{E})^{-1}\bar{P}\tilde{P}^{-1}N_{21}^E \qquad (14.5-6)$$

The following theorem follows directly from Thm. 11.4-1.

Theorem 14.5-1. *The overall system satisfies the robust performance condition $\mu(M) < 1, \forall\omega$ if the individual subsystems satisfy*

$$\sigma(\bar{H}) \le c_H \text{ or } \bar{\sigma}(\bar{E}) \le c_E \qquad \forall\omega \qquad (14.5-7)$$

where at each frequency c_H and c_E solve

$$\mu_{\hat{\Delta}}\begin{pmatrix} N_{11}^H & N_{12}^H\tilde{P}\bar{P}^{-1} \\ N_{21}^H c_H & -L_H c_H \end{pmatrix} = 1 \qquad (14.5-8)$$

$$\mu_{\hat{\Delta}}\begin{pmatrix} N_{11}^E & N_{12}^E \\ \bar{P}\tilde{P}^{-1}N_{21}^E c_E & L_E c_E \end{pmatrix} = 1 \qquad (14.5-9)$$

and μ is computed with respect to the structure $\hat{\Delta} = \text{diag}\{\Delta, C\}$.

We will first use an example to illustrate the construction of the LFT's. Then we will propose a design procedure based on Thm. 14.5-1.

Example 14.5-1 (Robust Performance with Input Uncertainty). In Sec. 11.3.3 we found that robust performance is achieved if and only if $\mu(M) < 1$, where

$$M = \begin{pmatrix} -\tilde{P}^{-1}\tilde{H}\tilde{P}\ell_I & -\tilde{P}^{-1}\tilde{H}w \\ \tilde{E}\tilde{P}\ell_I & \tilde{E}w \end{pmatrix} \qquad (14.5-10)$$

By inspection M may be written as LFT's in terms of $\tilde{H}$ and $\tilde{E}$

$$M = N_{11}^H + N_{12}^H \tilde{H} N_{21}^H = \begin{pmatrix} 0 & 0 \\ \tilde{P}\bar{\ell}_I & Iw \end{pmatrix} + \begin{pmatrix} -\tilde{P}^{-1} \\ -I \end{pmatrix} \tilde{H}(\tilde{P}\bar{\ell}_I \quad Iw) \quad (14.5-11)$$

$$M = N_{11}^E + N_{12}^E \tilde{E} N_{21}^E = \begin{pmatrix} -I\bar{\ell}_I & -\tilde{P}^{-1}w \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \tilde{P}^{-1} \\ I \end{pmatrix} \tilde{E}(\tilde{P}\bar{\ell}_I \quad Iw) \quad (14.5-12)$$

Thus c_H and c_E have to solve

$$\mu_{\hat{\Delta}} \begin{pmatrix} 0 & 0 & -\tilde{P}^{-1} \\ \tilde{P}\bar{\ell}_I & Iw & -\tilde{P}\bar{P}^{-1} \\ \tilde{P}\bar{\ell}_I c_H & Iwc_H & -L_H c_H \end{pmatrix} = 1 \quad (14.5-13)$$

and

$$\mu_{\hat{\Delta}} \begin{pmatrix} -I\bar{\ell}_I & -\tilde{P}^{-1}w & \tilde{P}^{-1} \\ 0 & 0 & I \\ \bar{P}\bar{\ell}_I c_E & \bar{P}\tilde{P}^{-1}wc_E & L_E c_E \end{pmatrix} = 1 \quad (14.5-14)$$

□

14.5.2 Design Procedure

Consistent with the other tests and techniques discussed in this section we will assume that each one of the "loops" is designed separately. We propose the following procedure: find a decentralized controller which yields individual loops ($\bar{H}$ and $\bar{E}$) which are stable and in addition satisfy

1. *Nominal Stability*: Satisfy $\bar{\sigma}(\bar{H}) < \mu^{-1}(L_H)$ (14.4–41) at *all* frequencies or satisfy $\bar{\sigma}(\bar{E}) < \mu^{-1}(L_E)$ (14.4–43) at *all* frequencies. It is *not* allowed to combine (14.4–41) and (14.4–43).
2. *Robust Performance*: At each frequency satisfy *either* $\bar{\sigma}(\bar{H}) \leq c_H$ *or* $\bar{\sigma}(\bar{E}) \leq c_E$ (14.5–7). Combining the two conditions over different frequency ranges is allowed.

Thus, in general, two separate conditions must be satisfied by the individual design: one for nominal stability and one for robust performance. Usually the bound $\bar{\sigma}(\bar{H}) \leq c_H$ is satisfied for high and the bound $\bar{\sigma}(\bar{E}) \leq c_E$ for low frequencies. In some rare cases a single bound can be satisfied over the whole frequency range. Assume $\bar{\sigma}(\bar{H}) \leq c_H, \forall\omega$. Because c_H solves (14.5–8), $\mu_\Delta(N_{11}^H) \leq 1, \forall\omega$. From (14.5–5) this is equivalent to $\mu_\Delta(M(\bar{H} = 0)) \leq 1, \forall\omega$. Thus, in order for $\bar{\sigma}(\bar{H}) \leq c_H, \forall\omega$ the performance requirements have to be such, that the robust performance condition is satisfied for $\bar{H} = 0$. This may be the case if we are interested in *robust stability* only.

Assume on the other hand $\bar{\sigma}(\bar{E}) \leq c_E, \forall\omega$. Because c_E solves (14.5–9), $\mu_\Delta(N_{11}^E) \leq 1, \forall\omega$. From (14.5–6) this is equivalent to $\mu_\Delta(M(\bar{E} = 0)) \leq 1, \forall\omega$. Thus, in order for $\bar{\sigma}(\bar{E}) \leq c_E, \forall\omega$, the uncertainty has to be so small, that the robust performance condition is satisfied for $\bar{E} = 0$. This may be the case if we are interested in *nominal performance* only.

In these extreme cases, it turns out that if the individual stable loops satisfy the robust performance bound (e.g., $\bar{\sigma}(\bar{H}) \leq c_H, \forall\omega$) for the overall system, then its nominal stability is implied automatically; Step 1 of the proposed design procedure can be omitted.

This again can be seen easily from the following argument. If $\bar{\sigma}(\bar{H}) \leq c_H, \forall\omega$, then c_H solves (14.5–8) for all values of ω and $\mu_c(L_H c_H) \leq 1$ or $c_H \leq \mu_c^{-1}(L_H), \forall\omega$. This implies that the robust performance constraint is tighter than the robust stability constraint (14.4–41). A similar derivation involving the sensitivity $\bar{E}$ is possible.

We want to stress that, in general, when neither of the *individual* bounds in (14.5–7) holds over the *whole* frequency range, then nominal stability is *not* implied by robust performance: for nominal stability either one of the two bounds (14.4–41 or 14.4–43) has to be satisfied for *all* frequencies. These bounds cannot be combined over different frequency ranges.

14.5.3 Example

We continue Example 14.5-1 on robust performance with diagonal input uncertainty. Consider the following plant

$$\hat{P} = \frac{1}{1+75s}\begin{pmatrix} -0.878\frac{1-0.2s}{1+0.2s} & 0.014 \\ -1.082\frac{1-0.2s}{1+0.2s} & -0.014\frac{1-0.2s}{1+0.2s} \end{pmatrix} \qquad (14.5-15)$$

Physically, this may correspond to a high-purity distillation column using distillate (D) and boilup (V) as manipulated inputs to control top and bottom composition (Appendix). We want to design a decentralized (diagonal) controller for this plant such that robust performance is guaranteed when there is 10% uncertainty on each manipulated input. The uncertainty and performance weights are

$$\bar{\ell}_I(s) = 0.1 \qquad (14.5-16)$$

$$w(s) = 0.25\frac{7s+1}{7s} \qquad (14.5-17)$$

The weight (14.5–17) implies that we require integral action ($w(0) = \infty$) and allow an amplification of disturbances at high frequencies by a factor of four at most ($w_P(\infty) = 0.25$). A particular sensitivity function which matches the performance bound (14.5–17) exactly for low frequencies and satisfies it easily for high frequencies is $E = \frac{28s}{28s+1}I$. This corresponds to a first order response with time constant 28 min.

Nominal Stability (NS). The nominal model has $\mu(L_H(0)) = 1.11$. Consequently, it is impossible to satisfy the NS-condition (14.4–41).

The NS-condition (14.4–43) for $\bar{\sigma}(\bar{E})$ cannot be satisfied either. Firstly, $\hat{P}$ has one RHP-zero, while the diagonal plant has two. Secondly, the plant is clearly not diagonal dominant at high frequencies, and $\mu(L_E(i\omega))$ is larger than one for $\omega > 4\ \mathrm{min}^{-1}$. The simplest way to get around this problem is to treat the RHP-zeros as uncertainty. This is actually not very conservative, since RHP-zeros limit the achievable performance anyway. To this end define the following new nominal model

$$\tilde{P} = \frac{1}{1+75s}\begin{pmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{pmatrix} \qquad (14.5-18)$$

and include the RHP-zeros in the input uncertainty by using the following new uncertainty weight

$$\bar{\ell}_I(s) = 0.1\frac{5s+1}{0.25s+1} \qquad (14.5-19)$$

$|\bar{\ell}_I(i\omega)|$ reaches a value at one at about $\omega = 2\ \mathrm{min}^{-1}$. This includes the neglected RHP-zeros since the multiplicative uncertainty introduced by replacing $\frac{1-0.2s}{1+0.2s}$ by 1 is $|1 - \frac{1-0.2s}{1+0.2s}|$, which reaches a value of one at about $\omega = 3\ \mathrm{min}^{-1}$.

With the new model (14.5–8) we still cannot satisfy the NS-condition (14.4–41) for $\bar{\sigma}(\bar{H})$. However, the NS-condition (14.4–43) for $\bar{\sigma}(\bar{E})$ is easily satisfied since $\tilde{P}$ and $\bar{P}$ have the same number of RHP-zeros (none), and $\mu(L_E) = 0.743$ at all frequencies. The only restriction this imposes on $\bar{E}$ is that the maximum peaks of $|\bar{\epsilon}_1|$ and $|\bar{\epsilon}_2|$ must be less than $1/0.743 = 1.35$. This is easily satisfied since both $\bar{p}_{11} = \frac{-0.878}{1+75s}$ and $\bar{p}_{22} = \frac{-0.014}{1+75s}$ are minimum phase.

In the remainder of this example the model of the plant ($\tilde{P}$) is assumed to be given by (14.5–18) and the uncertainty weight ($\bar{\ell}_I$) by (14.5–19).

Nominal Performance (NP). From the overall performance requirements

$$NP \Leftrightarrow \bar{\sigma}(\tilde{E}) \leq |w|^{-1} \qquad \forall\omega \qquad (14.5-20)$$

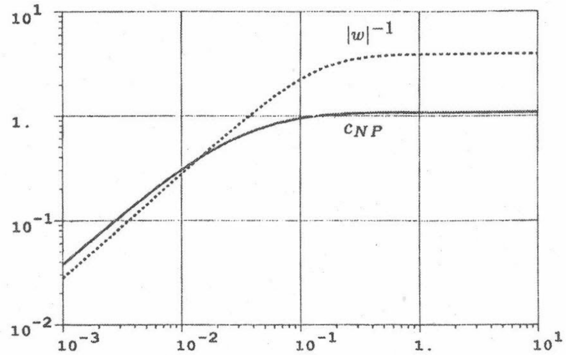

Figure 14.5-1. NP is satisfied if and only if $\bar{\sigma}(\tilde{E}) \leq |w|^{-1}$ which is satisfied if $\bar{\sigma}(\bar{E}) \leq c_{NP}$.

we would expect that the individual loops have to satisfy *at least*, $\bar{\sigma}(\tilde{E}) \leq |w|^{-1}$. However, this is not necessarily the case, as illustrated by the example: Condition (14.5–20) is equivalent to $\mu_\Delta(M) \leq 1$ with $M = w\tilde{E}$ and $\Delta = \Delta_P$ (Δ_P is a full matrix). From (14.5–7) we find the following sufficient condition for NP in terms of $\bar{E}$:

$$NP \Leftarrow \bar{\sigma}(\bar{E}) \leq c_{NP} \qquad \forall\omega \qquad (14.5-21)$$

where c_{NP} solves at each frequency

$$\mu_{\hat{\Delta}}\begin{pmatrix} 0 & wI \\ c_{NP}\bar{P}P^{-1} & c_{NP}L_E \end{pmatrix} = I \qquad (14.5-22)$$

and $\hat{\Delta} = diag\{\Delta_P, C\}$. In our example Δ_P is a "full" 2×2 matrix, and C is a diagonal 2×2 matrix. The bound c_{NP} is shown graphically in Fig. 14.5-1 and is seen to be *larger* than $|w|^{-1}$ for low frequencies. Consequently, the performance of the overall system may be *better* than that of the individual loops — that is, the interactions may improve the performance.

Robust Performance (RP). Bound on $\bar{\sigma}(\bar{H})$. The bound c_H defined by (14.5–8) is shown graphically in Fig. 14.5-2 (μ in (14.5–8) is computed with respect to the structure $\hat{\Delta} = \text{diag}\{\Delta_I, \Delta_P, C\}$, where Δ_I is a diagonal 2×2 matrix, Δ_P is a full 2×2 matrix and C is a diagonal 2×2 matrix). It is clearly *not* possible to satisfy the bound $\bar{\sigma}(\bar{H}) < c_H$ at all frequencies. In particular, we find $c_H < 0$ for $\omega < 0.03\ \text{min}^{-1}$. The reason is that the performance weight $|w| > 1$ in

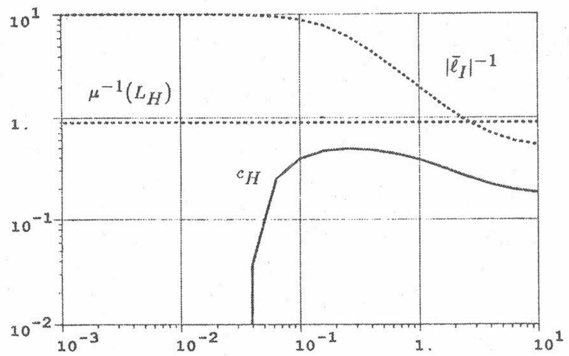

Figure 14.5-2. Bounds $\mu^{-1}(L_H)$ and c_H on $\bar{\sigma}(\bar{H})$.

this frequency range, which means that feedback is required (i.e., $\bar{H} = 0$ is not possible).

Bound on $\bar{\sigma}(\bar{E})$. The bound c_E defined by (14.5–9) is shown graphically in Fig. 14.5-3 (μ is computed with respect to the same structure as above). Again it is not possible to satisfy this bound at all frequencies. In particular, we find $c_E \leq 0$ for $\omega > 2$ min^{-1}. The reason is that the uncertainty weight $|\bar{\ell}_I| > 1$ in this frequency range, which means that perfect control ($\bar{E} = 0$) is not allowed.

Combining bounds on $\bar{\sigma}(\bar{H})$ and $\bar{\sigma}(\bar{E})$. The bound on $\bar{\sigma}(\bar{E})$ is easily satisfied at low frequencies, and the bound on $\bar{\sigma}(\bar{H})$ is easily satisfied at high frequencies. The difficulty is to find an $\bar{E} = I - \bar{H}$ which satisfies either one of the conditions in the frequency range from 0.1 to 1 min^{-1}. The following design is successful (Fig. 14.5-4).

$$\bar{\eta}_1 = \bar{\eta}_2 = \frac{1}{7.5s + 1} \quad , \quad \bar{\epsilon}_1 = \bar{\epsilon}_2 = \frac{7.5s}{7.5s + 1} \qquad (14.5-23)$$

The bound on $|\bar{\epsilon}_i|$ is satisfied for $\omega < 0.3$ min^{-1}, and the bound on $|\bar{\eta}_i|$ for $\omega > 0.23$ min^{-1}. The controller which yields (14.5–23) is

$$C = k\frac{(1 + 75s)}{s}\begin{pmatrix} \frac{-1}{0.878} & 0 \\ 0 & \frac{-1}{0.014} \end{pmatrix}, \quad k = 0.133 \qquad (14.5-24)$$

Because the bounds c_H and c_E are almost flat in the cross-over region, the result is fairly insensitive to the particular choice of controller gain: for $0.06 < k < 0.25$ the design satisfies either $\bar{\sigma}(\bar{E}) < c_E$ or $\bar{\sigma}(\bar{H}) < c_H$ at each frequency and thus has RP.

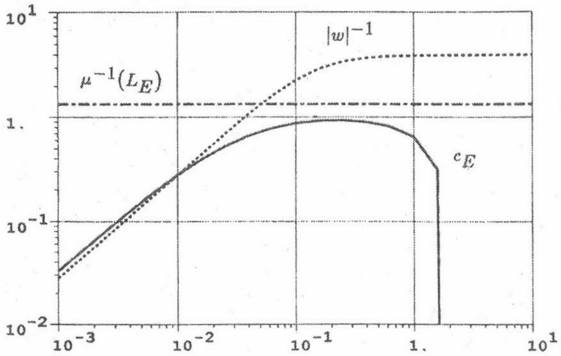

Figure 14.5-3. Bounds $\mu^{-1}(L_E)$ and c_E on $\bar{\sigma}(\bar{E})$.

Figure 14.5-5 shows that RP of the overall system is guaranteed ($\mu(M) \leq 0.63$) as expected from the design procedure. The fact that $\mu(M)$ is much smaller than one, demonstrates some of the conservativeness of conditions (14.5–7) which are only sufficient for RP.

14.6 Summary

Decentralized controllers are popular in practice because they involve fewer tuning parameters than full multivariable control systems and because it is easier to make them fault tolerant. Our approach is to design the controllers for the individual subsystems *independently* but subject to some constraint which guarantees the stability and (robust) performance of the overall system. In order for this to be possible with integral action in all channels the system has to be *Decentralized Integral Controllable* (DIC). The following conditions are *necessary* for DIC (Sec. 14.3.1) and can be used to identify attractive control structure candidates

$$\mathrm{Re}\{\lambda_i(P^+(0)\} \geq 0 \qquad \forall i$$

$$\mathrm{Re}\{\lambda_i(L(0))\} \geq -1 \qquad \forall i$$

$$\mathrm{RGA}: \ \lambda_{ii}(P(0)) \geq 0 \qquad \forall i$$

where

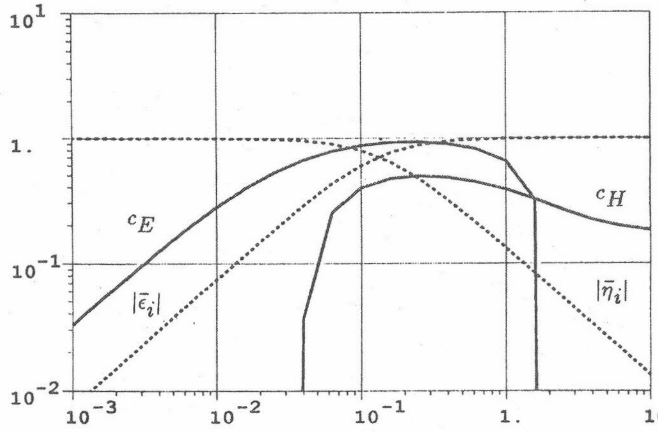

Figure 14.5-4. RP is guaranteed since $|\bar{\epsilon}_i| < c_E$ for $\omega < 0.3$ min^{-1} and $|\bar{\eta}_i| < c_H$ for $\omega > 0.23$ min^{-1}.

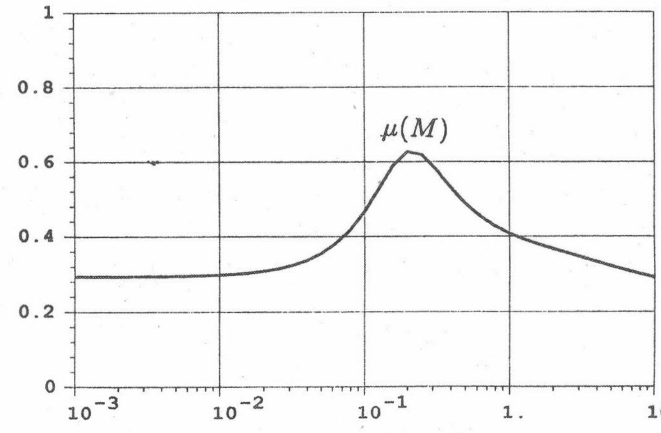

Figure 14.5-5. $\mu(M)$ as a function of frequency. RP of the overall system is guaranteed since $\mu(M) < 1$ for all frequencies.

$$L = (P - \bar{P})\bar{P}^{-1}$$

From different variations of the Small Gain Theorem we derived different bounds on the magnitude of the individual loop transfer functions which, when satisfied, guarantee the stability of the overall system. We call these bounds which constrain the individual designs *interaction measures*. For example, the IMC interaction measure for column dominance requires

$$|\bar{h}_j(i\omega)| < \left[\sum_{i \neq j} |p_{ij}(i\omega)/p_{jj}(i\omega)|\right]^{-1} \quad \forall j, \omega \qquad (14.4-31)$$

and general dominance requires

$$|\bar{h}_j(i\omega)| < \rho^{-1}(|L_H(i\omega)|) \quad \forall j, \omega \qquad (14.4-39)$$

The least conservative norm bound is provided by the μ-interaction measure

$$\bar{\sigma}(\bar{H}_j(i\omega)) < \mu^{-1}(L_H(i\omega)) \quad \forall j, \omega \qquad (14.4-41)$$

Given an uncertainty description in terms of a block diagonal Δ and an H_∞ performance specifications, bounds c_H and c_E of a similar type can be derived on $\bar{\sigma}(\bar{H}_j)$ and $\bar{\sigma}(\bar{E}_j)$

$$\bar{\sigma}(\bar{H}) \leq c_H \text{ or } \bar{\sigma}(\bar{E}) \leq c_E \qquad (14.5-7)$$

They guarantee robust performance of the overall system if the nominal system $\tilde{P}$ with the decentralized controller is stable. Here c_H and c_E are defined implicitly through (14.5–8) and (14.5–9).

14.7 References

14.1. A good discussion of the importance of the pairing problem was presented by Nett & Spang (1987).

14.2. Davison (1976) used a concept similar to integral controllability. The definition of decentralized integral controllability was proposed by Skogestad.

14.3. This section is based on the papers by Grosdidier, Morari & Holt (1985) and Morari (1983b, 1985). The determinant condition for IS (Thm. 14.3-1) was derived independently by Lunze (1982). Eigenvalue conditions similar to that in Thm. 14.3-2 were published by Locatelli et al. (1982) and Lunze (1983, 1985). Condition (c) for DIC was obtained by Mijares et al. (1986) in a very different setting and under more restrictive assumptions. Condition (a) for DIC was stated by Niederlinksi (1971) but the claims about necessity *and* sufficiency in that paper are incorrect. Example 14.3-4 is due to Koppel as cited by McAvoy (1983).

13.3.2. Postlethwaite & MacFarlane (1979) interpret the MIMO Nyquist criterion in terms of characteristic loci, which are used for the proof of Thm. 14.3-2.

14.4. This section follows closely the paper by Grosdidier & Morari (1986).

14.4.3. The stability condition based on the IMC interaction measure was derived by Economou & Morari (1986). Rosenbrock (1974) expresses an entirely equivalent stability condition in terms of "Gershgorin bands".

14.4.4. A proof of the Perron-Frobenius theorem is available from Seneta (1973). The concept of generalized diagonal dominance for control system design was introduced by Mees (1981) and Limebeer (1982).

Both the diagonal dominance and generalized diagonal dominance concept can be extended to block diagonal system (Feingold and Varga, 1962; Limebeer 1982). The bounds on $\|\bar{H}_i(i\omega)\|$ obtained by this approach are excessively conservative and therefore not very useful in most practical applications.

14.4.5. Grosdidier & Morari (1986) discuss how one or the other loop can be given preferential treatment (i.e., tightened) by employing weights in the computation of the μ-interaction measure.

14.4.6. The interaction measure κ_R was defined by Rijnsdorp (1965). Nett & Uthgenannt (1988) determine "optimal" weights for the μ-interaction measure for the 2×2 block case.

14.5. Skogestad & Morari (1988b) developed the design procedure for robust performance.