Part III

CONTINUOUS MULTI-INPUT MULTI-OUTPUT SYSTEMS

Chapter 10

FUNDAMENTALS OF MIMO FEEDBACK CONTROL

In the first section of this chapter some concepts from linear system theory are summarized. The singular value decomposition, which plays a key role in the rest of the book, is covered in detail. The control problem formulation introduced for SISO systems (model uncertainty description, input definition, performance objectives) is generalized to multivariable systems. The Nyquist stability criterion is extended to handle MIMO systems.

10.1 Definitions and Basic Principles

This introductory section is a self-contained summary. For the proofs and a deeper understanding the reader is referred to a basic text on linear systems.

10.1.1 Modeling

In the time domain a linear time invariant finite dimensional system can be described by the system of differential and algebraic equations

$$\dot{x} = Ax + Bu \qquad (10.1-1)$$

$$y = Cx + Du \qquad (10.1-2)$$

where $x \in \mathcal{R}^r, y \in \mathcal{R}^n$, and $u \in \mathcal{R}^m$ are the state, output and input vectors respectively and A, B, C, and D are constant matrices of appropriate dimensions. Taking the Laplace transform of (10.1–1) and (10.1–2) with zero initial conditions

$$sIx(s) = Ax(s) + Bu(s) \qquad (10.1-3)$$

or

$$x(s) = (sI - A)^{-1}Bu(s) \qquad (10.1-4)$$

$$y(s) = Cx(s) + Du(s) \qquad (10.1-5)$$

and substituting (10.1–4) into (10.1–5) we find

$$y(s) = (C(sI - A)^{-1}B + D)u(s) \qquad (10.1-6)$$

where

$$G(s) \triangleq C(sI - A)^{-1}B + D \qquad (10.1-7)$$

is referred to as the *system transfer matrix*. The elements $\{g_{ij}(s)\}$ of $G(s)$ are transfer functions expressing the relationship between specific inputs $u_j(s)$ and outputs $y_i(s)$. In this book, except for proofs and derivations we will use the transfer matrix rather than the state space description.

The matrix $G(s)$ will be assumed to be of full *normal rank* — i.e., rank $[G(s)] = \min\{m, n\}$ for every s in the set of complex numbers $\mathcal{C}$, except for a *finite* number of elements of $\mathcal{C}$.

In general, the elements $\{g_{ij}(s)\}$ will be allowed to include delays. The time domain realization of transfer matrices with delays is complex and will not be addressed here. In order to be physically realizable the transfer matrices have to be proper and causal.

Definition 10.1-1. *A system $G(s)$ is proper if all its elements $\{g_{ij}(s)\}$ are proper and strictly proper if all its elements are strictly proper. All systems which are not proper are improper.*

Definition 10.1-2. *A system $G(s)$ is causal if all its elements $\{g_{ij}(s)\}$ are causal. All systems which are not causal are noncausal.*

10.1.2 Poles

Definition 10.1-3. *The eigenvalues $\pi_i, i = 1, \ldots, n_p$, of the matrix A are called the poles of the system (10.1-1), (10.1-2). The pole polynomial $\pi(s)$ is defined as*

$$\pi(s) = \prod_{i=1}^{n_p}(s - \pi_i) \qquad (10.1-8)$$

Thus the poles are the roots of the *characteristic equation*

$$\pi(s) = 0 \qquad (10.1-9)$$

The poles determine the system's stability.

Theorem 10.1-1. *The system (10.1–1), (10.1–2) is stable if and only if all its poles $\{\pi_i\}$ are in the open left half plane.*

The following theorem allows us to determine the system poles directly from the transfer matrix $G(s)$ without performing a realization and constructing the matrix A first.

Theorem 10.1-2. *The pole polynomial $\pi(s)$ is the least common denominator of all non-identically-zero minors of all orders of $G(s)$.*

Example 10.1-1. Consider the matrix

$$G(s) = \frac{1}{(s+1)(s+2)(s-1)} \begin{pmatrix} (s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1) \end{pmatrix}$$

The minors of order 1 are the elements themselves. The minors of order 2 are

$$G^{1,2}_{1,2} = \frac{1}{(s+1)(s+2)} \qquad G^{1,2}_{1,3} = \frac{2}{(s+1)(s+2)}$$

$$G^{1,2}_{2,3} = \frac{-(s-1)}{(s+1)(s+2)^2}$$

(Here superscripts denote the rows and subscripts the columns used for computation of the minors). Considering the minors of all orders (i.e., orders 1 and 2) we find the least common denominator

$$\pi(s) = (s+1)(s+2)^2(s-1)$$

□

10.1.3 Zeros

Recall that if ζ is a zero of the SISO system $g(s)$ then $g(\zeta) = 0$. Furthermore, we know that ζ is a zero of $g(s)$ if and only if ζ is a pole of $g^{-1}(s)$. The following definition consistently extends this concept of a zero to MIMO systems.

Definition 10.1-4. *ζ is a zero of $G(s)$ if the rank of $G(\zeta)$ is less than the normal rank of $G(s)$.*

In other words, since $G(s)$ is assumed to be of full normal rank, the transfer matrix $G(s)$ becomes rank deficient at the zero $s = \zeta$. The zero polynomial $\zeta(s)$ is defined as

$$\zeta(s) = \prod_{i=1}^{n_z} (s - \zeta_i) \qquad (10.1-10)$$

where n_z is the number of finite zeros of $G(s)$. Thus the zeros are the roots of

$$\zeta(s) = 0 \qquad (10.1-11)$$

The following theorem provides a method for calculating the zeros.

Theorem 10.1-3. *The zero polynomial $\zeta(s)$ is the greatest common divisor of the numerators of all order-r minors of $G(s)$, where r is the normal rank of $G(s)$,* *provided that these minors have all been adjusted in such a way as to have the pole polynomial $\pi(s)$ as their denominator.*

Example 10.1-2. Consider the system from Ex. 10.1-1. and adjust the denominators of all the minors of order 2 to be $\pi(s)$

$$G_{1,2}^{1,2} = \frac{(s-1)(s+2)}{\pi(s)} \qquad G_{1,3}^{1,2} = \frac{2(s-1)(s+2)}{\pi(s)}$$

$$G_{2,3}^{1,2} = \frac{-(s-1)^2}{\pi(s)}$$

and so

$$\zeta(s) = (s-1)$$

□

As Exs. 10.1-1. and 10.1-2. show, MIMO systems can have zeros and poles at the *same location*. Therefore it is generally not possible to find all the zeros of a square system from the condition $\det G(s) = 0$ because, when forming the determinant, zeros and poles at the same location cancel.

Definition 10.1-5. *A system $G(s)$ is nonminimum phase (NMP) if its transfer matrix contains zeros in the RHP or there exists a common time delay term that can be factored out of every matrix element.*

Note that the zero locations of a MIMO system are in no way related to the zero location of the individual SISO transfer functions constituting the MIMO system. Thus, it is possible for a MIMO system to be NMP even when all the SISO transfer functions are MP and vice versa.

Example 10.1-3. The system

$$G(s) = \frac{1}{s+1}\begin{pmatrix} s+3 & 2 \\ 3 & 1 \end{pmatrix}$$

has one finite zero at $s = +3$ though all the SISO transfer functions are MP. □

10.1.4 Vector and Matrix Norms

Let E be a *linear space* over the *field* K (typically K is the field of real $\mathcal{R}$ or complex numbers $\mathcal{C}$). We say that a real valued function $\|\cdot\|$ is a *norm on* E if and only if

$$\|x\| > 0 \quad \forall x \in E, x \neq 0 \qquad (10.1-12a)$$

$$\|x\| = 0 \quad x = 0 \tag{10.1 - 12b}$$

$$\|\alpha x\| = |\alpha| \; \|x\| \quad \forall \alpha \in K, \quad \forall x \in E \tag{10.1 - 13}$$

$$\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in E \tag{10.1 - 14}$$

A norm is a single number measuring the "size" of an element of E. Given a linear space E there may be many possible norms on E. Given the linear space E and a norm $\|\cdot\|$ on E, the pair $(E, \|\cdot\|)$ is called a *normed space.*

In this section let the linear space E be $\mathcal{C}^n$. More precisely $x \in \mathcal{C}^n$ means that $x = (x_1, \; x_2, \; \ldots x_n)$ with $x_i \in \mathcal{C}, \forall i$. Three commonly used norms on $\mathcal{C}^n$ are given by

$$\|x\|_p \triangleq (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{\frac{1}{p}} \quad p = 1, 2, \infty \tag{10.1 - 15}$$

where $\|x\|_\infty$ is interpreted as $\max_i |x_i|$. The norm $\|x\|_2$ is the usual Euclidean length of the vector x.

Let $E = \mathcal{C}^{n \times n}$, the set of all $n \times n$ matrices with elements in $\mathcal{C}$. E is a linear space. The following are norms on $\mathcal{C}^{n \times n}$

$$\|A\|_1 = \max_j \sum_i |a_{ij}| \tag{10.1 - 16}$$

$$\|A\|_\infty = \max_i \sum_j |a_{ij}| \tag{10.1 - 17}$$

$$\|A\|_F = \left[\sum_i \sum_j |a_{ij}|^2\right]^{\frac{1}{2}} \quad \text{Frobenius or Euclidean norm} \tag{10.1 - 18}$$

$$\|A\|_2 = \max_i \lambda_i^{\frac{1}{2}}(A^H A) \quad \text{Spectral norm} \tag{10.1 - 19}$$

where the superscript H is used to denote complex conjugate transpose. The eigenvalues $\lambda_i(A^H A)$ are (necessarily) real and nonnegative. Some useful relationships involving the spectral and Frobenius norms are

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2 \tag{10.1 - 20}$$

where $A \in \mathcal{C}^{n \times n}$. These inequalities follow from the fact that $A^H A$ is positive semidefinite and

$$\max_i \lambda_i(A^H A) \leq \|A\|_F^2 = \text{trace}[A^H A] \leq n \max_i \lambda_i(A^H A) \tag{10.1 - 21}$$

Here we considered matrices as elements of a linear space. Next we shall consider matrices as representation of linear maps and shall relate the matrix norms to the vector norms of the domain and range spaces.

First let us adopt the following notation. From now on we shall use $|\cdot|$ for norms on $\mathcal{R}^n$ or $\mathcal{C}^n$ and $\|\cdot\|$ for norms on function spaces (see Sec. 10.1.6.) or for induced norms of linear operators.

Let $|\cdot|$ be a norm on E and let A be a linear map from E into E. Define the function

$$\|A\| \triangleq \sup_{x\neq 0} \frac{|Ax|}{|x|} \qquad (10.1-22)$$

or equivalently

$$\|A\| \triangleq \sup_{|x|=1} |Ax| \qquad (10.1-23)$$

The quantity $\|A\|$ is called the *induced norm of the linear map A* or the *operator norm induced by the norm* $|\cdot|$.

To interpret (10.1–23) geometrically, consider the set of all vectors of unit length — i.e., the unit sphere. Then $\|A\|$ is the least upper bound on the magnification of the elements of this set by the operator A.

It is easy to show from the definition (10.1–22) that any induced norm satisfies

$$|Ax| \leq \|A\| \cdot |x| \qquad (10.1-24)$$

$$\|\alpha A\| = |\alpha| \cdot \|A\| \qquad (10.1-25)$$

$$\|A+B\| \leq \|A\| + \|B\| \qquad (10.1-26)$$

$$\|AB\| \leq \|A\| \cdot \|B\| \qquad (10.1-27)$$

Any matrix norm $N(\cdot)$ which in addition to the axioms (10.1–12)-(10.1–14) satisfies

$$N(AB) \leq N(A)N(B) \qquad (10.1-28)$$

is called *compatible.* (It is "compatible with itself.") An induced norm is an example of a compatible norm. It can be shown that for every compatible matrix norm $N(\cdot)$ there exists a vector norm $|\cdot|$ such that

$$|Ax| \leq N(A)|x| \qquad (10.1-29)$$

We say that $N(\cdot)$ is an (operator) norm *compatible* with the (vector) norm $|\cdot|$.

It is left as an exercise for the reader to show that the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are operator norms induced by the vector norms $|\cdot|_1$, $|\cdot|_2$ and $|\cdot|_\infty$ respectively. The Frobenius norm $\|\cdot\|_F$ is not an induced norm but it is compatible with $|\cdot|_2$. Also, if $\lambda(A)$ is an eigenvalue of A and x is a corresponding eigenvector, then for compatible matrix and vector norms

$$|Ax| = |\lambda(A)| \cdot |x| \leq \|A\| \cdot |x| \qquad (10.1-30)$$

or

$$|\lambda(A)| \leq \|A\| \qquad (10.1-31)$$

Let $\rho(A)$ be the *spectral radius* of A — i.e.,

$$\rho(A) = \max_i |\lambda_i(A)| \qquad (10.1-32)$$

Because (10.1–31) holds for any eigenvalues of A

$$\rho(A) \leq \|A\| \qquad (10.1-33)$$

Thus the spectral radius forms a lower bound on any compatible matrix norm.

10.1.5 Singular Values and the Singular Value Decomposition

The *singular values* of a complex $n \times m$ matrix A, denoted $\sigma_i(A)$, are the k largest nonnegative square roots of the eigenvalues of $A^H A$ where $k = \min\{n, m\}$, that is

$$\sigma_i(A) = \sqrt{\lambda_i(A^H A)} \qquad i = 1, 2, \ldots, k \qquad (10.1-34)$$

where we assume that the σ_i are ordered such that $\sigma_i \geq \sigma_{i+1}$. In the last section we asked the reader to show that the maximum singular value is the matrix norm induced by the vector norm $|\cdot|_2$ — i.e., the spectral norm. We can define the maximum ($\bar{\sigma}$) and minimum ($\underline{\sigma}$) singular values alternatively by

$$\bar{\sigma}(A) = \max_{x \neq 0} \frac{|Ax|_2}{|x|_2} = \|A\|_2 \qquad (10.1-35)$$

$$\underline{\sigma}(A) = \left[\max_{x \neq 0} \frac{|A^{-1}x|_2}{|x|_2}\right]^{-1} = \|A^{-1}\|_2^{-1} \quad \text{if } A^{-1} \text{ exists} \qquad (10.1-36)$$

$$= \min_{x \neq 0} \left[\frac{|A^{-1}x|_2}{|x|_2}\right]^{-1} = \min_{x \neq 0} \frac{|x|_2}{|A^{-1}x|_2} = \min_{x \neq 0} \frac{|Ax|_2}{|x|_2}$$

Thus $\bar{\sigma}$ and $\underline{\sigma}$ can be interpreted geometrically as the least upper bound and the greatest lower bound on the magnification of a vector by the operator A.

The smallest singular value $\underline{\sigma}(A)$ measures how near the matrix A is to being singular or rank deficient (a matrix is rank deficient if *both* its rows *and* columns are linearly dependent). To see this, consider finding a matrix L of minimum spectral norm that makes $A + L$ rank deficient. Since $A + L$ must be rank deficient there exists a nonzero vector x such that $|x|_2 = 1$ and $(A + L)x = 0$. Thus by (10.1–35) and (10.1–36)

$$\underline{\sigma}(A) \leq |Ax|_2 = |Lx|_2 \leq \|L\|_2 = \bar{\sigma}(L) \qquad (10.1-37)$$

Therefore, L must have spectral norm of at least $\underline{\sigma}(A)$. Otherwise $A + L$ cannot be rank deficient. The property that

$$\underline{\sigma}(A) > \bar{\sigma}(L) \qquad (10.1-38)$$

implies that $A + L$ is nonsingular (assuming square matrices) and will be a basic inequality used in the formulation of various robustness tests.

Definition 10.1-6. *A complex matrix A is Hermitian if $A^H = A$.*

Definition 10.1-7. *A complex matrix A is unitary if $A^H = A^{-1}$.*

A convenient way of representing a matrix that exposes its internal structure is known as the Singular Value Decomposition (SVD). For an $n \times m$ matrix A, the SVD of A is given by

$$A = U\Sigma V^H = \sum_{i=1}^{k} \sigma_i(A) u_i v_i^H \qquad (10.1-39)$$

where U and V are unitary matrices with column vectors denoted by

$$U = (u_1, u_2, \ldots, u_n) \qquad (10.1-40a)$$

$$V = (v_1, v_2, \ldots, v_m) \qquad (10.1-40b)$$

and Σ contains a diagonal nonnegative definite matrix Σ_1 of singular values arranged in descending order as in

$$\Sigma = \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix}; \quad n \geq m$$

or

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 \end{pmatrix}; \quad n \leq m \qquad (10.1-41)$$

and

$$\Sigma_1 = \text{diag}\,\{\sigma_1, \sigma_2, \ldots, \sigma_k\}\,; \qquad k = \min\{m, n\} \qquad (10.1-42)$$

where

$$\bar{\sigma} = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k = \underline{\sigma}$$

It can be shown easily that the columns of V and U are unit eigenvectors of $A^H A$ and AA^H respectively. They are known as the *right* and *left singular vectors* of the matrix A. Trivially all unitary matrices have a spectral norm of unity. Thus by SVD an arbitrary matrix can be decomposed into a "rotation" (V^H) followed by scaling (Σ) followed by a "rotation" (U).

Example 10.1-4. The SVD of the matrix

$$A = \begin{pmatrix} 0.8712 & -1.3195 \\ 1.5783 & -0.0947 \end{pmatrix}$$

is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, V = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$$

It is interpreted geometrically in Fig. 10.1-1. □

Let $\underline{\lambda}(A)$ be the eigenvalue of minimum magnitude of A and $\underline{x}$ the associated eigenvector. Then from (10.1–36) we find

$$\underline{\sigma}(A) = \min_{x \neq 0} \frac{|Ax|_2}{|x|_2} \leq \frac{|A\underline{x}|_2}{|\underline{x}|_2} = |\underline{\lambda}(A)| \qquad (10.1-43)$$

Combining (10.1–33) and (10.1–43) we conclude that $\underline{\sigma}$ and $\bar{\sigma}$ bound the magnitude of the eigenvalues:

$$\underline{\sigma}(A) \leq |\lambda_i(A)| \leq \bar{\sigma}(A) \qquad (10.1-44)$$

If A is Hermitian then the singular values and the eigenvalues coincide.

Define $u_1 = \bar{u}, u_n = \underline{u}, v_1 = \bar{v}, v_m = \underline{v}$. Then if follows that

$$A\bar{v} = \bar{\sigma}\ \bar{u} \qquad (10.1-45)$$

$$A\underline{v} = \underline{\sigma}\ \underline{u} \qquad (10.1-46)$$

From a systems point of view the vector $\bar{v}(\underline{v})$ corresponds to the *input* direction with the largest (smallest) amplification. Furthermore $\bar{u}(\underline{u})$ is the *output* direction in which the inputs are most (least) effective.

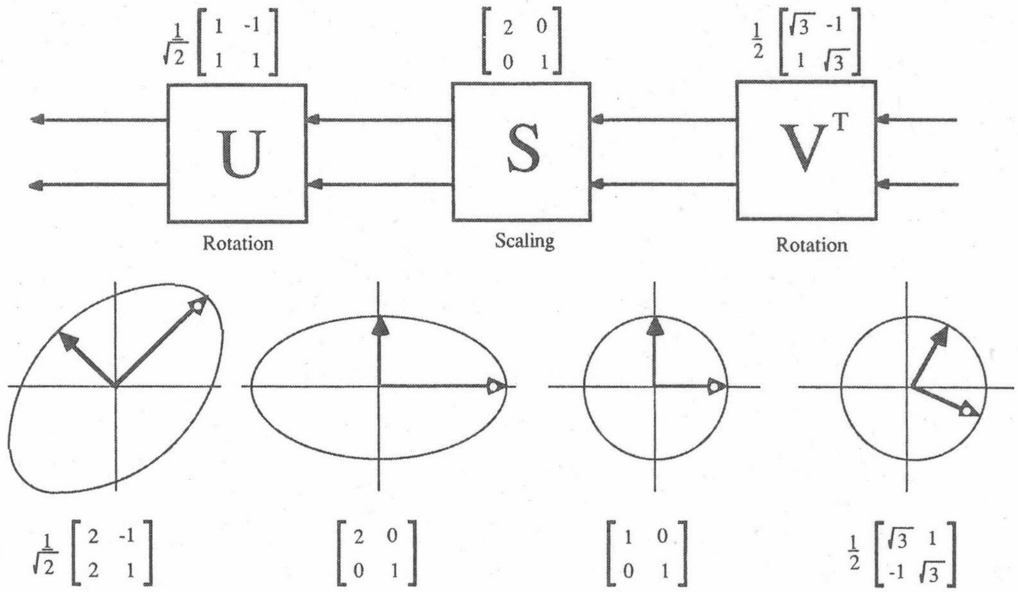

Figure 10.1-1. Geometric interpretation of the SVD for Ex. 10.1-4.

If A is square and nonsingular then

$$A^{-1} = V\Sigma^{-1}U^H \qquad (10.1-47)$$

is the SVD of A^{-1} but with the order of the singular values reversed. Let $\ell = n - j + 1$. Then it follows from (10.1–47) that

$$\sigma_j(A^{-1}) = 1/\sigma_\ell(A) \qquad (10.1-48)$$

$$u_j(A^{-1}) = v_\ell(A) \qquad (10.1-49a)$$

$$v_j(A^{-1}) = u_\ell(A) \qquad (10.1-49b)$$

and in particular

$$\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A) \qquad (10.1-50)$$

$$\bar{u}(A^{-1}) = \underline{v}(A) \qquad (10.1-51a)$$

$$\underline{u}(A^{-1}) = \bar{v}(A) \qquad (10.1-51b)$$

When $G(i\omega)$ is a transfer matrix we can plot the singular values $\sigma_i(G(i\omega))(i = 1, \ldots, k)$ as a function of frequency. These curves generalize the SISO amplitude-ratio Bode plot to MIMO systems. In the MIMO case the amplification of the input vector sinusoid $ue^{i\omega t}$ depends on the *direction* of the complex vector u: the amplification is at least $\underline{\sigma}(G(i\omega))$ and at most $\bar{\sigma}(G(i\omega))$.

10.1.6 Norms on Function Spaces

In this section we will illustrate the extension of the concept of a *norm* to linear spaces whose elements are functions. Let us first consider the vector valued function $y(s)$ of dimension n. We define the set L_2^n to be the set of all vector functions with dimension n, which are square-integrable on the imaginary axis — i.e., for which the following quantity is finite:

$$\|y\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} y(i\omega)^H y(i\omega)d\omega\right]^{\frac{1}{2}} \qquad (10.1-52)$$

Note that (10.1–52) defines the 2-norm of a function $y(s)$ through an *inner product*. For the special case when $y(s)$ has no poles in the closed RHP, Parseval's theorem yields an equivalent time domain expression for the 2-norm of $y(t)$ ($\|y\|_2$):

$$\|y\|_2 = \left[\int_0^\infty y(t)^T y(t) dt\right]^{\frac{1}{2}} \qquad (10.1-53)$$

Assume now that the function $G(s)$ is *matrix valued* with dimension $m \times n$. Then the definition (10.1–52) becomes

$$\|G\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[G(i\omega)^H G(i\omega)\right] d\omega\right]^{\frac{1}{2}} \qquad (10.1-54)$$

where $G(s)$ is in the set $L_2^{m\times n}$ of all matrix valued functions of dimension $m \times n$ for which (10.1–54) is finite. Equation (10.1–54) cannot be interpreted easily in a deterministic setting. We will discuss the implications later in our derivation of the Linear Quadratic Optimal Control problem.

Let us look next at the linear system

$$y(s) = G(s)u(s) \qquad (10.1-55)$$

and pose the following problem: given a bound on $\|u\|_2$ what is the least upper bound on $\|y\|_2$? In other words, we are looking for the operator norm $\|\cdot\|_{i2}$ induced by $\|\cdot\|_2$.

Theorem 10.1-4. *Let $u \in L_2^n$ and $G \in L_2^{m\times n}$. Then $y \in L_2^m$ and the norm of the operator G induced by $\|\cdot\|_2$ is*

$$\|G\|_{i2} = \sup_\omega \bar{\sigma}(G(i\omega)) \triangleq \|G\|_\infty \qquad (10.1-56)$$

where $\|G(i\omega)\|_\infty$ is the ∞-norm of the function G in the frequency domain.

Proof. We will sketch the proof for the SISO case ($y = gu$) and leave the rest as an exercise.

$$\|y\|_2^2 = \|g(i\omega)u(i\omega)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(i\omega)^H g(i\omega)u(i\omega)^H u(i\omega) d\omega$$

$$\leq \sup_\omega |g(i\omega)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} u(i\omega)^H u(i\omega) d\omega$$

Thus

$$\|y\|_2^2 \leq \|g\|_\infty^2 \|u\|_2^2$$

This proves that $\|g\|_\infty$ is an upper bound on the induced norm. To prove that it is in fact the least upper bound and thus equal to the induced norm we have to show that the bound can be reached for a specific u. The specific u is a "sinusoid" modified to be square integrable and occurring at the frequency where $|g(i\omega)|$ is maximum. □

To appreciate the difference between the *2-norm of an operator* and the *operator norm induced by the 2-norm* (∞-norm) we refer the reader back to our discussion of the SISO H_2- and H_∞-optimal control problems (Chap. 2). We found that for H_2-optimal control the performance for a *specific input signal* is optimized which implies the minimization of the weighted 2-norm of the sensitivity operator. For H_∞-optimal control the performance for a *set of 2-norm bounded signals* is optimized. This implies minimization of the norm of the sensitivity operator *induced* by the 2-norm, which we showed to be equal to the ∞-norm of the sensitivity function (Thm. 10.1-4).

10.2 Classic Feedback

10.2.1 Definitions

The block diagram of a typical classic feedback loop is shown in Fig. 10.2-1A. Here C denotes the controller and P the plant transfer function. The transfer function P_d describes the effect of the disturbance d' on the process output y. P_m symbolizes the measurement device transfer function. The measured variable y_m is corrupted by measurement noise n. The controller determines the process input (manipulated variable) u on the basis of the error e. The objective of the feedback loop is to keep y close to the reference (setpoint) r.

Commonly we will use the simplified block diagram in Fig. 10.2-1B. Here d denotes the effect of the disturbance on the output. Exact knowledge of the output y is assumed ($P_m = 1, n = 0$).

10.2.2 Multivariable Nyquist Criterion

Consider the closed loop system in Fig. 10.2-1B when P is square (dim u = dim y). Let the *open* loop transfer matrix $P(s)C(s)$ be described in state space by

$$\dot{x} = A_o x + B_o(-e) \qquad (10.2-1)$$

$$y = C_o x + D_o(-e) \qquad (10.2-2)$$

where

$$e = y - r \qquad (10.2-3)$$

Combining (10.2–1) – (10.2–3) we obtain

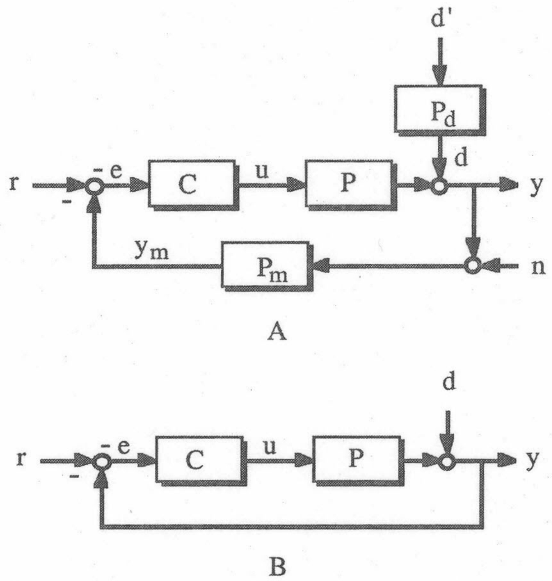

Figure 10.2-1. General (A) and simplified (B) block diagram of feedback control system.

$$\dot{x} = A_c x + B_c r \qquad (10.2-4)$$

$$y = C_c x + D_c r \qquad (10.2-5)$$

where

$$A_c = A_o - B_o(I + D_o)^{-1}C_o \qquad (10.2-6a)$$

$$B_c = B_o(I + D_o)^{-1} \qquad (10.2-6b)$$

$$C_c = (I + D_o)^{-1}C_o \qquad (10.2-6c)$$

$$D_c = (I + D_o)^{-1}D_o \qquad (10.2-6d)$$

We define the *open-loop characteristic polynomial (OLCP)*

$$\phi_{OL} = \det(sI - A_o) \qquad (10.2-7)$$

and the *closed-loop characteristic polynomial (CLCP)*

$$\phi_{CL} = \det(sI - A_c) \qquad (10.2-8)$$

Stability is determined by the zeros of the CLCP. We wish to express the CLCP in terms of $P(s)C(s)$. We define the *return difference operator* $F(s)$

$$F(s) = I + P(s)C(s) \qquad (10.2-9)$$

and state the following lemma.

Lemma 10.2-1 (Schur's formulae for partitioned determinants). *Let the square matrix G be partitioned as*

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

Then the determinant can be expressed as

$$\det G = \det G_{11} \cdot \det(G_{22} - G_{21}G_{11}^{-1}G_{12}) \quad \text{if} \quad \det G_{11} \neq 0 \qquad (10.2-10)$$

or

$$\det G = \det G_{22} \cdot \det(G_{11} - G_{12}G_{22}^{-1}G_{21}) \quad \text{if} \quad \det G_{22} \neq 0 \qquad (10.2-11)$$

The determinant of $F(s)$ can be expressed as

$$\det F(s) = \det(I + C_o(sI - A_o)^{-1}B_o + D_o)$$

or by using (10.2–10)

$$\det F(s) = \det\begin{pmatrix} sI - A_o & B_o \\ -C_o & I + D_o \end{pmatrix} \div \det(sI - A_o)$$

$$= \det\begin{pmatrix} I_r & -B_o(I + D_o)^{-1} \\ 0 & I_n \end{pmatrix} \cdot \det\begin{pmatrix} sI - A_o & B_o \\ -C_o & I + D_o \end{pmatrix} \div \det(sI - A_o)$$

because the first term is unity. Combining the first two matrices we find

$$\det F(s) = \det\begin{pmatrix} sI - A_o + B_o(I + D_o)^{-1}C_o & 0 \\ -C_o & I + D_o \end{pmatrix} \div \det(sI - A_o)$$

or

$$\det F(s) = \frac{\det(sI - A_o + B(I + D_o)^{-1}C_o)}{\det(sI - A_o)} \cdot \det(I + D_o)$$

and because $\lim_{s\to\infty} F(s) = I + D_o$

$$\det F(s) = \frac{\phi_{CL}}{\phi_{OL}} \det F(\infty) \qquad (10.2-12)$$

and finally

$$\phi_{CL} = \frac{\det F(s)}{\det F(\infty)} \phi_{OL} \qquad (10.2-13)$$

If the open-loop system is stable then all RHP zeros of ϕ_{CL} have to be RHP zeros of $\det F(s) = \det(I + P(s)C(s))$ and we can determine stability directly from $\det F(s) = 0$. If the open-loop system is unstable we can generally not do so, because by forming the determinant unstable poles and zeros might cancel as discussed in Sec. 10.1.3. Multiplication by ϕ_{OL} brings back any unstable zeros which are cancelled when $\det F(s)$ is computed. Just as in the SISO case we can apply the principle of the argument to (10.2–12) and derive the multivariable Nyquist Stability Criterion.

Theorem 10.2-1 (Nyquist Stability Criterion). *Let the map of the Nyquist D contour under $\det F(s) = \det(I + P(s)C(s))$ encircle the origin n_F times in the clockwise direction. Let the number of open-loop unstable poles of PC be n_{PC}. Then the closed-loop system is stable if and only if*

$$n_F = -n_{PC}$$

Recall that for SISO systems we generally count the encirclements of (-1,0) by $p(s)c(s)$. This number is equal to the number of encirclements of the origin (0,0) by $1 + p(s)c(s)$.

10.2.3 Internal Stability

The concept of internal stability introduced in Sec. 2.3. applies to MIMO systems as well. For internal stability all elements in the 2×2 block transfer matrix in (10.2–14) have to be stable

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} PC(I+PC)^{-1} & (I+PC)^{-1}P \\ C(I+PC)^{-1} & -C(I+PC)^{-1}P \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \qquad (10.2-14)$$

The Nyquist criterion developed in the last section is another test for internal stability. We will use both tests. Depending on the application one or the other allows us to make conclusions more easily.

10.2.4 Small Gain Theorem

Consider again the closed-loop system in Fig. 10.2-1B when P is square ($\dim u = dim\ y$) and the controller has been included in P so that we can set $C = I$.

Theorem 10.2-2 (Small Gain Theorem). *Assume that $P(s)$ is stable. Let $\rho(P(i\omega))$ be the spectral radius of $P(i\omega)$. Then the closed-loop system is stable if $\rho(P(i\omega)) < 1$, $\forall\omega$ or if $\|P(i\omega)\| < 1$, $\forall\omega$ where $\|\cdot\|$ denotes any compatible matrix norm.*

Proof. (By contradiction) Assume $\rho(P) < 1$, $\forall\omega$ and that the closed-loop system is unstable. We will employ the Nyquist stability criterion (Thm. 10.2-1). Instability implies that the image of $\det(I + P)$ encircles the origin as s traverses the Nyquist D contour. Because the image is closed there exists an $\epsilon \in [0, 1]$ and a frequency ω' such that

$$det(I + \epsilon P(i\omega')) = 0$$

(i.e., that the image goes through the origin).

$$\Leftrightarrow \prod_i \lambda_i(I + \epsilon P(i\omega')) = 0$$

$$\Leftrightarrow 1 + \epsilon\lambda_i(P(i\omega')) = 0 \quad \text{for some i}$$

$$\Leftrightarrow \lambda_i(P(i\omega')) = -\frac{1}{\epsilon} \quad \text{for some i}$$

$$\Rightarrow |\lambda_i(P(i\omega'))| \geq 1 \quad \text{for some i}$$

which is a contradiction because we assumed $\rho(P) < 1, \quad \forall\omega$. □

Theorem 10.2-2 states that for an open-loop stable system, a sufficient condition for stability is to keep the "loop gain" $\rho(P)$ or $\|P\|$ less than unity. It is the multivariable extension of the Bode stability criterion for SISO systems which requires $|p(i\omega)| < 1, \quad \forall\omega$ for closed-loop stability. The Small Gain Theorem provides only a sufficient condition for stability and is therefore potentially conservative. It is useful because it does not require detailed information about the system.

10.3 Formulation of Control Problem

As discussed in Ch. 2, the following essentials have to be specified for any design procedure to yield a control algorithm that works satisfactorily in a real environment.

- process model
- model uncertainty bounds
- type of inputs (i.e., setpoints and disturbances)
- performance objectives

For a MIMO system it is much more difficult to make meaningful specifications than for a SISO system. For SISO systems any design procedure addresses the trade-off between performance and robustness and/or control action. For MIMO systems there is also a performance trade-off among the different outputs as well as a control action trade-off among the different inputs. These trade-offs are affected in a complex manner by the specifications.

10.3.1 Process Model

Because all our *analysis* procedures (e.g., for robust stability and performance) are frequency domain oriented, any linear time invariant model can be handled with equal ease. Models of high order and/or with time delays do not cause any problems. On the other hand, in particular for continuous MIMO systems, controller *synthesis* procedures become extremely complex when time delays are

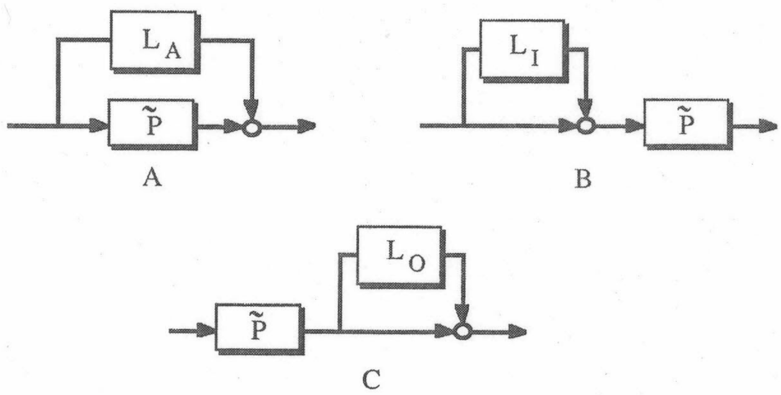

Figure 10.3-1. Additive (A), multiplicative input (B), and multiplicative output (C) uncertainty.

accounted for exactly. This added complexity is not justified by the final results: in practice the performance obtained with controllers based on Padé approximations is generally indistinguishable from that obtained with controllers based on irrational models. Thus while it is very convenient to describe many chemical processes with models involving time delays, it is unnecessarily cumbersome to account for these time delays explicitly during controller *synthesis*.

10.3.2 Model Uncertainty Description

Just as in the SISO case we will describe model uncertainty in the following manner: We will assume that the dynamic behavior of a plant is described not by a single linear time invariant model but by a *family* Π of linear time invariant models. While there are many different ways of parametrizing this family we concluded for SISO systems that a "Nyquist band" consisting of a union of disks of specified radius at each frequency was entirely adequate for most process control applications. This magnitude bounded additive (or multiplicative) uncertainty together with the H_∞-performance specification also allowed us to derive a very simple and exact condition for robust performance.

We can postulate similar uncertainty structures for MIMO systems (Fig. 10.3-1). For MIMO systems we have to distinguish multiplicative uncertainty at the plant input (L_I) and the plant output (L_O). The uncertainties L_I and L_O can be loosely interpreted as actuator and sensor uncertainty respectively. The plant P is related to the model $\tilde{P}$ and the uncertainty in the following manner.

$$P = \tilde{P} + L_A \qquad L_A = P - \tilde{P} \tag{10.3-1}$$

$$P = \tilde{P}(I + L_I) \qquad L_I = \tilde{P}^{-1}(P - \tilde{P}) \tag{10.3-2}$$

$$P = (I + L_O)\tilde{P} \qquad L_O = (P - \tilde{P})\tilde{P}^{-1} \tag{10.3-3}$$

We can state a frequency dependent magnitude bound on these uncertainties in terms of a matrix norm. In principle, any matrix norm defined in Sec. 10.1.4. could be used. As we will see later, however, we can derive *necessary and sufficient* conditions for robust stability and performance only if we use the spectral norm. Thus we can state the following uncertainty bounds:

$$\bar{\sigma}(L_A) \leq \bar{\ell}_A(\omega) \tag{10.3-4}$$

$$\bar{\sigma}(L_I) \leq \bar{\ell}_I(\omega) \tag{10.3-5}$$

$$\bar{\sigma}(L_O) \leq \bar{\ell}_O(\omega) \tag{10.3-6}$$

Note that contrary to the SISO case the three bounds are not equivalent and going from one uncertainty description to another generally increases the size of the family Π. For example, let us assume that we want to derive $\bar{\ell}_O$ from $\bar{\ell}_I$. From (10.3–2) and (10.3–3) we find

$$L_O = \tilde{P} L_I \tilde{P}^{-1} \tag{10.3-7}$$

$$\bar{\sigma}(L_O) = \bar{\sigma}(\tilde{P} L_I \tilde{P}^{-1}) \leq \bar{\sigma}(\tilde{P})\bar{\sigma}(\tilde{P}^{-1})\bar{\sigma}(L_I) \leq \kappa(\tilde{P})\bar{\ell}_I$$

where the first inequality follows from (10.1–27) and

$$\kappa(\tilde{P}) \triangleq \bar{\sigma}(\tilde{P})\bar{\sigma}(\tilde{P}^{-1}) = \frac{\bar{\sigma}(\tilde{P})}{\underline{\sigma}(\tilde{P})} \tag{10.3-8}$$

is the *condition number* of $\tilde{P}$. Thus

$$\bar{\ell}_O \leq \kappa(\tilde{P})\bar{\ell}_I \tag{10.3-9}$$

Note that if one wanted to derive $\bar{\ell}_I$ from $\bar{\ell}_O$, one could write (10.3–7) as

$$L_I = \tilde{P}^{-1} L_O \tilde{P} \tag{10.3-10}$$

and obtain the following upper bound for $\bar{\ell}_I$ by the same procedure:

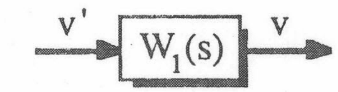

Figure 10.3-2. Input weight W_1 transforming normalized input v' to physical input v.

$$\bar{\ell}_I \leq \kappa(\tilde{P})\bar{\ell}_0 \qquad (10.3-11)$$

If the plant is unitary then $\kappa(\tilde{P}) = 1$ and $\bar{\ell}_O = \bar{\ell}_I$. If the plant is ill-conditioned — i.e., $\kappa(\tilde{P})$ is large — then the bounds formed by the RHS of (10.3–9) or (10.3–11) can be very conservative. Qualitatively, as we pass the uncertainty from the input through the ill-conditioned plant to the output it is strongly stretched in certain directions and the same happens when going from output to input uncertainty. Because it is not "round" anymore, the singular value bound describes it only in a conservative manner.

This example makes clear that for MIMO systems it is important to model uncertainty where it occurs and not necessarily where it is convenient mathematically. From this point of view all three types of uncertainty descriptions above are quite conservative because they "spread" the uncertainty (that might be caused by a single parameter or a single transfer matrix element) over the whole transfer matrix before defining a magnitude bound. In Chap. 11 we will introduce less conservative uncertainty descriptions which are closer to physical reality.

10.3.3 Input Specifications

As in the SISO case we will distinguish specific inputs and input sets. For notational convenience the inputs will be *normalized* (Fig. 10.3-2). It will be assumed that an input v entering the control loop is generated by passing the normalized input v' through a transfer matrix block $W_1(s)$, sometimes referred to as an *input weight* . The two types of *normalized* inputs of interest are

Specific input (vector of impulses):

$$v'(s) = \text{constant} \qquad (10.3-12)$$

Set of bounded inputs (all inputs with 2-norm bounded by unity):

$$\mathcal{V}' = \left\{ v' : \|v'\|_2^2 = \int_{-\infty}^{\infty} v'^H v' d\omega \leq 1 \right\} \qquad (10.3-13)$$

From these definitions follow the actual inputs v

Specific input:

$$v = W_1(s)v' \tag{10.3 – 14}$$

Set of bounded inputs:

$$\mathcal{V} = \left\{ v : \|W_1^{-1}v\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}(W_1^{-1}v)^H(W_1^{-1}v)d\omega \le 1 \right\} \tag{10.3 – 15}$$

In general, for MIMO systems the controller which optimizes performance for a specific input [(10.3–12) or (10.3–14)], is not unique. The reason is that for *one* specific input vector there are many different controller transfer *matrices* which give rise to the same *one* manipulated variable vector and thus to the same optimal output. As we will show later, we can often find a unique controller by requiring it to be optimal in some sense for n different input vectors v'.

The set (10.3–15) can be interpreted as follows: If the spectrum of v is narrow and concentrated near ω^* (i.e., the input looks almost like $\mathrm{Re}\left\{\bar{v}'e^{i\omega^* t}\right\}$, where $\bar{v}'$ is a constant complex vector), then the power of v is limited by $(W_1^{-1}(i\omega^*)\bar{v}')^H(W_1^{-1}(i\omega^*)\bar{v}') \le 1$. We expect W_1 to be large at low frequencies and small at high frequencies. Treating sets of inputs is attractive because at the design stage it is rarely possible to predict exactly what type of setpoint changes and disturbances are going to occur during actual operation. In principle, it is possible that if the input assumed for the design is not exactly equal to the input encountered in practice the performance could deteriorate significantly.

10.3.4 Control Objectives

In order for the controller to work well on the real plant the following objectives have to be met:

- Nominal stability
- Nominal performance
- Robust stability
- Robust performance

Nominal stability was treated in Sec. 10.2. The other objectives are going to be discussed next.

10.4 Nominal Performance

10.4.1 Sensitivity and Complementary Sensitivity Function

The most important relationships between the inputs and outputs in Fig. 10.2-1A ($P_d = P_m = I$) are

$$e = (I + PC)^{-1}(d - r); \quad \text{for } n = 0 \qquad (10.4-1)$$

$$y = PC(I + PC)^{-1}(r - n); \quad \text{for } d = 0 \qquad (10.4-2)$$

We define the sensitivity function

$$E(s) \triangleq (I + PC)^{-1} \qquad (10.4-3)$$

the complementary sensitivity function

$$H(s) \triangleq PC(I + PC)^{-1} \qquad (10.4-4)$$

and the "generic" external input

$$v \triangleq d - r \qquad (10.4-5)$$

For good performance it is desirable to make the sensitivity function as "small" as possible. This is only feasible over a finite frequency range because for strictly proper systems

$$\lim_{s \to +\infty} PC = 0 \qquad (10.4-6)$$

and therefore

$$\lim_{s \to +\infty} E(s) = \lim_{s \to +\infty} (I + PC)^{-1} = I \qquad (10.4-7)$$

Note that

$$E(s) + H(s) = I \qquad (10.4-8)$$

which explains the name *complementary sensitivity function.* Ideally for performance $H(s)$ should be unity but because of (10.4–7) and (10.4–8) this can be achieved only over a finite frequency range.

The trade-offs between good reference following and disturbance suppression ($E \approx 0$) on one hand and suppression of measurement noise on the other ($H \approx 0$) are apparent from (10.4–8). For SISO systems, in addition to measurement noise, multiplicative uncertainty imposes a bound on the complementary sensitivity. We will see later that for MIMO systems, depending on the type of uncertainty, the imposed bound usually takes a much more complex form.

10.4.2 Asymptotic Properties of Closed-Loop Response (System Type)

In analogy to SISO systems, we wish to characterize the asymptotic closed loop response for disturbances/setpoints of the polynomial type (s^{-k}). For MIMO systems the situation is potentially more complicated because we could classify the behavior in each one of the channels separately. We will not do so here but extend the SISO definitions from Sec. 2.4.3. directly.

Definition 10.4-1. *Let $G(s)$ be the $n \times n$ open-loop transfer matrix and let m be the largest integer for which*

$$rank\left[\lim_{s\to 0} s^m G(s)\right] = n$$

Then the system $G(s)$ is said to be of Type m. (Note that $G(s)$ has at least $n \times m$ poles at the origin.)

Theorem 10.4-1. *Let the open-loop system $G(s)$ be of Type m. Then the sensitivity operator $E(s) = (I + G(s))^{-1}$ satisfies*

Type m:

$$\lim_{s\to 0} s^{-k} E(s) = 0 \quad 1 \le k < m \qquad (10.4-9)$$

Assume that the closed loop system is stable. Then as $t \to \infty$ the closed loop system perfectly tracks setpoint changes (perfectly rejects disturbances) of the form $\Sigma_{k=0}^{m} a_k s^{-k}$ where a_k are real constant vectors.

Proof. Follows directly from the Final Value Theorem.

10.4.3 Linear Quadratic (H_2-) Optimal Control

In analogy to the SISO case we could minimize the 2-norm of the error vector

$$\|e\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e(i\omega)^H e(i\omega) d\omega \qquad (10.4-10)$$

for a particular input v. For MIMO systems, however, some modifications are required. First of all, some error components are usually more important than others. Also, we might be primarily interested in rejecting errors in a certain frequency range (for example, for low frequencies). This suggests the introduction of a frequency dependent (output) weight W_2 into the objective function (10.4-10)

$$\|e'\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}(W_2e)^H(W_2e)\, d\omega \qquad (10.4-11)$$

Furthermore, as we explained above, the controller, which solves this (weighted or unweighted) problem is not unique. Therefore we define an alternate problem: "Excite the system in *separate* experiments with n different linearly independent inputs v_i. Find the controller which minimizes the sum of squares of the 2-norms of the errors generated by the n experiments." From (10.4–1) we find that for one experiment the error is $e_i = Ev_i$. Let us define $W_1 = (v_1, v_2, \ldots, v_n)$. Then the columns of EW_1 are the errors from the n experiments. Consider now premultiplication by the output weight W_2 to generate W_2EW_1, the matrix whose columns are the weighted errors e'_i from the n experiments. Then the controller C which minimizes the sum of squares of the weighted error 2-norms is defined implicitly by

$$\min_C \|W_2EW_1\|_2^2 = \min_C \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[(W_2EW_1)^H(W_2EW_1)\right] d\omega \qquad (10.4-12)$$

The H_2-optimal control problem (10.4–12) can be interpreted as the minimization of the average magnitude or, in mathematical terms, the *minimization of the 2-norm of the sensitivity operator E with input weight W_1 and output weight W_2*. It should be compared with the equivalent formula for the SISO case in Sec. 2.4.4. The weighted sensitivity is illustrated by the block diagram in Fig. 10.4-1: The n normalized error vectors e'_i are generated by the n normalized input vectors v'_i. In v'_i only the i^{th} component is unity and all the other components are zero.

The definition of the MIMO H_2-objective must appear somewhat artificial. Though it is reasonable, it is certainty not something a control engineer would naturally formulate. The main motivation for this objective function is that powerful methods are available to minimize the weighted 2-norm of the sensitivity operator as defined by (10.4–12). Also (10.4–12) has a nice stochastic interpretation which will not be discussed in this book. Finally, more meaningful deterministic interpretations of (10.4–12) are available for special cases and are derived in Chap. 12.

The objective function (10.4–12) can be generalized to include, for example, a penalty term for excessive variations of the manipulated variables. This is well known and not discussed here.

The correct choice of weights for a particular practical problem is not trivial. Just as in the SISO case the weights should be regarded as tuning parameters which are chosen by the designer to achieve the best compromise between the conflicting objectives. The weight selection is guided by the expected system

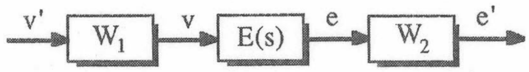

Figure 10.4-1. Sensitivity operator E with input weight W_1 and output weight W_2.

inputs and the relative importance of the outputs. If step setpoint changes for the different outputs are of primary importance then $W_1 = s^{-1}I$ is a reasonable weight. If regulation is more important then $W_1 = P_d$ should yield good performance for a vector d' of impulses. The error weight W_2 should reflect the relative importance of the errors as well as the relevant frequency range. A typical weight which penalizes low-frequency errors (i.e., offset) heavily would be $W_2 = s^{-1}\bar{W}_2$ where $\bar{W}_2$ is a constant diagonal matrix. Note, however, that there is no need to include the factor s^{-1} in *both* W_1 and W_2 if simply no offset to step-like inputs is desired.

10.4.4 H_∞-Optimal Control

The inputs v are assumed to belong to a set of norm-bounded functions with a frequency-dependent weight as discussed in Sec. 10.3.3.

$$\mathcal{V} = \{v : \|W_1^{-1}v\|_2^2 \leq 1\} \qquad (10.4-13)$$

Equivalently we can define the set of normalized inputs $v' = W_1^{-1}v$

$$\mathcal{V}' = \{v' : \|v'\|_2^2 \leq 1\} \qquad (10.4-14)$$

(We refer to Fig. 10.4-1 for an interpretation of the weights.) This input class is much more general than what we considered for the H_2-problem where $v'(t)$ was assumed to consist of impulses.

Each input $v \in \mathcal{V}$ gives rise to an error e. This error is processed through the output weight W_2 (Fig. 10.4-1) which reflects the relative importance of the individual error components and also the frequency range over which the error is to be made small. The controller is to be designed to minimize the worst *normalized* (i.e., weighted) error e' which can result from any input $v \in \mathcal{V}$.

$$\min_C \max_{v\in\mathcal{V}} \|e'\|_2 = \min_C \max_{v'\in\mathcal{V}'} \|W_2 E W_1 v'\|_2 \qquad (10.4-15)$$

From Thm. 10.1-4 we find for $\mathcal{V}'$ defined by (10.4-14)

$$\max_{v' \in \mathcal{V}'} \|W_2 E W_1 v'\|_2 = \sup_\omega \bar{\sigma}(W_2 E W_1(i\omega)) = \|W_2 E W_1\|_\infty \qquad (10.4-16)$$

With (10.4-16) the H_∞-optimal control problem becomes

$$\min_C \|W_2 E W_1\|_\infty = \min_C \sup_\omega \bar{\sigma}(W_2 E W_1(i\omega)) \qquad (10.4-17)$$

Thus, the H_∞-optimal controller minimizes the maximum magnitude or, in mathematical terms, *minimizes the* ∞*-norm* of the sensitivity function E with input weight W_1 and output weight W_2. According to this frequency domain interpretation the H_2-optimal controller minimizes the *average* value and the H_∞-optimal controller the *peak value* of the weighted sensitivity function.

Let us assume for simplicity that W_1 and W_2 are scalar and let the optimum value of the objective function (10-4–17) be k. Then for the optimal controller the sensitivity function satisfies:

$$\|E\|_\infty = \sup_\omega \bar{\sigma}(E(i\omega)) < k|W_1 W_2|^{-1} \qquad (10.4-18)$$

Inequality (10.4–18) implies that the maximum singular value of the sensitivity function lies below the bound $k|W_1W_2|^{-1}$. Typically this bound is selected by the designer to be low at low frequencies and to increase with frequency. Often the designer wishes to specify a minimum bandwidth and to limit the magnitude of the sensitivity operator in order to avoid excessive disturbance amplification. In the H_∞ formulation this can be done explicitly by specifying W_1W_2 accordingly. The H_2 optimal control objective is to minimize the *average* weighted sensitivity and large disturbance amplification is (in principle) possible at certain frequencies. In both cases (H_2 and H_∞), however, the weights are basically tuning parameters selected to reflect input types, relative importance of outputs and desired sensitivity function shapes.

The H_∞ performance requirement is usually written as

$$\|W_2 E W_1\|_\infty < 1 \qquad (10.4-19)$$

where it has been assumed that W_1 and W_2 have been scaled such that a unity bound on the RHS makes sense. For example, in the case (10.4-18), when the weights are scalars either W_1 or W_2 are specified to include k.

Note that for high frequencies $\bar{\sigma}(PC)$ is small and therefore

$$\bar{\sigma}(E) = \bar{\sigma}((I + PC)^{-1}) \cong 1 \qquad \omega \text{ large} \qquad (10.4-20)$$

Thus tight performance specifications are only meaningful in the low frequency range where PC is "large." Then the performance specification (10.4–18) with $k = 1$ reduces to

$$\underline{\sigma}(I + PC) \cong \underline{\sigma}(PC) > |W_1 W_2| \qquad \omega \text{ small} \qquad (10.4-21)$$

In analogy to the SISO case (Sec. 2.4) the smallest loop gain measured by $\underline{\sigma}(PC)$ has to be shaped to fall above the performance weight $|W_1W_2|$.

10.5 Summary

The singular values provide a practical framework for extending the concept of gain to MIMO systems. In particular the gain of the system G depends on the direction of the input vector u but it is bounded by the smallest and largest singular value.

$$\underline{\sigma}(G(i\omega))|u(i\omega)| \leq |G(i\omega)u(i\omega)| \leq \bar{\sigma}(G(i\omega))|u(i\omega)| \qquad (10.1-35,36)$$

The operator norm of G induced by the 2-norm ($\|\cdot\|_2$) is the ∞-norm of the transfer function matrix G:

$$\|G\|_{i2} = \sup_{\omega} \bar{\sigma}(G(i\omega)) \triangleq \|G\|_{\infty} \qquad (10.1-56)$$

If signal magnitude is measured by the 2-norm, then, by definition, a signal passing through the system G is amplified at most by $\|G\|_{\infty}$.

The closed-loop characteristic polynomial (ϕ_{CL}) can be obtained from the open loop characteristic polynomial (ϕ_{OL}) by

$$\phi_{CL} = \frac{\det F(s)}{\det F(\infty)} \phi_{OL} \qquad (10.2-13)$$

where $F(s)$ is the return difference operator

$$F(s) = I + PC(s) \qquad (10.2-9)$$

Closed loop stability can be determined by checking the encirclements of the origin by the map of the Nyquist D contour under $\det F(s)$ (Thm. 10.2-1).

An open-loop stable system is closed-loop stable if the loop gain ($\rho(PC(i\omega))$ or $\|PC(i\omega)\|$) is less than unity for all frequencies ω (Thm. 10.2-2).

For MIMO systems we have to distinguish uncertainty at the process input

$$P = \tilde{P}(I + L_I), \quad L_I = \tilde{P}^{-1}(P - \tilde{P}), \quad \bar{\sigma}(L_I) \leq \bar{\ell}_I(\omega) \qquad (10.2-2,5)$$

and the process output

$$P = (I + L_O)\tilde{P}, \quad L_O = (P - \tilde{P})\tilde{P}^{-1}, \quad \bar{\sigma}(L_O) \leq \bar{\ell}_O(\omega) \qquad (10.3-3,6)$$

In general, it is not possible to convert from one uncertainty description to the other without introducing conservatism.

The controller design techniques aim to make a measure of the sensitivity operator E

$$E = (I + PC)^{-1} \qquad (10.4-3)$$

small. The H_2-optimal controller minimizes the 2-norm ("average") of the sensitivity operator with input weight W_1 and output weight W_2

$$\min_C \|W_2 E W_1\|_2^2 = \min_C \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[(W_2 E W_1)^H (W_2 E W_1)\right] d\omega \qquad (10.4-12)$$

The H_∞-optimal controller minimizes the ∞-norm ("peak") of the weighted sensitivity function

$$\min_C \|W_2 E W_1\|_\infty = \min_C \sup_\omega \bar{\sigma}(W_2 E W_1(i\omega)) \qquad (10.4-17)$$

10.6 References

10.1. Most of these concepts are covered, for example, by Kwakernaak & Sivan (1972).

10.1.2 and 10.1.3. The definitions, theorems, and examples were taken from Postlethwaite & MacFarlane (1979). Alternate definitions of zeros are available and are summarized by Holt & Morari (1985a). Numerically it is most reliable to find the poles by computing the eigenvalues of A and the zeros by solving a generalized eigenvalue problem (Laub & Moore, 1978).

10.1.4–10.1.6. This material is covered comprehensively by Desoer & Vidyasagar (1975) who also prove Thm. 10.1-4. For a limited discussion in the matrix context the reader is referred to Gantmacher (1959) and Bellman (1970). A very good discussion on matrix and vector norms can be found in Stewart (1973), where the term "consistent" instead of "compatible" is used. The physical interpretation of SVD was adopted from Bruns & Smith (1982).

10.2.2. Lemma 10.2-1 can be found in Gantmacher (1959). The derivation of the closed-loop characteristic polynomial was adopted from Postlethwaite & MacFarlane (1979).

10.2.4. Desoer & Vidyasagar (1975) present the Small Gain Theorem in a general context.

10.3.2. The different types of uncertainty descriptions were used by Doyle & Stein (1981).

10.4.2. More general definitions of System Type were proposed by Sandell & Athans (1973) and Wolfe & Meditch (1977).

Chapter 11

ROBUST STABILITY AND PERFORMANCE

Many approaches can be taken to describe the uncertainty associated with a MIMO model for a physical system. It must be emphasized that however attractive an uncertainty description may seem from a practical point of view it is only useful if it permits the derivation of "tight" conditions for robust stability and robust performance. Two types of descriptions will be discussed in this chapter: "unstructured" and "structured" uncertainty. Both lead to "tight" (necessary and sufficient) robustness conditions. The necessity is only meaningful, however, if the assumed uncertainty is an accurate description of the true uncertainty. Otherwise the mathematically tight robustness conditions can be very conservative from a practical point of view (see the discussion of Thm. 2.5-1).

Unstructured Uncertainty. The uncertainty is expressed in terms of a specific *single* perturbation of the type introduced in Sec. 10.3.2. Similar to the SISO case, the conditions for robust stability can then be expressed as bounds on transfer matrices which are directly related to performance [e.g., $\bar{\sigma}(\tilde{H})$ or $\bar{\sigma}(\tilde{E})$]. Though the bounds derived using unstructured uncertainty are necessary and sufficient, they are generally *conservative* from a practical point of view since the actual uncertainty can rarely be lumped into a single norm-bounded perturbation without including many more possible plants than actually needed.

Structured Uncertainty. The individual sources of uncertainty are identified and represented directly – there is no need to lump them together. This generally leads to an uncertainty description with multiple perturbations (Δ_i's). By assuming norm bounds on these uncertainties (e.g., $\bar{\sigma}(\Delta_i) \leq 1$), it is possible to derive necessary and sufficient and, from a practical point of view, *non-conservative* conditions for robustness using the structured singular value μ. One disadvantage of this procedure is that the resulting conditions are not in terms of a simple bound on $\bar{\sigma}(\tilde{H})$ or $\bar{\sigma}(\tilde{E})$, but involve $\mu(M)$ where M may be a complicated function of $\tilde{E}$ and $\tilde{H}$.

To alleviate this problem, we will outline a general technique for deriving *sufficient* robustness conditions which can be expressed in terms of bounds on any arbitrary transfer matrix of interest. Though these conditions are mathematically conservative, they are appealing from an engineering point of view because they allow the designer to see how particular forms of uncertainty restrict, for example, the sensitivity operator.

11.1 Robust Stability for Unstructured Uncertainty

In this section, the uncertainty which may occur in different parts of the system is lumped into one single perturbation L. We refer to this uncertainty as "unstructured." More precisely, "unstructured" uncertainty means that *several* sources of uncertainty are described with a *single* perturbation which is a full matrix with the *same dimensions* as the plant P.

11.1.1 Uncertainty Description

Let $P \in \Pi$ be any member of the set of possible plants Π, and let $\tilde{P} \in \Pi$ denote the nominal model of the plant. To describe unstructured uncertainty the following four single perturbations are commonly used: additive (L_A), multiplicative output (L_O), multiplicative input (L_I), and inverse multiplicative output (L_E) perturbations (Fig. 11.1-1). Some of these were introduced in Sec. 10.3.2.

$$P = \tilde{P} + L_A \quad \text{or} \quad L_A = P - \tilde{P} \qquad (11.1-1)$$

$$P = (I + L_O)\tilde{P} \quad \text{or} \quad L_O = (P - \tilde{P})\tilde{P}^{-1} \qquad (11.1-2)$$

$$P = \tilde{P}(I + L_I) \quad \text{or} \quad L_I = \tilde{P}^{-1}(P - \tilde{P}) \qquad (11.1-3)$$

$$P = (I - L_E)^{-1}\tilde{P} \quad \text{or} \quad L_E = (P - \tilde{P})P^{-1} \qquad (11.1-4)$$

The conditions for robust stability are different depending on which single perturbation is chosen to describe the uncertainty.

In each of the cases above the magnitude of the perturbation L may be measured in terms of a bound on $\bar{\sigma}(L)$

$$\bar{\sigma}(L) \le \bar{\ell}(\omega) \qquad \forall\omega \qquad (11.1-5)$$

where

$$\bar{\ell}(\omega) = \max_{P\in\Pi} \bar{\sigma}(L)$$

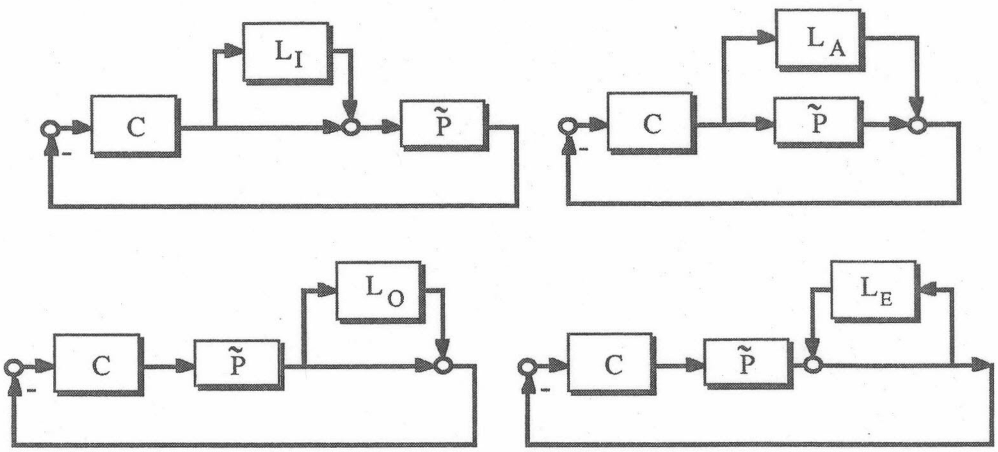

Figure 11.1-1. Four common uncertainty descriptions involving single perturbations: multiplicative input uncertainty (L_I); additive uncertainty (L_A); multiplicative output uncertainty (L_O); inverse multiplicative output uncertainty (L_E).

The bound $\bar{\ell}(\omega)$ can also be interpreted as a scalar *weight* on a normalized perturbation $\Delta(s)$

$$L(s) = \bar{\ell}(s)\Delta(s), \quad \bar{\sigma}(\Delta(i\omega)) \leq 1 \qquad \forall\omega \qquad (11.1-6)$$

Generally the magnitude bound $\bar{\ell}(\omega)$ will *not* constitute a tight description of the "real" uncertainty. This means that the set of plants satisfying (11.1-6) will be larger than the original set Π.

We will also assume that the set of uncertain plants is "connected." This implies that all plants in the set are obtained by continuously deforming the model in the frequency domain — just like the Nyquist bands were generated for SISO systems in Sec. 2.2.2.

11.1.2 General Robust Stability Theorem

When L is of the form (11.1–6) each one of the block diagrams in Fig. 11.1-1 can be put into the form shown in Fig. 11.1-2 where the perturbation Δ satisfies $\bar{\sigma}(\Delta) \leq 1$. (We will demonstrate this in detail in Secs. 11.1.3 through 11.1.5.) If the nominal system is stable then M is stable and Δ is a perturbation which can destabilize the system. The following theorem establishes conditions on M

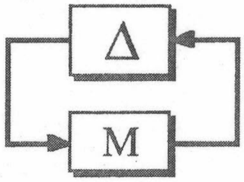

Figure 11.1-2. General $M - \Delta$ structure for robustness analysis.

so that it cannot be destabilized by Δ.

Theorem 11.1-1. *Assume that M is stable and that the perturbation Δ is of such a kind that the perturbed closed-loop system is stable if and only if the map of the Nyquist D contour under* $\det(I - M\Delta)$ *does not encircle the origin. Then the closed-loop system in Fig. 11.1-2 is stable for all perturbations Δ ($\bar{\sigma}(\Delta) \leq 1$) if and only if one of the following three equivalent conditions is satisfied:*

$$\det(I - M\Delta(i\omega)) \neq 0 \qquad \forall\omega, \forall\Delta \ni \bar{\sigma}(\Delta) \leq 1 \qquad (11.1-7)$$

$$\Leftrightarrow \rho(M\Delta(i\omega)) < 1 \qquad \forall\omega, \forall\Delta \ni \bar{\sigma}(\Delta) \leq 1 \qquad (11.1-8)$$

$$\Leftrightarrow \bar{\sigma}(M(i\omega)) < 1 \qquad \forall\omega \qquad (11.1-9a)$$

$$\Leftrightarrow \|M\|_\infty < 1 \qquad (11.1-9b)$$

Proof. Assume there exists a perturbation Δ' such that $\bar{\sigma}(\Delta') \leq 1$ and the image of $\det(I - M\Delta'(s))$ encircles the origin as s traverses the Nyquist contour. Because the Nyquist contour and its map are closed, there exists an $\epsilon \in [0,1]$ and an ω' such that $\det(I - M\epsilon\Delta'(i\omega')) = 0$. Since $\bar{\sigma}(\epsilon\Delta') = \epsilon\bar{\sigma}(\Delta') \leq 1$, $\epsilon\Delta'$ is just another perturbation from the set. Thus the closed-loop system is stable for all perturbations in the set if and only if (11.1–7) is satisfied.

Assume now there exists a perturbation Δ' and a frequency ω' such that $\rho(M\Delta'(i\omega')) < 1$ but that

$$\det(I - M\Delta'(i\omega')) = 0$$

$$\Leftrightarrow \prod_i \lambda_i(I - M\Delta'(i\omega')) = 0$$

$$\Leftrightarrow 1 - \lambda_i(M\Delta'(i\omega')) = 0 \quad \text{for some i}$$

$$\Rightarrow \rho(M\Delta'(i\omega')) \geq 1$$

which is a contradiction. Therefore (11.1–8) is sufficient for robust stability. Because of (10.1–33), (11.1–9) is also sufficient.

To prove necessity of (11.1–8) assume there is a Δ' for which $\bar{\sigma}(\Delta') \leq 1$ and $\rho(M\Delta') = 1$. Then $|\lambda_i(M\Delta')| = 1$ for some i. Δ' can always be chosen such that $\lambda_i(M\Delta') = +1$ and therefore $\det(I - M\Delta') = 0$. To prove necessity of (11.1–9) let $\bar{\sigma}(M) = 1$. Define $D = \text{diag}\{1, 0, \ldots, 0\}$ and $\Delta' = VDU^H$, where U and V are the matrices of the left and right singular vectors of $M (M = U\Sigma V^H)$. Clearly $\bar{\sigma}(\Delta') = 1$ and $\det(I - M\Delta') = \det(I - U\Sigma V^H VDU^H) = \det(I - U\Sigma DU^H) = \det(I - \Sigma D) = 0$. □

Theorem 11.1-1 states that if $\bar{\sigma}(M) < 1$, there is no perturbation Δ ($\bar{\sigma}(\Delta) \leq 1$) which makes $\det(I - M\Delta(s))$ encircle the origin as s traverses the Nyquist D contour. Note that we *assumed* that the absence of encirclements is necessary and sufficient for robust stability. This is the case, for example, when all perturbations Δ are stable or when all members P of the set Π of possible plants have the same number of RHP poles. We will generally assume one or the other. Using more complicated arguments it can be shown that the number of RHP poles may change as long as they appear and disappear by crossing the imaginary axis and not by moving away from or toward RHP zeros. This is also what we meant by a "connected" set of uncertain plants in Sec. 11.1.1. The connectedness condition is very difficult to check however.

In principle we could use a different norm to bound the uncertainty Δ. Assume $\|\Delta\| \leq 1$ where $\|\cdot\|$ is any compatible matrix norm. If Δ is stable, then it follows directly from the Small Gain Theorem (Thm. 10.2-2) that the closed-loop system in Fig. 11.1-2 is stable for all perturbations Δ ($\|\Delta\| \leq 1$) if $\|M\| < 1$. The Small Gain Theorem is only sufficient, however, and therefore potentially conservative. Thus even when $\|M\| = 1$, there is generally no Δ ($\|\Delta\| \leq 1$) which leads to instability. Our objective is to make all tests for robust stability and performance "tight" — i.e., necessary and sufficient. Therefore magnitude bounds on the uncertainty will always be given in terms of the spectral norm.

Next we will use Thm. 11.1-1 to derive conditions for robust stability for the different uncertainty descriptions (11.1–2)-(11.1–4). The derivation for the additive uncertainty is left as an exercise.

11.1.3 Multiplicative Output Uncertainty

Let

$$P = (I + L_O)\tilde{P} \text{ or } L_O = (P - \tilde{P})\tilde{P}^{-1} \qquad (11.1-2)$$

By comparing Fig. 11.1-1 and 11.1-2 we find

$$M = -\tilde{P}C(I + \tilde{P}C)^{-1}\bar{\ell}_O \qquad (11.1-10)$$

Corollary 11.1-1. *Under the assumption of Thm. 11.1-1 the closed-loop system is stable for all perturbations L_O ($\bar{\sigma}(L_O) \leq \bar{\ell}_O$) if and only if*

$$\bar{\sigma}(\tilde{P}C(I + \tilde{P}C)^{-1})\bar{\ell}_O = \bar{\sigma}(\tilde{H})\bar{\ell}_O < 1, \quad \forall\omega \quad \Leftrightarrow \quad \|\tilde{H}\bar{\ell}_O\|_\infty < 1 \qquad (11.1-11)$$

This result is a direct extension of the SISO result expressed through Thm. 2.5-1.

The robust stability condition (11.1-11) can always be satisfied for open loop stable systems since $\tilde{H} = 0$ (no feedback) is always possible. However, good disturbance rejection and good command following require $\tilde{H} \cong I$ (i.e., $\bar{\sigma}(\tilde{H}) \cong 1$). Condition (11.1-11) says that the system has to be "detuned" ($\bar{\sigma}(\tilde{H}) < 1$) at frequencies where $\ell_O(\omega) \geq 1$.

Note that for high frequencies $\tilde{P}C$ is "small"

$$\bar{\sigma}(\tilde{P}C(I + \tilde{P}C)^{-1}) = \underline{\sigma}^{-1}(I + (\tilde{P}C)^{-1}) \cong \bar{\sigma}(\tilde{P}C)$$

and therefore (11.1–11) becomes

$$\bar{\sigma}(\tilde{P}C) < \bar{\ell}_O^{-1} \qquad \omega \text{ large}$$

The design implication is that the controller gain for high frequencies is limited by uncertainty. In analogy to the SISO case (Sec. 2.5) the loop gain $\bar{\sigma}(\tilde{P}C)$ has to be "shaped" to fall below the uncertainty bound $\bar{\ell}_O^{-1}$.

11.1.4 Multiplicative Input Uncertainty

Let

$$P = \tilde{P}(I + L_I) \text{ or } L_I = \tilde{P}^{-1}(P - \tilde{P}) \qquad (11.1-3)$$

By comparing Figs. 11.1-1 and 11.1-2 we find

$$M = -(I + C\tilde{P})^{-1}C\tilde{P}\bar{\ell}_I \qquad (11.1-12)$$

Corollary 11.1-2. *Under the assumption of Thm. 11.1-1 the closed loop system is stable for all perturbations L_I ($\bar{\sigma}(L_I) \leq \bar{\ell}_I$) if and only if*

$$\bar{\sigma}(\tilde{H}_I)\bar{\ell}_I < 1, \quad \forall\omega \quad \Leftrightarrow \quad \|\tilde{H}_I\bar{\ell}_I\|_\infty < 1 \qquad (11.1-13)$$

where

$$\tilde{H}_I = (I + C\tilde{P})^{-1}C\tilde{P} \qquad (11.1-14)$$

$\tilde{H}_I$ is the nominal closed-loop transfer function as seen from the *input* of the plant. It is desirable to have this transfer function close to I in order to reject disturbances affecting the inputs to the plant. However, since performance is usually measured at the output of the plant it may be of interest to use (11.1–13) in order to derive a bound in terms of $\tilde{H}$. To derive this bound $\tilde{P}$ is assumed to be square and the inequality

$$\bar{\sigma}(\tilde{H}_I) = \bar{\sigma}(\tilde{P}^{-1}\tilde{H}\tilde{P}) \leq \bar{\sigma}(\tilde{P}^{-1})\bar{\sigma}(\tilde{H})\sigma(\tilde{P}) = \kappa(\tilde{P})\bar{\sigma}(\tilde{H})$$

is used; the bound for robust stability is:

$$\bar{\sigma}(\tilde{H})\bar{\ell}_I(\omega) < \frac{1}{\kappa(\tilde{P})} \qquad \forall\omega \qquad (11.1-15)$$

Condition (11.1–15) has been used to introduce the condition number $\kappa(\tilde{P})$ as a stability sensitivity measure with respect to input uncertainty, but this is misleading. The condition number enters the stability condition (11.1–15) mainly as the result of the conservative step introduced by going from an input (11.1–13) to an output uncertainty description (11.1–15). For $\kappa(\tilde{P})$ large, (11.1–15) may be arbitrarily conservative even though the uncertainty is tightly described in terms of a norm-bounded input uncertainty such that (11.1–13) is both necessary and sufficient.

Note that for high frequencies $C\tilde{P}$ is "small"

$$\bar{\sigma}(C(I + \tilde{P}C)^{-1}\tilde{P}) = \underline{\sigma}^{-1}(I + (C\tilde{P})^{-1}) \cong \bar{\sigma}(C\tilde{P})$$

and therefore (11.1–13) becomes

$$\bar{\sigma}(C\tilde{P}) < \bar{\ell}_I^{-1} \qquad \omega \text{ large}$$

The design implication is that the controller gain for high frequencies is limited by uncertainty. The loop gain $\bar{\sigma}(C\tilde{P})$, which is generally *not* equal to $\bar{\sigma}(\tilde{P}C)$ (see Sec. 11.1.3), has to be "shaped" to fall below the uncertainty bound $\bar{\ell}_I^{-1}$.

11.1.5 Inverse Multiplicative Output Uncertainty

Let

$$P = (I - L_E)^{-1}\tilde{P} \text{ or } L_E = (P - \tilde{P})P^{-1} \qquad (11.1-4)$$

By comparing Figs. 11.1-1 and 11.1-2 we find

$$M = (I + \tilde{P}C)^{-1}\bar{\ell}_E \qquad (11.1-16)$$

Corollary 11.1-3. *Under the assumption of Thm. 11.1-1 the closed loop system is stable for all perturbations L_E ($\bar{\sigma}(L_E) \leq \bar{\ell}_E$) if and only if*

$$\bar{\sigma}((I + \tilde{P}C)^{-1})\bar{\ell}_E = \bar{\sigma}(\tilde{E})\bar{\ell}_E < 1, \quad \forall\omega \quad \Leftrightarrow \quad \|\tilde{E}\bar{\ell}_E\|_\infty < 1 \qquad (11.1-17)$$

For minimum phase systems the nominal sensitivity function $\tilde{E}$ may be arbitrarily small ("perfect control") and (11.1–17) can always be satisfied. Therefore, condition (11.1–17) seems to imply that for minimum phase systems arbitrarily good nominal performance ($\tilde{E}$ small) is possible regardless of how large the uncertainty is. This is not quite true. The pitfall is that any real system has to be strictly proper, and $E = I$ as well as $\tilde{E} = I$ must be required as $\omega \to \infty$. Consequently, to satisfy (11.1–17) it is necessary that $\bar{\sigma}(L_E) = \bar{\sigma}((P - \tilde{P})P^{-1}) \leq 1$ as $\omega \to \infty$ for all possible P. This condition is usually violated in practice, because the relative order of the actual plant is higher than that of the model.

Corollaries 11.1-1 and 11.1-3 prescribe two fundamentally different ways of handling uncertainty: to guarantee robust stability Cor. 11.1-1 requires the system to be detuned (low gain), while Cor. 11.1-3 requires that the control be tightened (high gain). In practice, it is desirable to combine the two approaches: By tightening the control at low frequencies, better performance is obtained. Eventually, at higher frequencies, the system has to be detuned to guarantee robust stability. In fact, it can be shown that it is possible to combine Cor. 11.1-1 and 11.1-3 over different frequency ranges.

11.1.6 Example: Input Uncertainty for Distillation Column

Consider the distillation column described in the Appendix where the overhead composition is to be controlled at $y_D = 0.99$ and the bottom composition at $x_B = 0.01$ using the distillate D and boilup V as manipulated inputs. By linearizing the nonlinear model at steady state and by assuming that the dynamics may be approximated by a first-order system with time constant $\tau = 75$ min, the following linear model is derived

$$\tilde{P} = \frac{1}{\tau s + 1} \begin{pmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{pmatrix} \qquad (11.1-18)$$

A simple decentralized control system with two PI controllers is chosen

$$C(s) = \frac{1 + \tau s}{s} \begin{pmatrix} -0.15 & 0 \\ 0 & -7.5 \end{pmatrix} \qquad (11.1-19)$$

This controller can be shown to give acceptable nominal performance. Assume there is relative uncertainty of magnitude $w_I(s)$ on *each* manipulated variable:

$$w_I(s) = 0.2\frac{5s+1}{0.5s+1} \qquad (11.1-20)$$

This implies a relative uncertainty of up to 20% in the low frequency range which increases at high frequencies, reaching a value of 1 at $\omega \cong 1$ min^{-1}. This increase with frequency allows for a time delay of about one minute, and may represent the effect of the flow dynamics which were neglected when developing the model. This relative uncertainty can be written in terms of two *scalar* multiplicative perturbations Δ_D and Δ_V.

$$D = (1 + w_I(s)\Delta_D)D_c, \qquad |\Delta_D| \leq 1 \qquad \forall\omega \qquad (11.1-21a)$$

$$V = (1 + w_I(s)\Delta_V)V_c, \qquad |\Delta_V| \leq 1 \qquad \forall\omega \qquad (11.1-21b)$$

Here D and V are the actual inputs, while D_c and V_c are the desired values of the flow rates as computed by the controller. Equations (11.1–21) can be approximated by an "unstructured" single perturbation $L_I = w_I\Delta_I$, where Δ_I is a "full" 2×2 matrix

$$\begin{pmatrix} D \\ V \end{pmatrix} = (I + w_I(s)\Delta_I)\begin{pmatrix} D_c \\ V_c \end{pmatrix}, \qquad \bar{\sigma}(\Delta_I) \leq 1 \qquad \forall\omega \qquad (11.1-22)$$

with $\bar{\ell}_I(\omega) = |w_I(i\omega)|$. Inequality (11.1–13) indicates that robust stability is guaranteed if $\bar{\sigma}(\tilde{H}_I) < 1/\bar{\ell}_I(\omega) \quad \forall\omega$. From Fig. 11.1-3 it is seen that this condition is violated over a wide frequency range. By other means it can be shown, however, that the system is robustly stable. The reason for the conservativeness of condition (11.1–13) in this instance is that the use of unstructured uncertainty (11.1–22) includes plants not included in the "true" uncertainty description (11.1–21). These problems may be avoided by using the structured singular value $\mu(\tilde{H}_I)$ as discussed in Sec. 11.2.

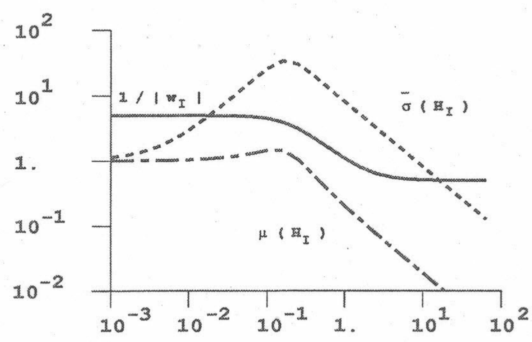

Figure 11.1-3. Robust stability for diagonal input uncertainty is guaranteed since $\mu(H_I) \leq 1/|w_I|$, $\forall\omega$. The use of unstructured uncertainty and $\bar{\sigma}(H_I)$ is conservative. (Reprinted with permission from *Chem. Eng. Sci.*, **42**, 1769 (1987), Pergamon Press, plc.)

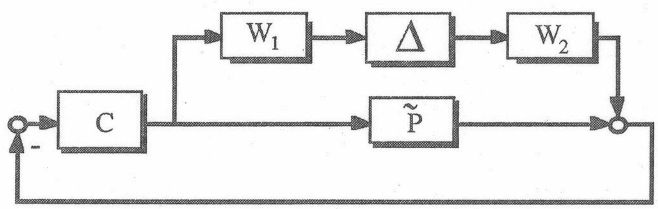

Figure 11.1-4. System with weighted additive uncertainty. Rearranging this system to fit Fig. 11.1-2 gives $M = W_1 C(I + \tilde{P}C)^{-1} W_2$.

11.1.7 Integral Control and Robust Stability

Because of the importance of integral action in the context of process control we will derive specifically conditions under which controllers with integral action can be designed in the presence of uncertainty. We will keep the uncertainty description as general as possible. We define Π_A as the set of plants which is generated by a single weighted additive norm perturbation (Fig. 11.1-4)

$$\Pi_A = \left\{P : P = \tilde{P} + L_A\right\}, \quad L_A = W_2 \Delta W_1, \quad \bar{\sigma}(\Delta) \leq 1 \quad \forall\omega \qquad (11.1-23)$$

Π_A includes additive uncertainty (11.1–1) ($W_1 = \bar{\ell}_A$, $W_2 = I$), multiplicative output uncertainty (11.1–2) ($W_1 = \tilde{P}\bar{\ell}_O$, $W_2 = I$) and multiplicative input uncertainty (11.1–3) ($W_1 = I$, $W_2 = \tilde{P}\ell_I$) as special cases.

Robust stability for a system under "perfect control" ($\tilde{H} = I$, $\forall\omega$) will be studied first. Though "perfect control" cannot be realized in practice it is a useful conceptual tool.

Theorem 11.1-2. (Perfect Control). *Let the set of plants be given by Π_A. Under the assumption of Thm. 11.1-1 robust stability may be achieved for the system under "perfect control" ($\tilde{H} = I$) if and only if*

$$\bar{\sigma}(W_1 \tilde{P}^{-1} W_2) < 1 \qquad \forall\omega \qquad (11.1-24)$$

Proof. Rearranging the block diagram in Fig. 11.1-4 into the form in Fig. 11.1-2 yields

$$M = -W_1 C(I + \tilde{P}C)^{-1} W_2 = -W_1 \tilde{P}^{-1} \tilde{H} W_2 \qquad (11.1-25)$$

□

(11.1–24) follows from (11.1–9) for $\tilde{H} = I$.

Corollary 11.1-4. *For specific choices of weighting matrices (11.1-24) is equivalent to the following:*

Additive Uncertainty:

$$\bar{\ell}_A < \underline{\sigma}(\tilde{P}) \qquad (11.1-26)$$

Multiplicative Uncertainty:

$$\bar{\ell}_O < 1 \text{ or } \bar{\ell}_I < 1 \qquad (11.1-27)$$

Arbitrary Weights:

$$\det P \neq 0 \quad \forall\omega, \quad \forall P \in \Pi_A \qquad (11.1-28)$$

Proof. (11.1–26) and (11.1–27) follow from (11.1–24) by substitution of the appropriate weights. (11.1–28) is a direct consequence of (11.1–7), (11.1–9) and (11.1–24). □

This corollary implies that robust "perfect control" is possible if and only if none of the plants P in the set Π_A has zeros on the imaginary axis (i.e., $\det P \neq 0$). The necessity of this condition is obvious since perfect control ($E = \tilde{E} = 0$) is impossible for plants with RHP zeros. Also, because of the particular norm bounded uncertainty description, RHP zeros can only arise from LHP zeros crossing the imaginary axis. Therefore, checking for zeros on the imaginary axis is sufficient to guarantee that there are no plants with RHP zeros in the set Π_A.

A general condition for robust stability for the set Π_A follows from (11.1–9) with M defined by (11.1–25)

$$\bar{\sigma}(W_1\tilde{P}^{-1}\tilde{H}W_2) < 1 \qquad (11.1-29)$$

For stable plants (11.1–29) can be satisfied simply by setting $\tilde{H} = 0$ (open loop). Integral action implies "perfect control" at steady state and imposes the performance requirement $\tilde{H}(0) = I$. If (11.1–29) is satisfied for $\omega = 0$ with $\tilde{H}(0) = I$ then a controller with sufficient roll-off ($\tilde{H}$ small enough) can always be found such that the system is robustly stable. Thus we have the following theorem.

Theorem 11.1-3 (Integral Control). *Assume all plants $P \in \Pi_A$ are stable. Then robust stability may be achieved for a system under integral control if and only if*

$$\bar{\sigma}(W_1\tilde{P}^{-1}W_2) < 1 \qquad \text{for } \omega = 0 \qquad (11.1-30)$$

or for specific choices of weighting matrices

Additive Uncertainty:

$$\bar{\ell}_A(0) < \underline{\sigma}(\tilde{P}(0)) \qquad (11.1-31)$$

Multiplicative Uncertainty:

$$\bar{\ell}_O(0) < 1 \text{ or } \bar{\ell}_I(0) < 1 \qquad (11.1-32)$$

Arbitrary Weights:

$$\det P(0) \neq 0, \qquad \forall P \in \Pi_A \qquad (11.1-33)$$

Proof. Follows directly from Thm. 11.1-2 and Cor. 11.1-4 for $\omega = 0$. □

Theorem 11.1-3 is the MIMO extension of Cor. 4.3-2. For MIMO systems the requirement that the multiplicative error must not exceed 100% is equivalent to the requirement that the gain matrix must remain nonsingular.

11.2 Robust Stability for Structured Uncertainty

11.2.1 Uncertainty Description

In this section, we will describe the uncertainty in a "structured" manner by identifying the sources and locations of uncertainty in the system. Usually, this leads to an uncertainty description with multiple perturbations (Δ_i). These perturbations may correspond to uncertainty in the model parameters, uncertainty with respect to the manipulated variables (input or actuator uncertainty) and the outputs (measurement uncertainty), etc. By using such a mechanistic approach, we can norm-bound each perturbation (e.g., $\bar{\sigma}(\Delta_i) \leq 1$) without introducing much conservativeness and get a "tight" description of the uncertainty set.

However, we should not necessarily describe the uncertainty as rigorously as possible. Rather, we should take an "engineering approach" and describe the uncertainty only as rigorously as necessary. This means, for example, that some sources of uncertainty (occurring at different places in the system) should be lumped into an "unstructured" multiplicative perturbation, if this does not add much conservativeness. This leads to a *practical uncertainty description*: some sources of uncertainty are described in a "structured" manner (e.g., parametric uncertainty), while the rest (usually uncertain high-frequency dynamics) is lumped into a single "unstructured" perturbation. This will be illustrated through an example later.

Consider the uncertainty as perturbations on the nominal system. Each perturbation Δ_i is assumed to be a *norm-bounded* transfer matrix

$$\bar{\sigma}(\Delta_i) \leq 1 \qquad \forall\omega \qquad (11.2-1)$$

Weighting matrices are used to normalize the uncertainty such that the bound is unity at all frequencies; that is, the actual perturbation L_i is

$$L_i = W_2 \Delta_i W_1 \qquad (11.2-2)$$

If Δ_i represents a real parameter variation we may restrict Δ_i to be real, but in general Δ_i may be any rational transfer matrix satisfying (11.2-1). Just like in Sec. 11.1 the choice of the singular value $\bar{\sigma}$ as the norm for bounding Δ_i is not arbitrary, but is needed to obtain the necessity in the theorems which follow.

The perturbations (uncertainties) which may occur at different places in the feedback system can be collected and placed into one large block diagonal perturbation matrix

$$\Delta = \text{diag}\{\Delta_1, \ldots, \Delta_m\} \qquad (11.2-3)$$

which satisfies

$$\bar{\sigma}(\Delta) \leq 1 \qquad \forall \omega \tag{11.2-4}$$

The blocks Δ_i in (11.2–3) can have any size and may also be repeated. For example, repetition can be needed in order to handle correlations between the uncertainties in different elements. The nominal closed loop system with no uncertainty ($\Delta = 0$) is assumed to be stable. The perturbations (uncertainty) give rise to stability problems because of the additional feedback paths created by the uncertainty. This is shown explicitly by writing the uncertainty as perturbations on the nominal system in the form shown in Fig. 11.1-2. M is the nominal closed-loop system "as seen from" the various uncertainties, and is stable since the nominal system is assumed stable. More precisely, M is the ***interconnection matrix***, the nominal transfer function from the output of the perturbations Δ_i to their inputs. Constructing M is conceptually straightforward, but may be tedious for specific problems. Many practical problems can be cast into the $M - \Delta$ form shown in Fig. 11.1-2 as we will demonstrate through examples. Indeed such a transformation is always possible when the plant is a *linear fractional transformation* of the Δ_i's.

In analogy to the well known scalar case P is a linear fractional transformation of Δ when it is of the form

$$\begin{aligned} P &= N_{11} + N_{12}\Delta(I - N_{22}\Delta)^{-1}N_{21} \\ &= N_{11} + N_{12}(I - \Delta N_{22})^{-1}\Delta N_{21} \end{aligned}$$

where the N_{ij}'s are matrices of appropriate dimension which do not involve Δ and Δ is block diagonal.

11.2.2 Structured Singular Value

Let X_ν be the set of all complex perturbations with a specific block diagonal structure and spectral norm less than ν:

$$X_\nu = \{\Delta = \text{diag}\{\Delta_1, \Delta_2, \ldots \Delta_m\} | \bar{\sigma}(\Delta_i) \leq \nu\} \tag{11.2-5}$$

By following the steps in the proof of Thm. 11.1-1 but with $\Delta \in X_\nu$, it can easily be shown that robust stability is guaranteed if and only if

$$\det(I - M\Delta) \neq 0 \qquad \forall \Delta \in X_\nu \tag{11.2-6}$$

$$\Leftrightarrow \rho(M\Delta) < 1 \qquad \forall \Delta \in X_\nu \tag{11.2-7}$$

or

$$\Leftarrow \nu < \bar{\sigma}^{-1}(M) \qquad (11.2-8)$$

Note that (11.2–8) is only a sufficient condition for (11.2–6). When we proved necessity of the similar condition (11.1–9) we made use of the fact that the perturbation set includes *all* $\Delta(\bar{\sigma}(\Delta) \leq 1)$. Here, however, we restrict the set of permissible Δ's to X_ν. In general (11.2–8) can be arbitrarily conservative. Therefore as an alternative to $\bar{\sigma}$ let us define the structured singular value which takes into account the structure of the perturbation Δ.

Definition 11.2-1. *The function $\mu(M)$, called the Structured Singular Value (SSV) is defined such that $\mu^{-1}(M)$ is equal to the smallest $\bar{\sigma}(\Delta)$ needed to make $(I - M\Delta)$ singular — i.e.*

$$\mu^{-1}(M) = \min_{\nu} \{\nu | \det(I - M\Delta) = 0 \text{ for some } \Delta \in X_\nu\} \qquad (11.2-9)$$

If no Δ exists such that $\det(I - M\Delta) = 0$*, then $\mu(M) = 0$.*

Condition (11.2–6) and Def. 11.2-1 yield the following theorem for robust stability.

Theorem 11.2-1. *Assume that the nominal system M is stable and that the perturbation Δ is of such a kind that the perturbed closed-loop system is stable if and only if the map of the Nyquist D contour under* $\det(I - M\Delta)$ *does not encircle the origin. Then the closed-loop system in Fig. 11.1-2 is stable for all perturbations $\Delta \in X_{\nu=1}$ if and only if*

$$\mu(M(i\omega)) < 1 \qquad \forall \omega \qquad (11.2-10)$$

Theorem 11.2-1 may be interpreted as a "generalized small gain theorem" which also takes into account the *structure* of Δ. The SSV is defined to obtain the tightest possible bound on M such that (11.2–6) is satisfied. It is important to note that $\mu(M)$ depends *both* on the matrix M *and* on the *structure* of the perturbation Δ. $\mu(M)$ is a generalization of the spectral radius $\rho(M)$ and maximum singular value $\bar{\sigma}(M)$: let the perturbations be of the form

$$X_1 = \{\Delta | \Delta = \delta I, |\delta| \leq 1\}$$

Then it is easy to show that $\mu(M) = \rho(M)$. If the perturbations are unstructured (Δ is a full matrix) then $\mu(M) = \bar{\sigma}(M)$ as we know from Thm. 11.1-1.

The definition of μ may be extended by restricting Δ to a smaller set — e.g., real Δ_i's or several identical Δ_i's ("repeated Δ's"). A detailed discussion of these issues is beyond the scope of this book.

Definition 11.2-1 is not in itself useful for computing μ since the optimization problem implied by it does not appear to be easily solvable. Fortunately, several properties of μ can be proven which make it a powerful tool for applications.

Properties of μ

1. $\mu(\alpha M) = |\alpha|\mu(M)$, where α is a scalar.

2. From the discussion above we conclude

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \qquad (11.2-11)$$

3. Let $\mathcal{U}$ be the set of all unitary matrices with the same block diagonal structure as Δ. If $U \in \mathcal{U}$ and $\Delta \in X$, then $U\Delta \in X$ and $\mu(MU) = \mu(M)$. Therefore from (11.2–11)

$$\rho(MU) \leq \mu(M) \qquad \forall U \in \mathcal{U} \qquad (11.2-12)$$

   Indeed it can be shown that

$$\max_{U \in \mathcal{U}} \rho(MU) = \mu(M) \qquad (11.2-13)$$

   This optimization problem is not convex.

4. Let $\mathcal{D}$ be the set of real positive diagonal matrices $D = \text{diag}\{d_i I_i\}$ where the size of each block (i.e., the size of I_i) is equal to the size of the blocks Δ_i. If $D \in \mathcal{D}$ and $\Delta \in X$, then $D\Delta D^{-1} \in X$ and $\mu(DMD^{-1}) = \mu(M)$. Therefore from (11.2–11)

$$\mu(M) \leq \bar{\sigma}(DMD^{-1}) \qquad \forall D \in \mathcal{D} \qquad (11.2-14)$$

   which suggests to determine an upper bound of $\mu(M)$ from

$$\mu(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) \qquad (11.2-15)$$

   It can be shown that the optimization problem is convex and that equality is reached in (11.2-15) for three or fewer blocks. Numerical evidence suggests that the bound (11.2-15) is tight for four or more blocks.

   Extensive numerical experimentation has shown that the minimization of $\|DMD^{-1}\|_F$ yields very good approximations for the optimal D which minimizes $\bar{\sigma}(DMD^{-1})$. This is theoretically justified from the property (10.1–20):

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \bar{\sigma}(A) \leq \|A\|_F$$

where n is the dimension of A. Clearly a significant reduction in $\|DMD^{-1}\|_F$ will result in a significant reduction of $\bar{\sigma}(DMD^{-1})$. Hence the D that minimizes $\|DMD^{-1}\|_F$ is usually a very good approximation of the D that minimizes $\bar{\sigma}(DMD^{-1})$.

Minimization of $\|DMD^{-1}\|_F$. Let m be the number of blocks in Δ. One of the scalars can be kept constant ($d_m = 1$) without loss of generality. Obtain the optimal $d_1, \ldots, d_{m-1}$ as follows: for a specific j $(1 \leq j \leq m-1)$ partition

$$D = \mathrm{diag}\left\{D_j^a, d_j I_j, D_j^c\right\}$$

$$M = \begin{pmatrix} M_j^{a_1} & M_j^{a_2} & M_j^{a_3} \\ M_j^{b_1} & M_j^{b_2} & M_j^{b_3} \\ M_j^{c_1} & M_j^{c_2} & M_j^{c_3} \end{pmatrix}$$

Then

$$DMD^{-1} = \begin{pmatrix} D_j^a M_j^{a_1}(D_j^a)^{-1} & D_j^a M_j^{a_2} d_j^{-1} & D_j^a M_j^{a_3}(D_j^c)^{-1} \\ d_j M_j^{b_1}(D_j^a)^{-1} & M_j^{b_2} & d_j M_j^{b_3}(D_j^c)^{-1} \\ D_j^c M_j^{c_1}(D_j^a)^{-1} & D_j^c M_j^{c_2} d_j^{-1} & D_j^c M_j^{c_3}(D_j^c)^{-1} \end{pmatrix}$$

$$\|DMD^{-1}\|_F^2 = \|D_j^a M_j^{a_1}(D_j^a)^{-1}\|_F^2 + \|D_j^a M_j^{a_3}(D_j^c)^{-1}\|_F^2 + \|M_j^{b_2}\|_F^2$$

$$+\|D_j^c M_j^{c_1}(D_j^a)^{-1}\|_F^2 + \|D_j^c M_j^{c_3}(D_j^c)^{-1}\|_F^2$$

$$+d_j^2\left[\|M_j^{b_1}(D_j^a)^{-1}\|_F^2 + \|M_j^{b_3}(D_j^c)^{-1}\|_F^2\right]$$

$$+d_j^{-2}\left[\|D_j^a M_j^{a_2}\|_F^2 + \|D_j^c M_j^{c_2}\|_F^2\right]$$

$$\triangleq \alpha_j^4 + d_j^2\beta_j^4 + d_j^{-2}\gamma_j^4$$

where $\alpha_j, \beta_j, \gamma_j$ are positive real numbers, independent of d_j.

The optimal D is determined iteratively. Start with some initial guesses for $d_1, \ldots, d_{m-1}$, e.g., $D = I$ or D equal to the optimal D for the previous ω that was considered and set $k = 0$.

For iteration k:

$$j := 1 + \mathrm{mod}(k, (m-1))$$
$$k := k + 1$$
$$d_j := \gamma_j / \beta_j$$

This procedure converges rapidly.

Example 11.2-1. We continue the distillation column example of Sec. 11.1.6. The input uncertainty is expressed through (11.1-21) or equivalently (11.1–22) where the perturbation matrix Δ_I is *diagonal.* The interconnection matrix $M = w_I(s)\tilde{H}_I$ and from Thm. 11.2-1 the system is robustly stable if and only if

$$\mu(\tilde{H}_I) < |w_I(i\omega)|^{-1} = \bar{\ell}_I^{-1}(\omega) \qquad \forall\omega \qquad (11.2-16)$$

where $\mu(\tilde{H}_I)$ is computed with respect to the diagonal matrix Δ_I. From Fig. 11.1-3 we see that (11.2–16) is satisfied and robust stability is guaranteed with the controller (11.1–19). □

11.2.3 Simultaneous Multiplicative Input and Output Uncertainty

Consider the system in Fig. 11.2-1 with both multiplicative input and output uncertainty. The possible plants are given by

$$P = (I + L_O)\tilde{P}(I + L_I) \qquad (11.2-17a)$$

$$L_I = W_{2I}\Delta_I W_{1I}, \qquad \bar{\sigma}(\Delta_I) \le 1 \qquad \forall\omega \qquad (11.2-17b)$$

$$L_O = W_{2O}\Delta_O W_{1O}, \qquad \bar{\sigma}(\Delta_O) \le 1 \qquad \forall\omega \qquad (11.2-17c)$$

The reader should verify that the plant is a linear fractional transformation of the uncertainty $\Delta = \mathrm{diag}\{\Delta_O, \Delta_I\}$:

$$P = N_{11} + N_{12}\Delta(I - N_{22}\Delta)^{-1}N_{21}$$

where

$$N_{11} = \tilde{P}$$
$$N_{12} = \begin{pmatrix} W_{1O} & \tilde{P}W_{1I} \end{pmatrix}$$
$$N_{21} = \begin{pmatrix} W_{2O}\tilde{P} \\ W_{2I} \end{pmatrix}$$
$$N_{22} = \begin{pmatrix} 0 & W_{2O}\tilde{P}W_{1I} \\ 0 & 0 \end{pmatrix}$$

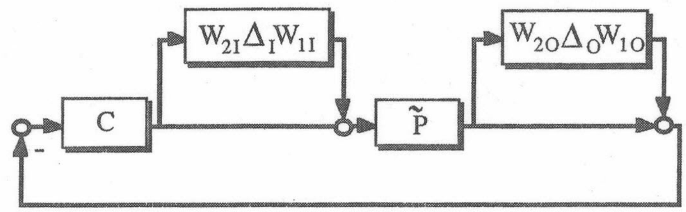

Figure 11.2-1. System with weighted multiplicative input and output uncertainty.

The perturbation block Δ_I represents the multiplicative input uncertainty. If its source is uncertainty in the manipulated variables, then

$$\Delta_I : \text{ diagonal}, \quad W_{1I} = \text{diag}\{w_{Ii}\}, \quad W_{2I} = I \qquad (11.2-18)$$

where w_{Ii} represents the relative uncertainty of each manipulated input.

The block Δ_O represents the multiplicative output uncertainty. If its source is uncertainty or neglected deadtimes involved in one or more of the measurements, then

$$\Delta_O : \text{ diagonal}, \quad W_{1O} = \text{diag}\{w_{Oi}\}, \quad W_{2O} = I \qquad (11.2-19)$$

where w_{Oi} represents the relative uncertainty for each measurement. These sources of input and output uncertainty are present in any plant. Δ_I and Δ_O are restricted to be *diagonal* matrices, since there is little reason to assume that the actuators or measurements influence each other. However, some of the unmodelled dynamics of the plant $\tilde{P}$ itself, which has cross terms, may be approximated by lumping them into Δ_I and Δ_O, thus making either one of them a full matrix.

To examine the constraints on the nominal system imposed by the robust stability requirement for this uncertainty description, let $\Delta = \text{diag}\{\Delta_I, \Delta_O\}$ and rearrange the system in Fig. 11.2-1 into the form in Fig. 11.1-2. The interconnection matrix M becomes:

$$M = \begin{bmatrix} -W_{1I}C\tilde{P}(I+C\tilde{P})^{-1}W_{2I} & -W_{1I}C(I+\tilde{P}C)^{-1}W_{2O} \\ W_{1O}\tilde{P}(I+C\tilde{P})^{-1}W_{2I} & -W_{1O}\tilde{P}C(I+\tilde{P}C)^{-1}W_{2O} \end{bmatrix}$$

$$= \begin{bmatrix} W_{1I} & 0 \\ 0 & W_{1O} \end{bmatrix} \begin{bmatrix} -\tilde{P}^{-1}\tilde{H}\tilde{P} & -\tilde{P}^{-1}\tilde{H} \\ \tilde{E}\tilde{P} & -\tilde{H} \end{bmatrix} \begin{bmatrix} W_{2I} & 0 \\ 0 & W_{2O} \end{bmatrix} \qquad (11.2-20)$$

and robust stability is guaranteed for all Δ such that $\bar{\sigma}(\Delta) < 1$ if and only if $\mu(M) \leq 1, \forall\omega$. μ is computed with respect to the structure of Δ which in turn depends on the structure assumed for Δ_I and Δ_O. Note that (11.1-11) and (11.1-13) follow as special cases when the weights are assumed to be scalar, Δ_I and Δ_O are full matrices and either $\Delta_I = 0$ or $\Delta_O = 0$.

11.2.4 Batch Reactor: Simultaneous Parametric and Unstructured Uncertainty

Consider a perfectly mixed batch reactor where an exothermic reaction is taking place. The reaction temperature T is controlled by the temperature T_c of the fluid in the cooling jacket (the fluid in the cooling jacket may be boiling, and T_c may be adjusted by changing the pressure). A heat balance for the batch reactor gives

$$C_p \frac{dT}{dt} = (-\Delta H_r) r - UA(T - T_c) \qquad (11.2-21)$$

where

T	reactor temperature (K)
T_c	coolant temperature (K)
r	reaction rate (function of T) (mol/s)
ΔH_r	heat of reaction (negative constant) (J/mol)
C_p	total heat capacity of fluid in reactor (J/K)
UA	overall heat transfer coefficient (J/sK)

Linearizing the reaction rate at the operating point T^0

$$r = r^0 + k_T(T - T^0)$$

results in a linear transfer function from T_c to T

$$T(s) = \frac{UA/C_p}{s+a} T_c(s) \qquad (11.2-22)$$

where

$$a = \frac{UA - (-\Delta H_r)k_T}{C_p} \qquad (11.2-23)$$

Two sources of uncertainty will be considered for the linear model (11.2–22): the effect of nonlinearity expressed as uncertainty in the pole location a and neglected high-frequency dynamics.

Pole Uncertainty (Δ_E). Most of the terms in (11.2-23) are nearly constant, except $k_T = \partial r / \partial T$ which is a strong function of temperature. From (11.2-23) we see that the reactor may be open-loop stable ($a > 0$) at low temperatures where k_T is small, and unstable at high temperatures where the reactor is more temperature sensitive. To describe the effect of temperature on a, let

$$|a - \tilde{a}| \leq r_a \tilde{a}$$

where $\tilde{a}$ is the nominal pole location and r_a the relative "uncertainty" of the real constant a. If $r_a > 1$ the plant may be stable or unstable. Equivalently, the possible a's may be written in terms of a norm-bounded perturbation Δ_E

$$a = \tilde{a}(1 + r_a \Delta_E), \qquad |\Delta_E| \leq 1, \qquad \Delta_E \quad \text{real} \qquad (11.2-24)$$

This uncertainty may be expressed as an inverse multiplicative perturbation $(I + w_E \Delta_E)^{-1}$ on the plant

$$\frac{1}{s+a} = \frac{1}{s+\tilde{a}} \cdot \frac{1}{1 + w_E(s)\Delta_E}, \quad w_E(s) = \frac{r_a}{1 + s/\tilde{a}} \qquad (11.2-25)$$

Neglected Dynamics (Δ_O). Uncertainty in the high frequency dynamics cannot be modelled in a "structured" manner using parametric uncertainty. It is most conveniently expressed as multiplicative uncertainty, for example output multiplicative uncertainty $(I + w_O \Delta_O)$. Physically, this uncertainty may account for neglected (and unknown) dynamics for changing the cooling temperature T_c (if T_c is manipulated indirectly with pressure), neglected actuator dynamics (the valve used to control pressure) and neglected dynamics introduced by the heat capacity of the walls.

The following considerations can assist in arriving at a choice for w_0. Naturally $|w_0|$ should be small at low frequencies and increase with frequency. One could view the neglected dynamics as an unknown delay with upper bound $\bar{\delta}$. This would lead to a multiplicative uncertainty of the form (4.6–4) which in turn can be approximated by $w_0 = 2\bar{\delta}s(\bar{\delta}s + 2)^{-1}$ (see Fig. 4.4-2).

A block diagram representation of the uncertainty is depicted in Fig. 11.2-2. Note that in general both blocks (Δ_E and Δ_O) are needed: We *cannot* lump the pole uncertainty (Δ_E) into the output uncertainty (Δ_O) if the pole is allowed to cross the imaginary axis. This would result in $|w_O(j\omega)| \to \infty$ at $\omega = 0$. Similarly we *cannot* lump the output uncertainty into the pole uncertainty. The reason is that the inverse multiplicative uncertainty description (Δ_E) cannot be used to model neglected or uncertain RHP zeros (this would require an unstable perturbation Δ_E). It is therefore not suited for handling neglected high-frequency dynamics which most certainly include RHP zeros.

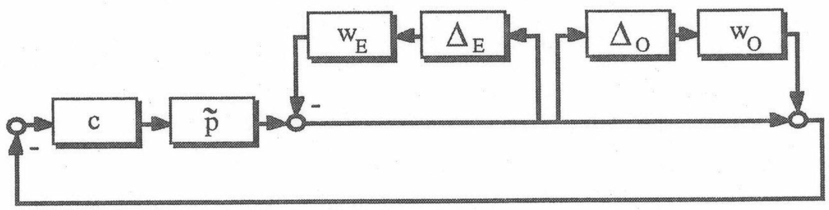

Figure 11.2-2. Reactor control loop with parametric uncertainty represented as inverse multiplicative output uncertainty Δ_E and unstructured multiplicative output uncertainty Δ_O.

Combining the two scalar perturbations into one block perturbation $\Delta = \text{diag}\{\Delta_E, \Delta_O\}$ and rearranging Fig. 11.2-2 to match Fig. 11.1-2 yields the following interconnection matrix:

$$M = \begin{bmatrix} w_E\tilde{\epsilon} & -w_O\tilde{\eta} \\ w_E\tilde{\epsilon} & -w_O\tilde{\eta} \end{bmatrix} \qquad (11.2-26)$$

If, in addition to the real Δ_E's, all *complex* Δ_E's with $|\Delta_E| < 1$ are considered possible, then robust stability is guaranteed if and only if $\mu(M) < 1$ or using Def. 11.2-1 if and only if

$$|w_E\tilde{\epsilon}| + |w_O\tilde{\eta}| < 1 \qquad (11.2-27)$$

Because of the identity $\tilde{\eta} + \tilde{\epsilon} = 1$, this bound is *impossible* to satisfy if $|w_E|$ and $|w_O|$ are both "large" (that is, close to one or larger) over the same frequency range. For $r_a > 1$ the pole may cross the imaginary axis, and $|w_E| > 1$ for $\omega < \omega^* = \tilde{a}\sqrt{r_a^2 - 1}$ and $|w_E| < 1$ for $\omega > \omega^*$. In that situation, robust stability is guaranteed only if the unstructured relative uncertainty given in terms of $|w_O(j\omega)|$ reaches one at a frequency *higher* than ω^*.

If pole uncertainty were the only source of uncertainty ($w_O = 0$), the robust stability bound would be $|\tilde{\epsilon}| < |w_E|^{-1}$. Since the plant is minimum phase, this bound could always be satisfied by increasing the gain and making $\tilde{\epsilon}$ small, regardless of the size of r_a.

In summary, the pole location uncertainty is handled by "tightening" the control at low frequencies. Indeed, $\tilde{\epsilon}$ small ("tight" control) is needed in order to stabilize an unstable plant. To realize robust stability in face of uncertain high-frequency dynamics, however, it is necessary to detune the system and make $\tilde{\eta}$ small ($\tilde{\epsilon} \cong 1$) at frequencies where $w_O(\omega)$ is larger than one. Thus we cannot stabilize an unstable plant if there are RHP-zeros or model uncertainty in the same frequency range as the location of the unstable pole.

The reactor example in this section is meant primarily to illustrate the modelling of uncertainty and the implications of different types of uncertainty on controller tuning. The approximate analysis performed here does not guarantee in any way stability of the nonlinear system (11.2–21).

11.2.5 Independent Uncertainty in the Transfer Matrix Elements

In many cases the uncertainty is most easily described in terms of uncertainties of the individual transfer matrix elements. This kind of uncertainty description may arise from an experimental identification of the system. In general, it is *not* a good representation of the actual sources of uncertainty, but it is included here because it has been proposed in the literature on several occasions.

Let us assume that each element p_{ij} in the plant P is independent, but confined to a disk with radius $a_{ij}(\omega)$ centered at $\tilde{p}_{ij}$ in the Nyquist plane

$$|p_{ij} - \tilde{p}_{ij}| \leq a_{ij} \qquad (11.2-28)$$

or equivalently

$$|p_{ij} - \tilde{p}_{ij}| \leq r_{ij}|\tilde{p}_{ij}| \qquad (11.2-29)$$

where a_{ij} and r_{ij} are the additive and multiplicative (relative) uncertainty respectively. The main limitations of these uncertainty descriptions is that correlations between the elements cannot be handled, which is potentially *very* conservative. Defining the scalar complex perturbation Δ_{ij} (11.2–28) becomes

$$p_{ij} - \tilde{p}_{ij} = \Delta_{ij}a_{ij}, \qquad |\Delta_{ij}| \leq 1 \qquad (11.2-30)$$

or equivalently, in matrix form

$$P - \tilde{P} = \begin{pmatrix} \Delta_{11}a_{11} & \Delta_{12}a_{12} & \cdots \\ \Delta_{21}a_{21} & \cdots & \cdots \\ \cdots & \cdots & \Delta_{nn}a_{nn} \end{pmatrix} \qquad (11.2-31)$$

Introducing weighting matrices W_1 and W_2 it is possible to rewrite (11.2-31) in terms of the "large" diagonal perturbation matrix Δ_e

$$P - \tilde{P} = W_{e2}\Delta_e W_{e1} \qquad (11.2-32)$$

where $W_{e2} \in \mathcal{R}^{n\times n^2}$, $W_{e1} \in \mathcal{R}^{n^2\times n}$ and $\Delta_e \in \mathcal{C}^{n^2\times n^2}$ are defined as

$$W_{e2} = (I \quad I \ldots I), \quad W_{e1} = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \quad a_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix} \qquad (11.2-33)$$

$$\Delta_e = \mathrm{diag}\{\Delta_{11}, \Delta_{21}, \ldots, \Delta_{nn}\}, \qquad |\Delta_{ij}| \le 1$$

A block diagram representation of (11.2–32) is given in Fig. 11.1-4 with $W_2 = W_{e2}$ and $W_1 = W_{e1}$. The interconnection matrix M (Fig. 11.1-2) is $M = -W_{e1}C(I+\tilde{P}C)^{-1}W_{e2} = -W_{e1}\tilde{P}^{-1}\tilde{H}W_{e2}$. Thus we have robust stability for the uncertainty (11.2–30) if and only if

$$\mu(W_{e1}\tilde{P}^{-1}\tilde{H}W_{e2}) < 1 \qquad \forall\omega \qquad (11.2-34)$$

For the special case $\tilde{H} = \tilde{\eta}I$, (11.2-34) provides an explicit bound on the complementary sensitivity

$$\bar{\sigma}(\tilde{H}) = |\tilde{\eta}| < \mu^{-1}(W_{e1}\tilde{P}^{-1}W_{e2}) \qquad \forall\omega \qquad (11.2-35)$$

11.2.6 Condition Number and Relative Gain Array as Sensitivity Measures

We would like to learn for what *class of systems* independent element uncertainty as discussed in the preceding section, imposes severe constraints on the complementary sensitivity [(11.2–34) and (11.2–35)]. We can then determine for each system *a priori* if a detailed analysis of independent element uncertainty and its effect on robust stability is justified. The proofs of all results in this section are omitted because they are straightforward but tedious.

Theorem 11.2-2. (Condition number criterion.) *Assume the nominal response is decoupled, $\tilde{H} = \mathrm{diag}\{\tilde{\eta}_i\}$. Under the assumption of Thm. 11.2-1 robust stability is guaranteed for element uncertainty (11.2–29) if*

$$|\tilde{\eta}_i| < \frac{1}{r_{max}\sqrt{n}\kappa^*(\tilde{P})} \qquad \forall\omega, \quad \forall i \qquad (11.2-36)$$

Here we have used the following definitions:

Maximum relative uncertainty:

$$r_{max} = \max_{ij} r_{ij} \qquad (11.2-37)$$

Minimized condition number:

$$\kappa^*(G) = \min_{D_1,D_2} \kappa(D_1 G D_2) \tag{11.2-38}$$

where D_1 and D_1 are real diagonal matrices. Note that because r_{ij} and r_{max} are independent of the scaling of the system inputs and outputs and because κ^* is obtained by minimizing over all scaling matrices, the inequality (11.2–36) is *scaling invariant.* It indicates that for systems with high condition number κ^* only small errors r_{ij} are allowed. Otherwise robust stability cannot be guaranteed. Because (11.2–36) is only sufficient, a comparison with the necessary and sufficient condition (11.2–35) yields

$$\mu(W_{e1}\tilde{P}^{-1}W_{e2}) \leq r_{max}\sqrt{n}\kappa^*(\tilde{P}) \tag{11.2-39}$$

Numerical experience suggests that this inequality is quite tight. An exact condition is available for 2×2 systems at steady state, as we will show next.

The uncertainty description (11.2–30) assumes that Δ_{ij} are *complex* scalars. This may be reasonable at non-zero frequencies, but does not make any physical sense at steady state ($\omega = 0$) where $\tilde{P}, P$ and Δ_{ij} must be *real.* Conditions (11.2–35) and (11.2–36) may therefore be conservative at $\omega = 0$ where complex perturbations cannot occur. If all perturbations are real and all bounds are equal ($r_{ij} = r \quad \forall i,j$) we find for 2×2 systems:

$$\mu_{real}(W_{e1}\tilde{P}^{-1}W_{e2}) = r\kappa^*(\tilde{P}) \quad (\omega = 0) \tag{11.2-40}$$

We know from Thm. 11.1-3 that for robust integral control for stable systems it is necessary and sufficient to bound the steady state uncertainty. Thus we have the following theorem.

Theorem 11.2-3 (2×2 systems). *Assume that the uncertainties of the elements in $\tilde{P}(0)$ are independent and real and have equal relative magnitude bounds r. Then for open loop stable systems, robust stability and integral control may be achieved if and only if*

$$\kappa^*(\tilde{P}(0)) < r^{-1} \tag{11.2-41}$$

If the magnitude bounds on the relative uncertainties are not equal, and r is replaced by r_{max}, then (11.2–41) provides a sufficient condition for robust stability and integral control. A comparison with (11.1–33) indicates that (11.2–41) implies nonsingularity at steady state.

Theorems 11.2-2 and 11.2-3 give clear interpretations of the minimized condition number as a sensitivity measure: $\kappa^*(\tilde{P}(0))$ and $\kappa^*(\tilde{P}(j\omega))$ are good measures of sensitivity only if the plant uncertainties are given in terms of *independent* (uncorrelated) norm-bounded elements with *equal relative error bounds.* For other uncertainty structures the minimized condition number may be misleading, and bounds on the uncertainties such as (11.2–41) may be arbitrarily conservative. This will be illustrated by a subsequent example.

Conditions (11.2–36) and (11.2–41) provide some insight into the effects of plant ill-conditioning but from a numerical point of view they are hardly more convenient than the general condition (11.2–35) because they involve nonconvex optimization problems. Fortunately, accurate bounds on κ^* can be obtained from the Relative Gain Array.

Relative Gain Array (RGA). The RGA Λ of a matrix M is defined as

$$\Lambda(M) = M \times (M^{-1})^T \qquad (11.2-42)$$

where $\times$ denotes the element-by-element (Schur) product. If M is a transfer matrix then $\Lambda(M)$ is a function of frequency. It can be easily shown that $\Lambda(M)$ has the following properties: all rows and columns of Λ sum to one

$$\sum_i \lambda_{ij} = \sum_j \lambda_{ij} = 1 \qquad (11.2-43)$$

and Λ is independent of scaling

$$\Lambda(D_1 M D_2) = \Lambda(M) \qquad (11.2-44)$$

where D_1 and D_2 are arbitrary nonsingular diagonal matrices. Also a permutation of rows (columns) of M leads to the same permutation of rows (columns) of $\Lambda(M)$. When the argument M is omitted in $\Lambda(M)$ we generally mean the RGA of the plant, i.e., $\Lambda = \Lambda(P)$. When we speak of a "system M with a large RGA" we mean that some norm of $\Lambda(M)$ is large.

The following inequalities show that plants with large elements in the RGA are *always* ill-conditioned

$$\kappa(P) \geq \kappa^*(P) \geq \|\Lambda\|_m - 1/\kappa^*(P) \geq \|\Lambda\|_m - 1 \qquad (11.2-45)$$

where

$$\|\Lambda\|_m = 2 \cdot \max\{\|\Lambda\|_1, \|\Lambda\|_\infty\} \qquad (11.2-46)$$

Vice versa, a large value of $\kappa^*(P)$ implies large elements in the RGA. At least for 2×2 systems we have

$$\kappa^*(P) \leq \|\Lambda\|_m \qquad (11.2-47)$$

and it is conjectured that a similar inequality holds for larger systems. We can combine (11.2–45) and (11.2–47) to show the close relationship between κ^* and Λ at least for 2×2 systems

$$\|\Lambda\|_m - \frac{1}{\kappa^*(P)} \le \kappa^*(P) \le \|\Lambda\|_m \qquad (11.2-48)$$

Example 11.2-2. Let us examine again the distillation column introduced in Sec. 11.1.6 but with reflux L and boilup V as manipulated inputs. The steady-state gain matrix is

$$\tilde{P}(0) = \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \qquad (11.2-49)$$

and

$$\lambda_{11} = 35.07, \quad \|\Lambda\|_m = 138.275, \quad \kappa^*(\tilde{P}) = 138.268, \quad \kappa(\tilde{P}) = 141.7$$

From the high condition number $\kappa^*(\tilde{P})$, one might conclude that the plant may become singular for very small perturbations. This would be true if the uncertainty had the form of independent element errors, but not necessarily otherwise. To illustrate this point consider conditions for using integral control ($\tilde{H}(0) = I$) under two different assumptions about the uncertainty.

Case 1. The elements are assumed independent and norm bounded with equal relative error r. Theorem 11.1-3 and (11.2–35) imply that robust stability with integral control may be achieved if and only if $\mu(W_{e1}\tilde{P}^{-1}W_{e2}) < 1$ for $\omega = 0$, where μ is computed with respect to the *real* perturbation matrix Δ_e.

$$W_{e2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad W_{e1} = r\begin{pmatrix} 0.878 & 0 \\ 1.082 & 0 \\ 0 & 0.864 \\ 0 & 1.096 \end{pmatrix}$$

$$W_{e1}\tilde{P}^{-1}W_{e2} = r\begin{pmatrix} 35.07 & -27.65 & 35.07 & -27.65 \\ 34.07 & -27.65 & 34.07 & -27.65 \\ 43.22 & -34.07 & 43.22 & -34.07 \\ 43.22 & -35.07 & 43.22 & -35.07 \end{pmatrix}$$

This gives $\mu_{real}(W_{e1}\tilde{P}^{-1}W_{e2}) = 138.268r$ which is equal to $r\kappa^*(P)$ as expected from (11.2–40). From Thm. 11.2-3 robust stability with integral action is possible if and only if $r < \kappa^*(P)^{-1} = 0.0072$. In practice, the variation in each element (mainly due to nonlinearities) is much larger than 0.7%, and integral control does not seem to be possible for this distillation column according to this analysis.

Case 2. A more realistic uncertainty description for this high purity distillation column is the following additive uncertainty

$$P - \tilde{P} = \begin{pmatrix} d & -d \\ -d & d \end{pmatrix}$$

which may be written as in Fig. 11.1-4 in terms of one real scalar Δ-block with

$$P - \tilde{P} = W_2 \Delta W_1, \quad W_2 = |d| \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad W_1 = (1 \quad -1), \quad |\Delta| \le 1$$

This structure of the uncertainty arises from the material balance constraints which cannot be violated. Using Thm. 11.1-3, robust stability and integral control ($\tilde{H}(0) = I$) are possible if and only if $\bar{\sigma}(W_1 \tilde{P}^{-1} W_2) < 1$ for $\omega = 0$. Here $W_1 \tilde{P}^{-1} W_2 = 0 \cdot |d|$ and therefore robust stability and integral control are possible for *any value of* d and the elements may even change sign without causing stability problems. Thus, despite the high condition number, the system is not at all sensitive to this physically-motivated model error. □

11.3 Robust Performance

11.3.1 H_∞-Performance Objective

We require that the performance objective defined in Sec. 10.4.4 be satisfied for all plants P in the uncertainty set Π

$$\|W_2 E(P) W_1\|_\infty = \sup_\omega \bar{\sigma}(W_2 E(P) W_1) < 1 \quad \forall P \in \Pi \qquad (11.3-1)$$

Note that W_1 and W_2 are the *performance weights* and are entirely unrelated to the *uncertainty weights* in Sec. 11.2 for which the same symbols were used. In order to be able to evaluate (11.3–1) we assume the uncertainty to be norm bounded and of the form introduced in Sec. 11.2.1. Thus after appropriate scaling it can be expressed as a block diagonal matrix

$$\Delta_u = \text{diag}\{\Delta_1, \ldots, \Delta_m\} \qquad (11.3-2)$$

which satisfies

$$\bar{\sigma}(\Delta_u) \le 1 \qquad \forall \omega \qquad (11.3-3)$$

(Here the subscript u stands for "uncertainty.") In a procedure similar to the one for constructing M in Sec. 11.2.1 we can construct the matrix G shown in Fig. 11.3-1A. The input vector consists of the outputs from the uncertainty block Δ_u and the normalized inputs v'. The output vector is formed by the inputs to the

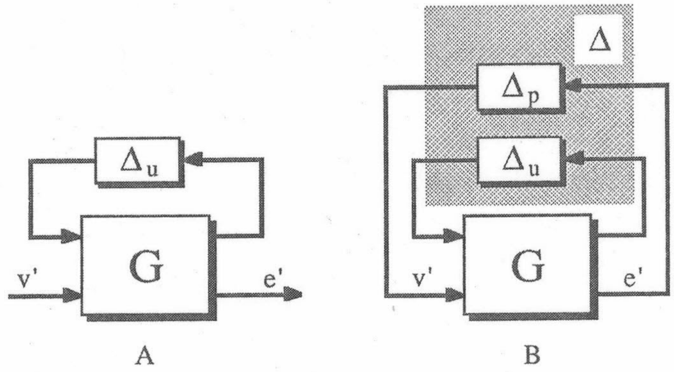

Figure 11.3-1. Block diagram structure for checking robust performance. Full perturbation matrix Δ_p in (B) leads to robust performance test via SSV.

uncertainty block Δ_u and the normalized outputs e'. If we partition G into four blocks consistent with the dimensions of the two input and two output vectors we can identify G_{11} as the matrix M shown in Fig. 11.1-2 and G_{22} as the weighted *nominal* sensitivity function $W_2E(\tilde{P})W_1$.

The robust performance objective (11.3–1) can now be expressed in terms of G.

$$\|F(G,\Delta_u)\|_\infty = \sup_\omega \bar{\sigma}(F(G,\Delta_u)) < 1 \qquad (11.3-4)$$

where the transfer matrix from v' to e'

$$e' = F(G,\Delta_u)v' \qquad (11.3-5)$$

is described by the *Linear Fractional Transformation* (LFT)

$$F(G,\Delta_u) = G_{22} + G_{21}\Delta_u(I - G_{11}\Delta_u)^{-1}G_{12} \qquad (11.3-6)$$

Comparing condition (11.1–9) for robust stability and the formally identical condition (11.3–4) for robust performance we conclude: *the system $F(G,\Delta_u)$ satisfies the robust performance condition (11.3–4)* **if and only if** *it is robustly stable for the norm bounded matrix perturbation Δ_p* ($\bar{\sigma}(\Delta_p) \leq 1$). (Here the subscript p stands for "performance.") We have expressed this equivalence between robust performance and robust stability in Fig. 11.3-1B: conditions (11.3–1) and (11.3–4) are satisfied if and only if the system G is robustly stable with respect to the block diagonal perturbation

$$\Delta = \operatorname{diag}\{\Delta_u, \Delta_p\}, \qquad \bar{\sigma}(\Delta) \leq 1 \tag{11.3-7}$$

Δ_p is generally a full matrix of appropriate dimensions. A necessary and sufficient condition for robust stability in the presence of norm bounded block diagonal perturbations can be expressed in terms of the structured singular value μ (Thm. 11.2-1).

Theorem 11.3-1. *The nominally stable system G (Fig. 11.3-1) subjected to the block diagonal uncertainty Δ_u ($\bar{\sigma}(\Delta_u) \leq 1$) satisfies the robust performance condition $\|F(G, \Delta_u)\|_\infty < 1$ if and only if*

$$\mu_\Delta(G) < 1 \qquad \forall\omega \tag{11.3-8}$$

where μ is computed with respect to the block diagonal perturbation $\Delta = \operatorname{diag}\{\Delta_u, \Delta_p\}$.

Theorem 11.3-1 is probably the main reason for measuring performance in terms of the ∞-norm and bounding uncertainty in the same manner. It is then possible to express robust performance in terms of robust stability and to test for either one in a *nonconservative* manner by calculating μ. Indeed, if the uncertainty is modeled *exactly* by Δ_u — i.e., if all plants in this norm-bounded set do actually occur in practice, the conditions for robust stability and performance are necessary and sufficient.

Some care is necessary to interpret the robust performance test correctly when $\mu(\omega) = \beta(\omega) > 1$. It means that if each one of the uncertainty blocks is *reduced* by a factor β^{-1} then the *relaxed* performance specification $\bar{\sigma}(W_2 E W_1) \leq \beta$ can be met. $\mu > 1$ does *not* give any explicit information on how much the performance violates the specification (11.3–1) for the uncertainty Δ_u.

Because $\Delta_1 = \operatorname{diag}\{\Delta_u, 0\}$ and $\Delta_2 = \operatorname{diag}\{0, \Delta_p\}$ are special cases of Δ ($\bar{\sigma}(\Delta) \leq 1$) we find

$$\mu_\Delta(G) \geq \max\{\mu_{\Delta_u}(G_{11}), \mu_{\Delta_p}(G_{22}) = \bar{\sigma}(G_{22})\} \tag{11.3-9}$$

Inequality (11.3–9) implies that for robust performance ($\mu_\Delta(G) < 1$) it is necessary that the system is robustly stable ($\mu_{\Delta_u}(G_{11}) < 1$) and satisfies the performance specifications in the absence of uncertainty ($\bar{\sigma}(G_{22}) < 1$), which is not very surprising. This suggests that *robust performance* might not always be a very important issue: If both the nominal performance and the robust stability condition are satisfied with some margin then the robust performance condition should also be satisfied. The next two sections will shed some light on this issue.

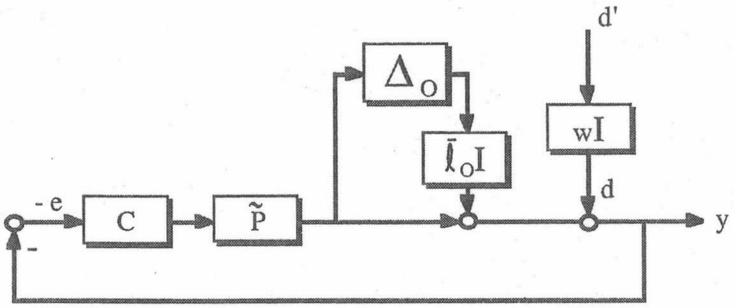

Figure 11.3-2. System with multiplicative output uncertainty and performance weight w.

11.3.2 Multiplicative Output Uncertainty

Consider the robust performance problem for the system depicted in Fig. 11.3-2. The set of plants is described by

$$\Pi = \{P = (I + \bar{\ell}_O \Delta_O)\tilde{P};\ \bar{\sigma}(\Delta_O) \leq 1\} \qquad (11.3-10)$$

The H_∞ performance specification places a bound on the sensitivity operator

$$\bar{\sigma}(Ew) < 1 \qquad \forall \omega, \forall P \in \Pi \qquad (11.3-11)$$

where w is a scalar weight. Defining $v' = d' = w^{-1}d$ and $e' = e$ we can put the block diagram in Fig. 11.3-2 into the form shown in Fig. 11.3-1 with

$$G = \begin{pmatrix} -\tilde{H}\bar{\ell}_O & -\tilde{H}w \\ \tilde{E}\bar{\ell}_O & \tilde{E}w \end{pmatrix} \qquad (11.3-12)$$

According to Thm. 11.3-1, the robust performance condition (11.3–11) is met if and only if $\mu(G) < 1$ where μ is evaluated with respect to the block diagonal matrix $\Delta = \text{diag}\{\Delta_u, \Delta_p\}$ and Δ_u and Δ_p are full. Alternatively, we can start from (11.3–11) and derive a sufficient condition. Straightforward algebra yields for the multiplicative output uncertainty (11.1–2), (11.1–6)

$$E = \tilde{E}(I + \bar{\ell}_O \Delta_O \tilde{H})^{-1} \qquad (11.3-13)$$

We substitute (11.3–13) into (11.3–11)

$$\bar{\sigma}(w\tilde{E}(I + \bar{\ell}_O \Delta_O \tilde{H})^{-1}) < 1$$

$$\Leftarrow \bar{\sigma}(w\tilde{E})\bar{\sigma}(I + \bar{\ell}_O \Delta_O \tilde{H})^{-1} < 1$$

$$\Leftrightarrow \bar{\sigma}(w\tilde{E}) < \underline{\sigma}(I + \bar{\ell}_O \Delta_O \tilde{H})$$

$$\Leftrightarrow \bar{\sigma}(w\tilde{E}) < 1 - \bar{\sigma}(\bar{\ell}_O \Delta_O \tilde{H})$$

$$\Leftrightarrow \bar{\sigma}(w\tilde{E}) + \bar{\sigma}(\bar{\ell}_O \tilde{H}) < 1 \qquad (11.3-14)$$

For the specific case of a multiplicative output uncertainty (11.3–14) is a sufficient condition for robust performance and therefore an upper bound on μ.

$$\mu(G) \leq \bar{\sigma}(w\tilde{E}) + \bar{\sigma}(\bar{\ell}_O \tilde{H}) \qquad (11.3-15)$$

A comparison with Thm. 2.6-1 or an examination of the steps leading to (11.3–14) reveals that (11.3–15) is an *equality* for SISO systems. Because the "robust stability term" ($\bar{\sigma}(\bar{\ell}_0 \tilde{H})$) and the "nominal performance term" ($\bar{\sigma}(w\tilde{E})$) appear *additively*, robust performance can be easily achieved by satisfying both robust stability and nominal performance with some margin ($\bar{\sigma}(\bar{\ell}_O \tilde{H}) \leq \alpha$, $\bar{\sigma}(w\tilde{E}) \leq 1 - \alpha$, $\alpha < 1$). This is exact for SISO systems but can be somewhat conservative for MIMO systems. Thus, "robust performance" is not very critical for SISO systems or MIMO systems with multiplicative output uncertainty: an examination of robust stability and nominal performance suffices as an approximate check for robust performance.

11.3.3 Multiplicative Input Uncertainty

Next we study the robust performance problem for the system shown in Fig. 11.3-3. The set of plants is described by

$$\Pi = \{P = \tilde{P}(I + \bar{\ell}_I \Delta_I),\ \bar{\sigma}(\Delta_I) \leq 1\} \qquad (11.3-16)$$

The performance specification is again given by (11.3–11). The interconnection matrix G (Fig. 11.3-1A) derived from the block diagram in Fig. 11.3-3 is

$$G = \begin{pmatrix} -\tilde{P}^{-1}\tilde{H}\tilde{P}\bar{\ell}_I & -\tilde{P}^{-1}\tilde{H}w \\ \tilde{E}\tilde{P}\bar{\ell}_I & \tilde{E}w \end{pmatrix} \qquad (11.3-17)$$

and robust performance is guaranteed according to Thm. 11.3-1 if and only if $\mu(G) < 1$ for G defined by (11.3–17) and μ evaluated with respect to the block diagonal matrix $\Delta = \text{diag}\{\Delta_u, \Delta_p\}$ where Δ_u and Δ_p are full. Similarly as in the last section we can start from the requirement (11.3–11). Comparing Fig. 11.3-2 and 11.3-3 we find

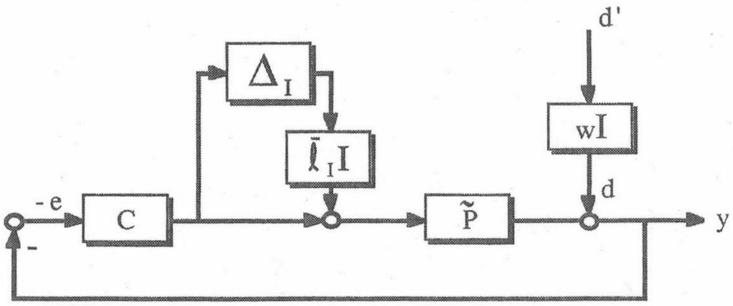

Figure 11.3-3. System with multiplicative input uncertainty and performance weight w.

$$\bar{\ell}_O \Delta_O = \bar{\ell}_I \tilde{P} \Delta_I \tilde{P}^{-1} \qquad (11.3-18)$$

We substitute (11.3–18) into (11.3–13) to obtain the condition for robust performance

$$\bar{\sigma}(w\tilde{E}(I + \bar{\ell}_I \tilde{P} \Delta_I \tilde{P}^{-1} \tilde{H})^{-1}) < 1 \qquad (11.3-19)$$

$$\Leftarrow \bar{\sigma}(w\tilde{E}) < \underline{\sigma}(I + \bar{\ell}_I \tilde{P} \Delta_I \tilde{P}^{-1} \tilde{H})$$

$$\Leftarrow \bar{\sigma}(w\tilde{E}) < 1 - \bar{\sigma}(\bar{\ell}_I \tilde{P} \Delta_I \tilde{P}^{-1} \tilde{H})$$

$$\Leftarrow \bar{\sigma}(w\tilde{E}) + \bar{\sigma}(\tilde{P})\bar{\sigma}(\tilde{P}^{-1})\bar{\sigma}(\bar{\ell}_I \tilde{H}) < 1$$

$$\Leftrightarrow \bar{\sigma}(w\tilde{E}) + \kappa(\tilde{P})\bar{\sigma}(\bar{\ell}_I \tilde{H}) < 1 \qquad (11.3-20)$$

We can rewrite (11.3–19) as

$$\bar{\sigma}(w(I + \bar{\ell}_I \tilde{H} C^{-1} \Delta_I C)^{-1} \tilde{E}) < 1 \qquad (11.3-21)$$

and follow the same steps as above to find

$$\bar{\sigma}(w\tilde{E}) + \kappa(C)\bar{\sigma}(\bar{\ell}_I \tilde{H}) < 1 \qquad (11.3-22)$$

We leave it to the reader to show that conditions similar to (11.3–20) and (11.3–22) but involving the sensitivity and complementary sensitivity (11.1–14) at the plant input can be derived, which are sufficient for robust performance.

$$\kappa(\tilde{P})\bar{\sigma}(w\tilde{E}_I) + \bar{\sigma}(\bar{\ell}_I\tilde{H}_I) < 1 \qquad (11.3-23)$$

$$\kappa(C)\bar{\sigma}(w\tilde{E}_I) + \bar{\sigma}(\bar{\ell}_I\tilde{H}_I) < 1 \qquad (11.3-24)$$

We will concentrate the following discussion on (11.3–20) and (11.3–22). Conditions (11.3–23) and (11.3–24) can be interpreted similarly.

Note first that even when robust stability and nominal performance are satisfied with a reasonable margin ($\bar{\sigma}(\bar{\ell}_I\tilde{H}) < 1$ and $\bar{\sigma}(w\tilde{E}) < 1$) the robust performance condition can be violated by an *arbitrarily* large amount if either the controller C or the plant $\tilde{P}$ is ill-conditioned. On the other hand if either $\kappa(\tilde{P})$ or $\kappa(C)$ is small, the input uncertainty can be treated more or less like output uncertainty and it is not necessary to pay special attention to robust performance. It should be emphasized, however, that both (11.3–20) and (11.3–22) are only *sufficient.* If the plant is ill-conditioned, any controller designed for good nominal performance will also be ill-conditioned because it tends to invert the plant. Under these circumstances (11.3–20) and (11.3–22) can be *arbitrarily* conservative compared to the exact condition involving μ.

Nevertheless, (11.3–20) and (11.3–22) give rough guidelines for controller design to avoid robust performance problems. For a well-conditioned plant a simple decoupling (inverse-based) controller is also well conditioned and should give good robust performance. For a badly conditioned plant decoupling should be avoided and for robust performance much attention has to be paid to the modelling of the uncertainty and the control system design.

11.3.4 H_2-Performance Objective

We wish to evaluate a bound on the 2-norm of the weighted sensitivity

$$\|W_2EW_1\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}[(W_2EW_1)^H(W_2EW_1)]d\omega \qquad (11.3-25)$$

for a family Π of plants. Assume that a bound $\beta_0(\omega)$ can be found such that

$$\sup_{P\in\Pi} \bar{\sigma}(W_2EW_1) = \beta_0(\omega) \qquad (11.3-26)$$

Then, because for $A \in \mathcal{C}^{n\times n}$, $\text{trace}(A^HA) \leq n\bar{\sigma}^2(A)$

$$\sup_{P\in\Pi} \|W_2EW_1\|_2^2 \leq \frac{n}{2\pi}\int_{-\infty}^{\infty} \beta_0^2(\omega)d\omega \qquad (11.3-27)$$

where n is the maximum rank of $W_2EW_1(\omega)$. For the type of norm-bounded uncertainty introduced in Sec. 11.2.1 the bound $\beta_0(\omega)$ can be found from μ as

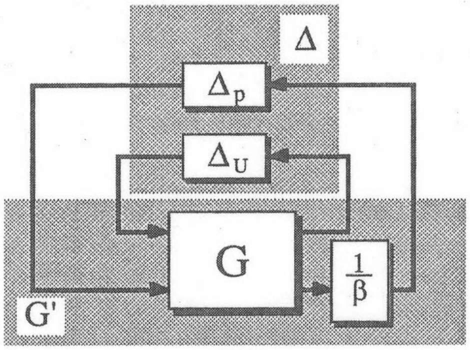

Figure 11.3-4. Robust performance block diagram with additional block $\beta^{-1}I$.

follows. Modify the block diagram in Fig. 11.3-1 by introducing an additional block $\beta^{-1}I$ ($\beta > 0$) as shown in Fig. 11.3-4. Define

$$G' = \begin{pmatrix} G_{11} & G_{12} \\ \beta^{-1}G_{21} & \beta^{-1}G_{22} \end{pmatrix} \qquad (11.3-28)$$

Then

$$\mu(G'(\beta)) = 1 \Leftrightarrow \beta = \beta_0 \qquad (11.3-29)$$

defines a function $\beta_0(\omega)$ such that

$$\sup_{\bar{\sigma}(\Delta_u)\leq 1} \bar{\sigma}(F(G, \Delta_u)) = \beta_0(\omega) \qquad (11.3-30)$$

where F is the perturbed weighted sensitivity described by the LFT (11.3-6). Equation (11.3–26) follows directly from (11.3–30).

Inequality (11.3–27) provides only a bound for the H_2 objective, which can be conservative. Alternatively we can compute for *one* specific input v the worst ISE that can result from any plant in the set Π or equivalently for any $\Delta \in X$. This can be done *exactly* without conservatism as shown next.

The 2-norm of the weighted error for a specific input v ($\|W_2Ev\|_2$) is given by (11.3–25) with

$$W_1 = (v \quad 0) \qquad (11.3-31)$$

where the 0-matrix is chosen to make W_1 square. Because W_2EW_1 is now of rank one and trace$(A^H A) = \bar{\sigma}^2(A)$ when A is of rank one, (11.3–27) becomes the *equality*

$$\sup_{P\in\Pi} \|W_2Ev\|_2^2 = \sup_{P\in\Pi} \|W_2EW_1\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \beta_0^2(\omega)d\omega \qquad (11.3-32)$$

For the specific case of an SISO system with multiplicative uncertainty, $W_2 = 1$, $W_1 = v$, and we can find G' from (11.3–12) or (11.3–17)

$$G' = \begin{pmatrix} -\tilde{\eta}\bar{\ell}_m & -\tilde{\eta}v \\ \beta^{-1}\tilde{\epsilon}\bar{\ell}_m & \beta^{-1}\tilde{\epsilon}v \end{pmatrix} \qquad (11.3-33)$$

Here we know from Sec. 11.3.2

$$\mu(G') = |\tilde{\eta}\bar{\ell}_m| + |\beta^{-1}\tilde{\epsilon}v| \qquad (11.3-34)$$

Setting $\mu(G') = 1$ and solving for β we find

$$\beta_0 = |\tilde{\epsilon}v|(1 - |\tilde{\eta}\bar{\ell}_m|)^{-1} \qquad (11.3-35)$$

Substituting this expression for β_0 in (11.3–32) we find the same result as in Section 2.6.1.

Definition (11.3–29) implies that at each frequency that value of β has to be found which makes the SSV unity. It is obvious that $\mu(G'(\beta))$ is a monotonic function of β: as β increases the destabilizing effect of the uncertainty decreases. More precisely, if the system in Fig. 11.3-6 is stable for β_1 and $\Delta_i, \bar{\sigma}(\Delta_i) \leq 1$, then it is also stable for any $\beta_2 > \beta_1$. Therefore $\mu(G'(\beta_2)) \leq \mu(G'(\beta_1))$. In the computations we usually employ the upper bound of μ rather than μ itself. The following theorem makes the iterations necessary to solve (11.3–29) in terms of the upper bound very simple.

Theorem 11.3-2. *Let*

$$M^x = \begin{pmatrix} M_{11} & M_{12} \\ xM_{21} & xM_{22} \end{pmatrix} \qquad (11.3-36)$$

where x is a positive scalar and let $D = \text{diag}\{D_1, D_2\}$. Then the upper bound of the SSV $\mu(M^x)$, $\inf_{D\in\mathcal{D}} \bar{\sigma}(DM^xD^{-1})$, (see (11.2–15)), is a non-decreasing function of x.

Proof. Let $0 < x_2 \leq x_1$. Then we can write $x_2 = \alpha x_1$ where $0 < \alpha \leq 1$. From (11.3–36) we have

$$DM^{x_2}D^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \alpha I \end{pmatrix} M^{x_1}D^{-1} = \begin{pmatrix} I & 0 \\ 0 & \alpha I \end{pmatrix} DM^{x_1}D^{-1}$$

$$\Rightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \leq \bar{\sigma} \begin{pmatrix} I & 0 \\ 0 & \alpha I \end{pmatrix} \bar{\sigma}(DM^{x_1}D^{-1})$$

$$\Leftrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \leq \bar{\sigma}(DM^{x_1}D^{-1}) \qquad \forall D \in \mathcal{D}$$

$$\Rightarrow \inf_{D\in\mathcal{D}} \bar{\sigma}(DM^{x_2}D^{-1}) \leq \inf_{D\in\mathcal{D}} \bar{\sigma}(DM^{x_1}D^{-1})$$

11.3.5 Application: High-Purity Distillation

Consider the distillation column described in the Appendix where the overhead composition is to be controlled at $y_D = 0.99$ and the bottom composition at $x_B = 0.01$ using the reflux L and boilup V as manipulated variables. After linearization the model is

$$\tilde{P} = \frac{1}{75s+1} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \qquad (11.3-37)$$

In a similar manner as in Sec. 11.1.6 we will assume a full block input uncertainty with weight

$$w_I(s) = 0.2\frac{5s+1}{0.5s+1} \qquad (11.3-38)$$

The performance specification is simply

$$\bar{\sigma}(E) < |w_P|^{-1} \qquad \forall P \in \Pi,\ \forall \omega \qquad (11.3-39)$$

where

$$w_P = 0.5\frac{10s+1}{10s} \qquad (11.3-40)$$

The performance weight $w_P(s)$ implies that we require integral action ($w_P(0) = \infty$). It allows an amplification of disturbances at high frequencies by a factor of two at most ($\lim_{\omega\to\infty} |w_P(i\omega)|^{-1} = 2$). A particular sensitivity function which exactly matches the performance bound (11.3–40) at low frequencies and satisfies it easily at high frequencies is $E = \frac{20s}{20s+1}I$. This corresponds to a first order response with time constant 20 min.

For robust performance

Table 11.3-1. State space realization of "μ-optimal" controller, $C_\mu(s) = C(sI - A)^{-1}B + D$.

$$A = \begin{pmatrix} -1.002 \cdot 10^{-7} & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.272 \cdot 10^{-6} & 0 & 0 & 0 & \\ 0 & 0 & -0.1510 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9.032 & 0 & 0 \\ 0 & 0 & 0 & 0 & -538.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -586.8 \end{pmatrix}$$

$$B = \begin{pmatrix} -65.13 & -90.09 \\ 72.24 & 90.31 \\ 5.492 & -4.394 \\ -90.86 & -113.6 \\ 1867 & -1494 \\ 672.2 & 840.3 \end{pmatrix}$$

$$C = \begin{pmatrix} 0.6564 & 0.7171 & 4.949 & 5.033 & -1691 & -311.2 \\ 0.6555 & 0.5425 & 4.941 & -5.040 & -1689 & 311.6 \end{pmatrix}$$

$$D = \begin{pmatrix} 5866 & -3816 \\ 5002 & -4878 \end{pmatrix}$$

$$\mu(G) < 1 \qquad \forall \omega \qquad (11.3-41)$$

where G is defined by (11.3–17) with $\bar{\ell}_I = w_I$ and $w = w_P$. We will consider three different controllers: an inverse-based controller $C_1(s)$ (in this case equivalent to a steady state decoupler with PI controllers), a diagonal PI-controller $C_2(s)$ and a "μ-optimal" controller $C_\mu(s)$, found by approximate minimization of the LHS of (11.3–41).

$$C_1(s) = c_1(s) G_{LV}^{-1}(s) \qquad (11.3-42a)$$

$$c_1(s) = 0.7s^{-1} \qquad (11.3-42b)$$

$$C_2(s) = c_2(s) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (11.3-43a)$$

$$c_2(s) = 2.4(75s + 1)s^{-1} \qquad (11.3-43b)$$

$$C_\mu(s): \ \text{Table 11.3-1}$$

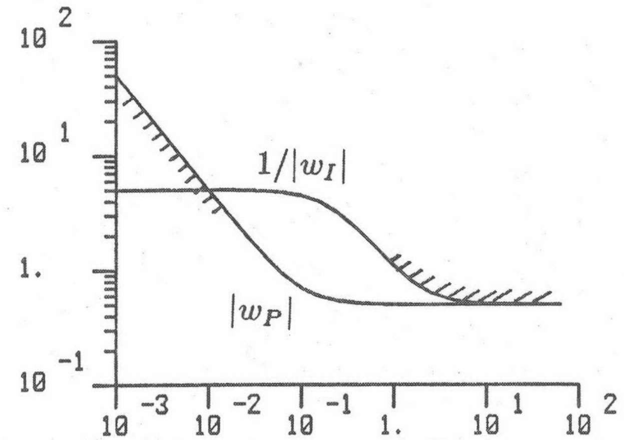

Figure 11.3-5. Performance and robustness bounds for multivariable loop shaping.

One way of designing controllers which meet the nominal performance and robust stability specifications is to use multivariable loop shaping. For nominal performance, $\underline{\sigma}(\tilde{P}C)$ must be above $|w_P|$ for low frequencies (Sec. 10.4.4). For robust stability with input uncertainty, $\bar{\sigma}(C\tilde{P})$ must lie below $1/|w_I|$ for high frequencies (Sec. 11.1.4) (Fig. 11.3-5).

For the inverse-based controller (11.3–42) we get $\bar{\sigma}(C_1\tilde{P}) = \underline{\sigma}(\tilde{P}C_1) = |c_1|$ and it is trivial to choose a $c_1(s)$ to satisfy these conditions. The choice $c_1(s) = 0.7s^{-1}$ yields a controller which has much better nominal performance than required, and which can allow about two times more uncertainty than assumed. This is also seen from Fig. 11.3-6 and 11.3-7 where the nominal performance and robust stability conditions (10.4–19) and (11.1–13) are displayed graphically.

For the diagonal controller (11.3–43) we find $\bar{\sigma}(C_2\tilde{P}) = 1.972|c_2|$ and $\underline{\sigma}(\tilde{P}C_2) = 0.0139|c_2|$, and the difference between these two singular values is so large that no choice of c_2 is able to satisfy both nominal performance and robust stability. This is shown in Figs. 11.3-6 and 11.3-7 for $c_2(s)$ defined by (11.3–43b).

The sufficient conditions for robust performance (11.3–20) and (11.3–22) suggest that the ill-conditioned controller C_1 ($\kappa(C_1) = 141.7$) for the ill-conditioned plant $\tilde{P}$ ($\kappa(\tilde{P}) = 141.7$) may give very poor robust performance even though both the nominal performance ($\bar{\sigma}(w_P\tilde{E}) < 1$) and robust stability conditions ($\bar{\sigma}(w_I\tilde{H}_I) < 1$) are individually satisfied. On the other hand, for a controller with a low condition number ($\kappa(C_2) = 1$) we expect to get robust performance for

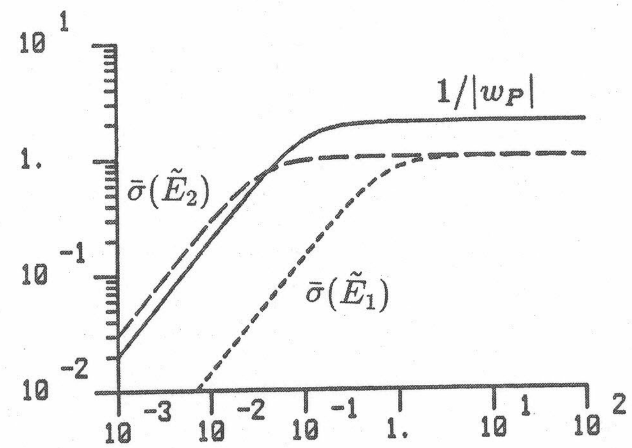

Figure 11.3-6. Nominal performance test for controllers C_1 and C_2.

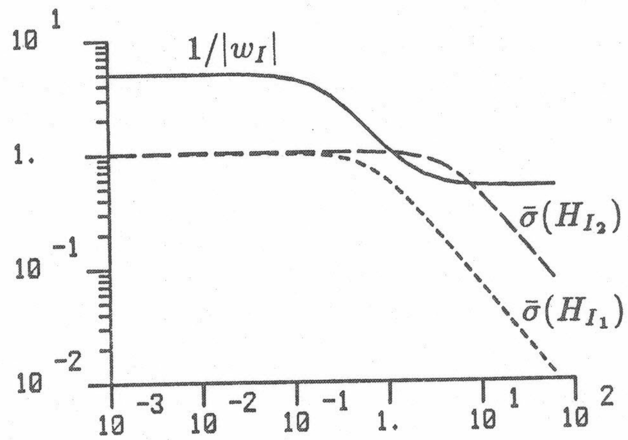

Figure 11.3-7. Robust stability test for controllers C_1 and C_2.

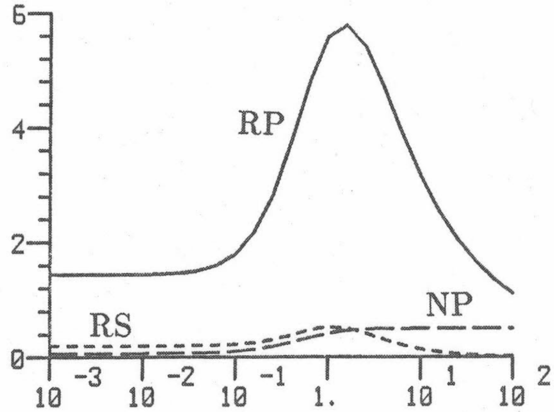

Figure 11.3-8. μ-plots for inverse-based controller $C_1(s)$.

"free" *provided* nominal performance and robust stability are satisfied. However, as we saw for C_2 in Fig. 11.3-6 it is rarely possible to achieve good nominal performance for an ill-conditioned plant with a scalar controller.

The exact test for robust performance (11.3–41) is plotted in Figs. 11.3-8 and 11.3-9 for C_1 and C_2. As expected, the inverse-based controller $C_1(s)$ is far from meeting the robust performance requirements (μ_{RP} is about 5.8), even though the controller was shown to achieve both nominal performance and robust stability. On the other hand, the performance of the diagonal controller $C_2(s)$ is much less affected by uncertainty ($\mu_{RP} = 1.71$).

The μ-synthesis method used to design the "μ-optimal" controller gives controllers of very high order, but by employing model reduction, we were able to find a "μ-optimal" controller with six states (Table 11.3-1). The robust performance test for this controller is shown in Fig. 11.3-10. (The μ-plot is not quite flat as it should be for the truly optimal case.) The peak value for μ is 1.06, which means that this controller almost satisfies the robust performance condition. This value for μ_{RP} is significantly lower than for the diagonal PI controller C_2.

The time responses in Fig. 11.3-11 confirm the predictions by μ. Note in particular the poor performance of C_1 in the presence of model uncertainty. The large value of $\mu(0)$ for the diagonal PI controller leads to a very sluggish approach to steady state when compared to the μ-optimal controller.

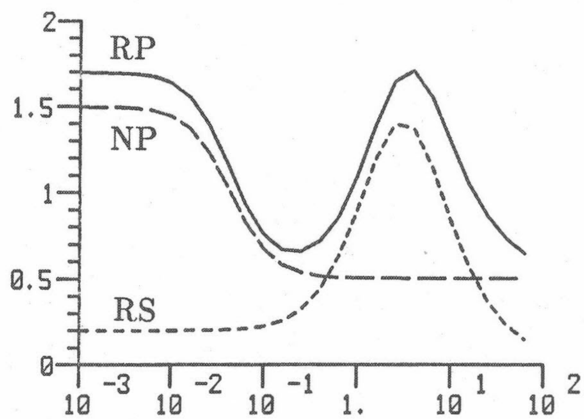

Figure 11.3-9. μ-plots for diagonal controller $C_2(s)$.

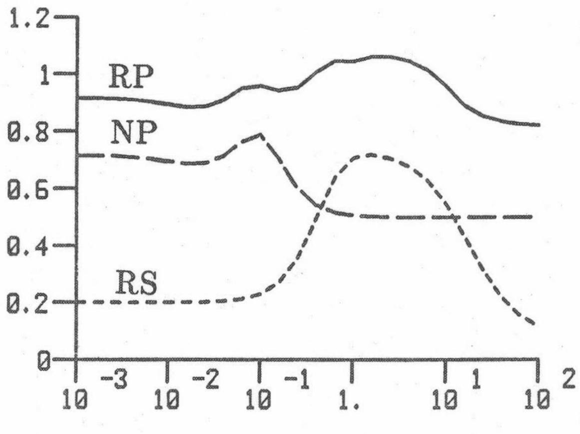

Figure 11.3-10. μ-plots for "μ-optimal" controller $C_\mu(s)$.

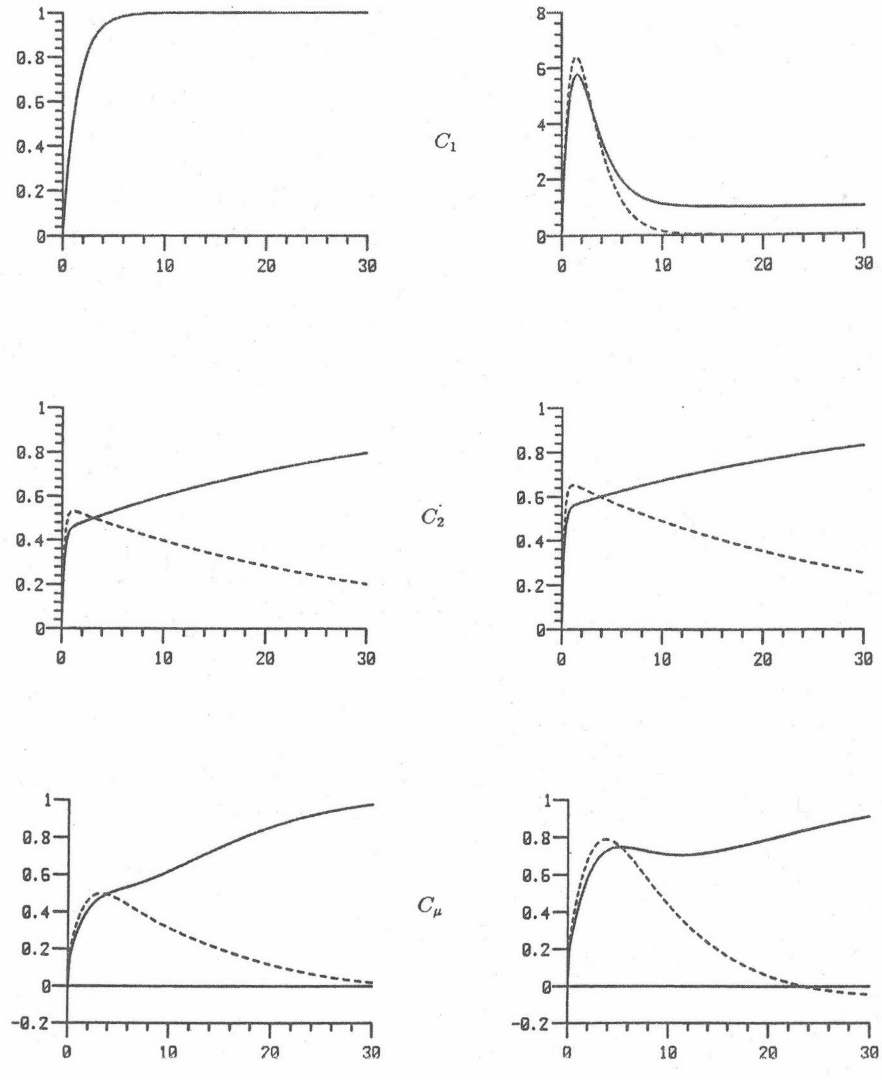

Figure 11.3-11. Time responses for the three controllers for a setpoint change $r = (1,0)^T$. Left column: no model error, right column $L_I = \text{diag}\{0.2, -0.2\}$.

11.4 Robustness Conditions in Terms of Specific Transfer Matrices

In Secs. 11.2 and 11.3 we derived necessary and sufficient conditions of the form

$$\mu_\Delta(M) < k(\omega), \quad \forall\omega \tag{11.4-1}$$

for robust stability and performance. The implications of (11.4–1) may not be easy to understand for the engineer. A simple robustness bound of the form $\bar{\sigma}(T) < k'(\omega)\ \forall\omega$ may provide more insight, where T denotes a transfer matrix of engineering significance — e.g., the sensitivity $\tilde{E}$ — the complementary sensitivity $\tilde{H}$ or the loop gain $\tilde{P}C$. The goal of this section is to derive such bounds.

More specifically we will show first that in all cases of practical interest M can be related to the transfer matrix T of engineering significance by a *Linear Fractional Transformation* (LFT)

$$M = N_{11} + N_{12}T(I - N_{22}T)^{-1}N_{21} \tag{11.4-2}$$

(Sometimes a superscript on N — e.g., N^T — will be used to denote the dependence of N on the particular choice of T.) Then we will derive from N a bound on $\bar{\sigma}(T)$ which guarantees that (11.4–1) is satisfied. Since one objective is to assist the engineer with the bound in the controller design, it is important that N be independent of C.

11.4.1 How to find the LFT

In many cases the LFT (11.4-2) can be found by inspection. In other cases the following three-step procedure may be used (Fig. 11.4-1).

1) Write M as a LFT of C:

$$M = G_{11} + G_{12}C(I - G_{22}C)^{-1}G_{21} \tag{11.4-3}$$

The matrix G is easy to construct by inspection of the block diagram.

2) Write the controller C as a LFT of the transfer matrix of interest (T).

$$C = J_{11} + J_{12}T(I - J_{22}T)^{-1}J_{21} \tag{11.4-4}$$

J is most easily found by solving the expression $T(C)$ for C.

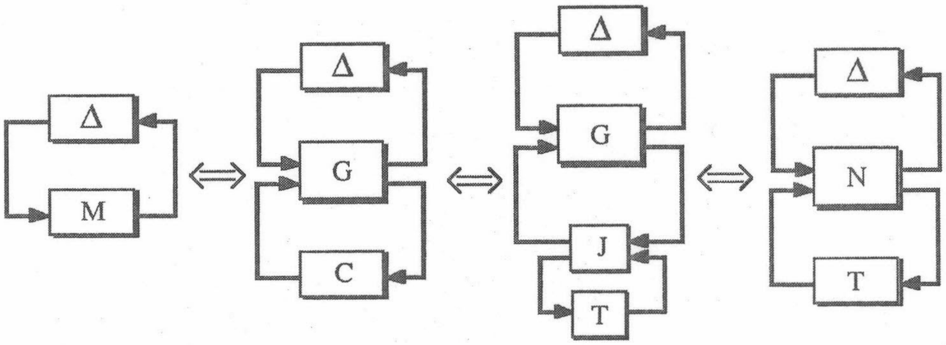

Figure 11.4-1. Equivalent representations of system M with perturbation Δ.

3) Given G and J, N follows easily because any interconnection of LFT's is again an LFT

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} G_{11} + G_{12}J_{11}(I - G_{22}J_{11})^{-1}G_{21} & G_{12}(I - J_{11}G_{22})^{-1}J_{12} \\ J_{21}(I - G_{22}J_{11})^{-1}G_{21} & J_{22} + J_{21}G_{22}(I - J_{11}G_{22})^{-1}J_{12} \end{pmatrix} \quad (11.4-5)$$

For the special case $J_{11} = 0$ this reduces to

$$N = \begin{pmatrix} G_{11} & G_{12}J_{12} \\ J_{21}G_{21} & J_{22} + J_{21}G_{22}J_{12} \end{pmatrix} \quad (11.4-6)$$

Without proof we remark that when T is a *closed-loop* transfer function then $N_{22} = 0$.

If N^H is known, then it is easy to derive N for other closed-loop transfer functions T. Note that $\tilde{H}$ is an LFT of $\tilde{E}$:

$$\tilde{H} = I - \tilde{E} \quad (11.4-7)$$

Let

$$\tilde{H} = J_{11} + J_{12}\tilde{E}(I - J_{22}\tilde{E})^{-1}J_{21} \quad (11.4-8)$$

then we find by comparing (11.4–7) and (11.4–8)

$$J = \begin{pmatrix} I & -I \\ I & 0 \end{pmatrix} \quad (11.4-9)$$

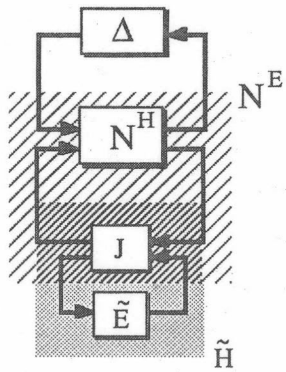

Figure 11.4-2. Illustration for finding N^E from N^H.

The transformations are interpreted in the block diagram in Fig. 11.4-2. The matrix N^E can be found from (11.4–5) with N^H instead of G, and J defined by (11.4–9)

$$N^E = \begin{pmatrix} N_{11}^H + N_{12}^H N_{21}^H & -N_{12}^H \\ N_{21}^H & 0 \end{pmatrix} \qquad (11.4-10)$$

Here we have set $N_{22}^H = 0$ as explained above.

Example 11.4-1. Consider the system with simultaneous multiplicative input and output uncertainty studied in Sec. 11.2.3. For $W_{1I} = W_{1O} = wI$ and $W_{2I} = W_{2O} = I$, (11.2–20) becomes

$$M = w \begin{pmatrix} -C\tilde{P}(I + C\tilde{P})^{-1} & -C(I + \tilde{P}C)^{-1} \\ \tilde{P}(I + C\tilde{P})^{-1} & -\tilde{P}C(I + \tilde{P}C)^{-1} \end{pmatrix} \qquad (11.4-11)$$

Recall that $\Delta = \text{diag}\{\Delta_I, \Delta_O\}$. Let us express M as a LFT of $\tilde{H}$ using the three-step procedure

(1) It is easier to find the matrix G directly from Fig. 11.2-1 than from (11.4–11):

$$G = \begin{pmatrix} 0 & 0 & -I \\ P & 0 & -P \\ P & I & -P \end{pmatrix} \qquad (11.4-12)$$

Note that G_{11} is the upper left 2×2 block of G corresponding to Δ.

(2) Solving

$$\tilde{H} = \tilde{P}C(I + \tilde{P}C)^{-1} \tag{11.4 – 13}$$

for C yields

$$C = \tilde{P}^{-1}\tilde{H}(I - \tilde{H})^{-1} \tag{11.4 – 14}$$

Comparing (11.4–14) with (11.4–4) we find

$$J = \begin{pmatrix} 0 & \tilde{P}^{-1} \\ I & I \end{pmatrix} \tag{11.4 – 15}$$

Substituting (11.4–12) and (11.4–15) into (11.4–5) yields

$$N^H_{11} = \begin{pmatrix} 0 & 0 \\ \tilde{P} & 0 \end{pmatrix}, \quad N^H_{12} = \begin{pmatrix} -\tilde{P}^{-1} \\ -I \end{pmatrix}, \quad N^H_{21} = (\tilde{P} \quad I), \quad N^H_{22} = 0 \tag{11.4 – 16}$$

To find M as a LFT of $\tilde{E}$, use N^H and (11.4–10) to get:

$$N^E_{11} = \begin{pmatrix} -I & -\tilde{P}^{-1} \\ 0 & -I \end{pmatrix}, \quad N^E_{12} = \begin{pmatrix} \tilde{P}^{-1} \\ I \end{pmatrix}, \quad N^E_{21} = (\tilde{P} \quad I), \quad N^E_{22} = 0 \tag{11.4 – 17}$$

□

11.4.2 New Properties of μ

The results in this section apply to any complex matrices although in our case these will be transfer matrices.

Theorem 11.4-1. *Let M be written as a LFT of T*

$$M = N_{11} + N_{12}T(I - N_{22}T)^{-1}N_{21} \tag{11.4 – 18}$$

and let k be a given constant. Assume $\mu_\Delta(N_{11}) < k$ and $\det(I - N_{22}T) \neq 0$. Then

$$\mu_\Delta(M) < k \tag{11.4 – 19}$$

if

$$\bar{\sigma}(T) \leq c_T \tag{11.4 – 20}$$

where c_T solves

$$\mu_{\tilde{\Delta}}\begin{pmatrix} N_{11} & N_{12} \\ kc_T N_{21} & kc_T N_{22} \end{pmatrix} = k \qquad (11.4-21)$$

and $\tilde{\Delta} = \text{diag}\{\Delta, T\}$.

Proof. Assume that T is defined such that $\bar{\sigma}(T) \leq c_T \; \forall T$. Then at each frequency holds

$$\mu_{\Delta}(M) < k \qquad \forall T$$

$$\Leftrightarrow \det(I + M\Delta) \neq 0 \qquad \forall \Delta \ni \bar{\sigma}(\Delta) \leq 1/k, \quad \forall T$$

$$\Leftrightarrow \det\begin{pmatrix} I + N_{11}\Delta & -N_{12}T \\ N_{21}\Delta & I - N_{22}T \end{pmatrix} \neq 0 \quad \forall \Delta \ni \bar{\sigma}(\Delta) \leq 1/k, \quad \forall T$$

The last step follows from (11.4–18) and Schur's formula (Lemma 10.2-1).

$$\Leftrightarrow \det\left[I + \begin{pmatrix} \frac{1}{k}N_{11} & c_T N_{12} \\ \frac{1}{k}N_{21} & c_T N_{22} \end{pmatrix}\begin{pmatrix} k\Delta & 0 \\ 0 & -\frac{1}{c_T}T \end{pmatrix}\right] \neq 0 \quad \forall \Delta \ni \bar{\sigma}(\Delta) \leq 1/k, \quad \forall T$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{pmatrix} \frac{1}{k}N_{11} & c_T N_{12} \\ \frac{1}{k}N_{21} & c_T N_{22} \end{pmatrix} < 1, \quad \forall T$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{pmatrix} N_{11} & kc_T N_{12} \\ N_{21} & kc_T N_{22} \end{pmatrix} < k, \qquad \forall T$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{pmatrix} N_{11} & N_{12} \\ kc_T N_{21} & kc_T N_{22} \end{pmatrix} < k, \quad \forall T$$

□

In general, c_T can be found numerically using the implicit expression (11.4–21). This search is straightforward because of Thm. 11.3-2. In the special case when $N_{11} = N_{22} = 0$, c_T can be computed explicitly.

Theorem 11.4-2. *Assume* $N_{11} = N_{22} = 0$. *Then* c_T *satisfying (11.4–21) is given by*

$$c_T = k\mu_{\tilde{\Delta}}^{-2}\begin{pmatrix} 0 & N_{12} \\ N_{21} & 0 \end{pmatrix} \qquad (11.4-22)$$

Proof. Let $\tilde{\Delta} = \text{diag}\{\Delta_1, \Delta_2\}$, $\bar{\sigma}(\Delta_i) \leq 1$; Δ_i may have additional structure

$$\mu_{\tilde{\Delta}}\begin{pmatrix} 0 & N_{12} \\ kc_T N_{21} & 0 \end{pmatrix} < k$$

$$\Leftrightarrow \det\left[I + \frac{1}{k}\begin{pmatrix} 0 & N_{12} \\ kc_T N_{21} & 0 \end{pmatrix}\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}\right] \neq 0$$

$$\Leftrightarrow \det\begin{pmatrix} I & \frac{1}{k}N_{12}\Delta_2 \\ c_T N_{21}\Delta_1 & I \end{pmatrix} \neq 0$$

$$\Leftrightarrow \det(I - \frac{c_T}{k}N_{12}\Delta_2 N_{21}\Delta_1) \neq 0$$

$$\Leftrightarrow \det\begin{pmatrix} I & \sqrt{\frac{c_T}{k}}N_{12}\Delta_2 \\ \sqrt{\frac{c_T}{k}}N_{21}\Delta_1 & I \end{pmatrix} \neq 0$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{pmatrix} 0 & N_{12} \\ N_{21} & 0 \end{pmatrix} = \sqrt{\frac{k}{c_T}}$$

□

Let us first discuss the assumptions made in Thm. 11.4-1. In general, the bound (11.4–19) results from a robust stability or robust performance condition and T is a particular transfer matrix of interest (e.g., $\tilde{H}$ or $\tilde{E}$). In this case M is (internally) stable and $\det(I - N_{22}T) \neq 0 \ \forall\omega$ as required by the assumption. Furthermore since $\mu_\Delta(M) \geq \mu_\Delta(N_{11})$, the condition $\mu_\Delta(N_{11}) < k$ is necessary for a solution of (11.4–21) to exist. If $\mu(M) < k(\omega)$ is a robust stability (performance) condition, then the condition $\mu(N_{11}) < k(\omega)$ is equivalent to requiring that the robust stability (performance) condition be satisfied for $T = 0$ at this frequency.

Condition (11.4–20) is necessary *and* sufficient for (11.4–19) if (11.4–19) is to be satisfied for *all* T's satisfying $\bar{\sigma}(T) \leq c_T$. (This follows directly from the proof of the theorem.) In most cases we are interested only in a specific M (and a specific T), and condition (11.4–20) is only sufficient for (11.4–19).

However, the value of c_T solving (11.4–21) provides the *least conservative* bound which may be derived on $\bar{\sigma}(T)$ such that (11.4–19) is satisfied.

Note that (11.4–21) is computed based on the structure of Δ *and of* T. The least restrictive bound on $\bar{\sigma}(T)$ (c_T large) is found when $T = tI$ is assumed, where t is a scalar, and the most restrictive bound (c_T small) when T is a full matrix. The reason is that by requiring $T = tI$, the class of perturbations is restricted, and the magnitude of the perturbations is allowed to be larger.

Theorem 11.4-1 may be used to derive a bound on any transfer matrix T which is related to M through a linear fractional transformation (LFT). Note that these bounds (e.g., on $\bar{\sigma}(\tilde{H})$ and $\bar{\sigma}(\tilde{E})$) may be *combined* over different frequency ranges since Thm. 11.4-1 applies on a frequency-by-frequency basis. This provides a powerful method for deriving simple robustness bounds.

Table 11.4-1.

Case		$\mu^2_{\tilde{\Delta}}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$
1	Δ and T full	$\bar{\sigma}(A)\bar{\sigma}B$
2	$T = tI$	$\mu_\Delta(AB)$
3	$\Delta = \delta I$	$\mu_T(BA)$
4	$\Delta = \delta I, T = tI$	$\rho(AB) = \rho(BA)$
5	$B = I$	$\mu_{T\Delta}(A)$

The theorem below which follows directly from Thms. 11.4-1 and 11.4-2 is useful in specific applications.

Theorem 11.4-3: *Let* $\tilde{\Delta} = \text{diag}\{\Delta, T\}$. *Then*

$$\mu_\Delta(ATB) \leq \bar{\sigma}(T)\mu^2_{\tilde{\Delta}}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \qquad (11.4-23)$$

(Note that ATB *and* T *are square matrices, while* A *and* B *may be non-square.) In special cases* $\mu^2_{\tilde{\Delta}}\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ *may be evaluated in terms of other quantities as shown in Table 11.4-1.*

Proof. From Thms. 11.4-1 and 11.4-2 for the case $N_{11} = N_{22} = 0$:

$$\mu_\Delta(N_{12}TN_{21}) < k \quad \text{if} \quad \bar{\sigma}(T)\mu^2_{\tilde{\Delta}}\begin{pmatrix} 0 & N_{12} \\ N_{21} & 0 \end{pmatrix} < k \qquad (11.4-24)$$

Since (11.4–24) holds for *any* choice of k it is equivalent to

$$\mu_\Delta(N_{12}TN_{21}) \leq \bar{\sigma}(T)\mu^2_{\tilde{\Delta}}\begin{pmatrix} 0 & N_{12} \\ N_{21} & 0 \end{pmatrix}$$

Inequality (11.4–23) follows by choosing $N_{12} = A, N_{21} = B$.

The relations in Table 11.4-1 are proved next. Let Δ_1 and Δ_2 have the same *structure* as Δ and T in the theorem. Define $\tilde{\Delta} = \text{diag}\{\Delta_1, \Delta_2\}$, $\bar{\sigma}(\Delta_i) \leq 1$. Then

$$\mu^2_{\tilde{\Delta}}\begin{pmatrix}0 & A\\ B & 0\end{pmatrix} < 1/k$$

$$\Leftrightarrow \mu_{\tilde{\Delta}}\begin{pmatrix}0 & kA\\ B & 0\end{pmatrix} < 1$$

$$\Leftrightarrow \det\left[I + \begin{pmatrix}0 & kA\\ B & 0\end{pmatrix}\begin{pmatrix}\Delta_1 & 0\\ 0 & \Delta_2\end{pmatrix}\right] \neq 0 \qquad \forall \Delta_1, \Delta_2$$

$$\Leftrightarrow \det\begin{pmatrix}I & kA\Delta_2\\ B\Delta_1 & I\end{pmatrix} \neq 0 \qquad \forall \Delta_1, \Delta_2$$

$$\Leftrightarrow \det(I - kA\Delta_2 B\Delta_1) = \det(I - kB\Delta_1 A\Delta_2) \neq 0 \qquad \forall \Delta_1, \Delta_2$$

$$\Leftrightarrow \mu_{\Delta_1}(A\Delta_2 B) < \frac{1}{k}, \qquad \forall \Delta_2 \qquad (11.4-25)$$

$$\Leftrightarrow \mu_{\Delta_2}(B\Delta_1 A) < \frac{1}{k}, \qquad \forall \Delta_1 \qquad (11.4-26)$$

$$\Leftrightarrow \rho(A\Delta_2 B\Delta_1) = \rho(B\Delta_1 A\Delta_2) < \frac{1}{k} \qquad \forall \Delta_1, \Delta_2 \qquad (11.4-27)$$

1: From the basic properties of norms $\rho(A\Delta_2 B\Delta_1) \leq \bar{\sigma}(A)\bar{\sigma}(B)$. We now have to show that this holds as an equality for some choice of Δ_1, Δ_2. Let $A = U_A\Sigma_A V_A^H$ and $B = U_B\Sigma_B V_B^H$. Since Δ_1 and Δ_2 are full we may choose them such that $\Delta_2 U_B = V_A$ and $V_B^H\Delta_1 = U_A^H$. Then $\rho(A\Delta_2 B\Delta_1) = \rho(U_A\Sigma_A\Sigma_B U_A^H) = \rho(\Sigma_A\Sigma_B) = \bar{\sigma}(A)\bar{\sigma}(B)$.

2: From (11.4-25); 3: From (11.4-26); 4,5: From (11.4-27). □

11.4.3 Examples

Example 11.4-2 (Input Uncertainty.) If there is only multiplicative input uncertainty of magnitude $\bar{\sigma}(\Delta_I) < w(\omega)$ we find [see (11.4–11)] the necessary and sufficient condition for robust stability

$$\mu(\tilde{P}^{-1}\tilde{H}\tilde{P}) \leq w^{-1}(\omega) \qquad (11.4-28)$$

Here μ is computed with respect to the structure of Δ_I which may be a diagonal matrix. The least conservative bound on $\bar{\sigma}(\tilde{H})$ which may be derived from (11.4–28) is found using Thm. 11.4-3:

$$\bar{\sigma}(\tilde{H}) \leq 1/\mu^2\begin{pmatrix}0 & \tilde{P}^{-1}\\ \tilde{P} & 0\end{pmatrix} w(\omega) \Rightarrow \text{Robust Stability} \qquad (11.4-29)$$

μ in (11.4–29) is computed with respect to the structure diag$\{\Delta_I, H\}$. Note the following special cases:

(i) Δ_I and H are both full matrices: $\mu^2\begin{pmatrix} 0 & \tilde{P}^{-1} \\ \tilde{P} & 0 \end{pmatrix} = \kappa(\tilde{P})$, where $\kappa(\tilde{P})$ is the condition number of the plant (compare Sec. 11.1.4).

(ii) $H = hI : \mu^2\begin{pmatrix} 0 & P^{-1} \\ P & P \end{pmatrix} = 1$

□

Example 11.4-3 (Robust Performance for SISO Plant.) For multiplicative uncertainty ($|\ell(i\omega)| \leq w_O$) we find the necessary and sufficient condition for robust performance ($\bar{\sigma}(Ew_p) < 1$)

$$\mu(M) < 1 \qquad (11.4-30)$$

where (see Sec. 11.3.2)

$$M = \begin{pmatrix} w_O\tilde{H} & w_O\tilde{H} \\ w_P\tilde{E} & w_P\tilde{E} \end{pmatrix} \qquad (11.4-31)$$

The SSV in (11.4-30) is computed with respect to the diagonal 2×2-matrix diag$\{\Delta_O, \Delta_P\}$. Bounds on $\bar{\sigma}(\tilde{H}) = |\tilde{H}|$ and $\bar{\sigma}(\tilde{E}) = |\tilde{E}|$ are easily derived using Thm. 11.4-1 with $k = 1$. Write M as a LFT of H:

$$N_{11}^H = \begin{pmatrix} 0 & 0 \\ w_P & w_P \end{pmatrix} \quad N_{12}^H = \begin{pmatrix} w_O \\ -w_P \end{pmatrix} \quad N_{21}^H = (1 \quad 1) \quad N_{22}^H = 0$$

Theorem 12.4-1 provides the sufficient condition for robust performance: $|\tilde{H}| \leq c_H$, $\forall\omega$ where c_H solves at each frequency

$$\mu\begin{pmatrix} 0 & 0 & w_O \\ w_P & w_P & -w_P \\ c_H & c_H & 0 \end{pmatrix} = 1 \qquad (11.4-32)$$

The SSV μ in (11.4–32) is computed with respect to the structure diag$\{\Delta_O, \Delta_P, \tilde{H}\}$ — i.e., a diagonal 3×3 matrix. Note that (11.4–32) is independent of the plant model $\tilde{P}$. However, M (and therefore $\tilde{H}$) must be *stable*, and this implicitly makes the allowable $\tilde{H}$'s dependent on $\tilde{P}$. An analytic expression for c_H may be derived for this simple case.

$$|\tilde{H}| \leq c_H = \frac{1 - |w_P|}{|w_O| + |w_P|} \Rightarrow \text{Robust Performance} \qquad (11.4-33)$$

Similarly, a condition in terms of $\tilde{E}$ is derived

$$|\tilde{E}| \leq c_E = \frac{1 - |w_O|}{|w_O| + |w_P|} \Rightarrow \text{Robust Performance} \qquad (11.4-34)$$

The expressions for c_H and c_E in (11.4–33) and (11.4–34) are most easily derived from the identity (see Sec. 11.3.2)

$$\mu \begin{pmatrix} w_O\tilde{H} & w_O\tilde{H} \\ w_P\tilde{E} & w_P\tilde{E} \end{pmatrix} = |w_O\tilde{H}| + |w_P\tilde{E}| \qquad (11.4-35)$$

combined with the triangle inequality (e.g., use $|\tilde{E}| = |1 - \tilde{H}| \leq 1 + |\tilde{H}|$ to derive (11.4–33)). Note that (11.4–33) is impossible to satisfy at low frequencies where tight performance is desired and $|w_P|$ is larger than one (corresponds to $\mu(N_{11}) > k$ in Thm. 11.4-1). Similarly, (11.4–34) is impossible to satisfy at high frequencies where the uncertainty exceeds 100% and $|w_O|$ is larger than one. However, we may combine the bounds: (11.4–30) is satisfied if (11.4–34) is satisfied at low frequencies and (11.4-33) at high frequencies. The bounds (11.4–33) and (11.4–34) (even when combined) tend to be conservative around cross-over where $|\tilde{H}|$ and $|\tilde{E}|$ have similar magnitude. This means that there will be systems which satisfy (11.4–30), but do not satisfy (11.4–33) and (11.4–34).

Conditions (11.4–33) and (11.4–34) are shown graphically in Fig. 11.4-3A for the choice $w_O(s) = 0.2(0.5s+1)$ and $w_P(s) = 0.5(1+s^{-1})$. Assume that the plant is minimum phase such that $\tilde{H} = (s+1)^{-1}$ is an allowable (stable) closed-loop transfer function. This corresponds to a nominal first-order response with time constant one. This choice is seen to satisfy (11.4–33) for $\omega > 1.2$ and (11.4–34) for $\omega < 2$ (Fig. 11.4-3B). Consequently, (11.4–30) is satisfied at all frequencies and robust performance is guaranteed. □

11.5 Summary

The simplest uncertainty description for MIMO systems is in terms of a single norm-bounded perturbation matrix with the same dimensions as the plant. Typical examples are multiplicative output (L_O) and multiplicative input (L_I) uncertainty:

$$P = (I + L_O)\tilde{P}; \quad L_O = (P - \tilde{P})\tilde{P}^{-1}; \quad \bar{\sigma}(L_O(i\omega)) \leq \bar{\ell}_O(\omega), \forall\omega \qquad (11.1-2,5)$$

$$P = \tilde{P}(I + L_I); \quad L_I = \tilde{P}^{-1}(P - \tilde{P}); \quad \bar{\sigma}(L_I(i\omega)) \leq \bar{\ell}_I(\omega), \forall\omega \qquad (11.1-3,5)$$

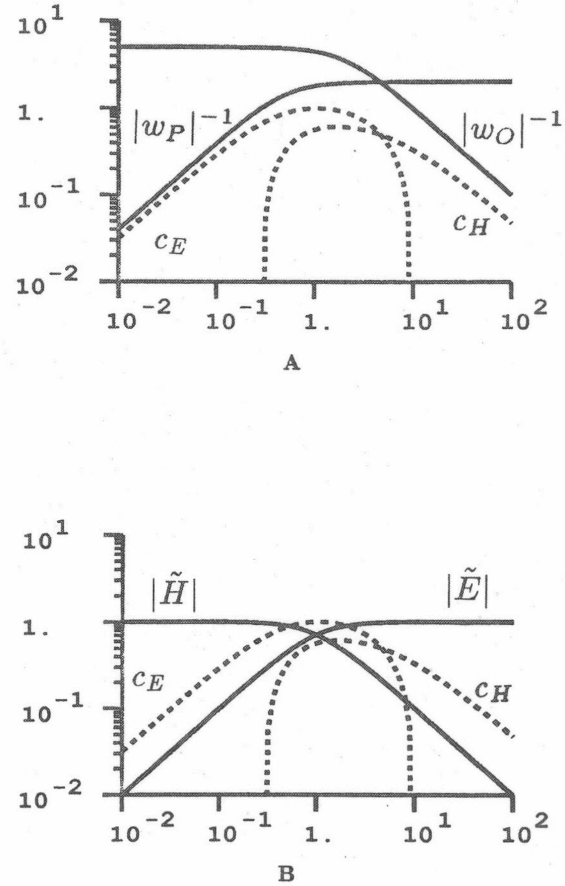

Figure 11.4-3. Graphical representation of conditions (11.4–33) and (11.4–34). Robust performance is guaranteed since $|\tilde{E}| < c_E$ for $\omega < 2$ and $|\tilde{H}| < c_H$ for $\omega > 1.4$.

The advantage of these uncertainty descriptions is that they lead to very simple necessary and sufficient robust stability conditions:

$$\bar{\sigma}(\tilde{P}C(I+\tilde{P}C)^{-1})\bar{\ell}_O = \bar{\sigma}(\tilde{H})\bar{\ell}_O < 1 \quad \forall\omega \qquad (11.1-11)$$

$$\bar{\sigma}(C(I+\tilde{P}C)^{-1}\tilde{P})\bar{\ell}_I = \bar{\sigma}(\tilde{H}_I)\bar{\ell}_I < 1 \quad \forall\omega \qquad (11.1-13)$$

It follows directly that robust integral control is possible for an open loop stable system if and only if $\bar{\ell}_O(0) < 1$ or $\bar{\ell}_I(0) < 1$. This implies that the gain matrix must remain nonsingular for all perturbations (Thm. 11.1-3).

It is often difficult to model the physical uncertainty accurately and non-conservatively with single perturbations. Therefore an uncertainty description involving multiple norm-bounded perturbations Δ_i was introduced. Employing weighting matrices W_1 and W_2 the actual perturbation is

$$L_i = W_2\Delta_i W_1 \qquad (11.2-2)$$

where Δ_i may be any rational transfer matrix satisfying $\bar{\sigma}(\Delta_i) \leq 1$, $\forall\omega$. The individual perturbations are combined into one large block diagonal perturbation matrix

$$\Delta = \text{diag}\{\Delta_1, \ldots, \Delta_m\}\,;\; \bar{\sigma}(\Delta) \leq 1 \qquad (11.2-3,4)$$

Many practical uncertainty problems can be cast into the $M-\Delta$ structure shown in Fig. 11.1-2 where Δ is of the form (11.2–3,4). Assuming that M is stable a necessary and sufficient condition for robust stability can be established via the Structured Singular Value (SSV) μ

$$\mu_\Delta(M(i\omega)) < 1 \qquad \forall\omega \qquad (11.2-10)$$

where the subscript Δ indicates that μ is computed with respect to the ***structure*** of Δ. Among the problems which can be treated in this framework are simultaneous input and output uncertainty (Sec. 11.2.3), and independent uncertainty in the transfer matrix elements (Sec. 11.2.5). The latter type imposes severe constraint on performance if the plant is ill-conditioned: if the nominal response is decoupled — i.e., $\tilde{H} = \text{diag}\{\tilde{\eta}_i\}$ — then robust stability is guaranteed if

$$|\tilde{\eta}_i| < \frac{1}{r_{max}\sqrt{n}\kappa^*(P)} \qquad \forall\omega, \forall i \qquad (11.2-36)$$

where r_{max} is the maximum relative element uncertainty, n the dimension of the system and κ^* the minimized condition number. Tighter conditions can be derived for 2×2 systems. The minimized condition number κ^* is closely related to the Relative Gain Array Λ:

$$\kappa^*(P) \cong \|\Lambda\|_m \qquad \text{for } \kappa^*(P) \text{ large}$$

where

$$\|\Lambda\|_m = 2 \cdot \max\{\|\Lambda\|_1,\ \|\Lambda\|_\infty\} \qquad (11.2-46)$$

Thus, systems with large RGA are very sensitive to *independent* element uncertainty. In practice, however, the variations of the transfer matrix elements are usually highly *correlated*.

The major advantage of the H_∞ performance objective is that it allows us to express the robust *performance* test as a robust *stability* test in the presence of a structured perturbation (Thm. 11.3-1): the nominally stable system G (Fig. 11.3-1) subjected to the block diagonal uncertainty Δ_u $(\bar{\sigma}(\Delta_u) \leq 1)$ satisfies the H_∞ robust performance condition if and only if

$$\mu_\Delta(G) < 1 \qquad \forall\omega \qquad (11.3-8)$$

For multiplicative output uncertainty the SSV in (11.3–8) can be approximated by

$$\mu_\Delta(G) \leq \bar{\sigma}(w\tilde{E}) + \bar{\sigma}(\bar{\ell}_O\tilde{H}) \qquad (11.3-15)$$

and for multiplicative input uncertainty by

$$\mu_\Delta(G) \leq \bar{\sigma}(w\tilde{E}) + \kappa(\tilde{P})\bar{\sigma}(\bar{\ell}_O\tilde{H}) \qquad (11.3-20)$$

or

$$\mu_\Delta(G) \leq \bar{\sigma}(w\tilde{E}) + \kappa(C)\bar{\sigma}(\bar{\ell}_O\tilde{H}) \qquad (11.3-22)$$

Thus, multiplicative output uncertainty does not cause any robust performance difficulties: if both the nominal performance $(\bar{\sigma}(w\tilde{E}) < 1)$ and the robust stability $(\bar{\sigma}(\bar{\ell}_O\tilde{H}) < 1)$ conditions are satisfied with some margin, then robust performance is guaranteed automatically. On the other hand input uncertainty causes robust performance problems when either the plant or the controller is ill-conditioned.

Sometimes, it is attractive to express robust stability and performance conditions in terms of bounds on transfer functions of direct engineering significance (e.g., $\tilde{H}$ or $\tilde{E}$) rather than in an implicit manner (11.3–8). This is possible if the transfer matrix G is related to the transfer matrix T of interest through a linear fractional transformation (Sec. 11.4). However, contrary to condition (11.3–8) these bounds are only sufficient for robust stability and performance.

11.6 References

Sections 11.1 through 11.3 closely follow the paper by Skogestad & Morari (1987b). All the examples in these sections are also taken from that paper.

11.1.2. The general robust stability theorem is covered in the paper by Doyle & Stein (1981). They emphasize, in particular, the *necessity* of the theorem. The proof presented here is patterned after that by Lehtomaki (1981). A detailed explanation of what is meant by a "connected" set of plants is provided by Vidyasagar, et al. (1982) and Postlethwaite & Foo (1985).

11.1.5. Postlethwaite & Foo (1985) show that Cor. 11.1-1 and 11.1-3 can be combined over different frequency ranges.

11.1.7. A condition for robust integral control similar to (11.1–33) was proved by Garcia & Morari (1985a).

11.2.2. The Structured Singular Value was introduced by Doyle (1982) who also discussed its properties and a generalized gradient search procedure to minimize its upper bound. Osborne (1960) developed the iterative scheme to minimize $\|DMD^{-1}\|_F$. An efficient procedure for computing the SSV based on its lower bound was proposed by Fan & Tits (1986).

11.2.5. Alternative *sufficient* stability conditions in the presence of element by element uncertainty were derived by Kantor & Andres (1983) and Kouvaritakis & Latchman (1985). The claim about necessity in the latter paper is incorrect.

11.2.6. Morari (1983a), Morari et al. (1985) and Grosdidier, Morari & Holt (1985) argued in a somewhat qualitative manner that for robust stability the minimized condition number is a measure of sensitivity with respect to uncertainty. Bristol (1966), in his original paper on the RGA, pointed out its similarity with the condition number. A quantitative relationship between the two was first established by Grosdidier et al. (1985) and then extended by Nett & Manousiouthakis (1987). Tighter but more complicated conditions for robust stability involving the condition number were derived by Skogestad & Morari (1987b).

11.3. Doyle & Wall (1982) and Doyle (1984) pointed out the equivalence between robust stability and robust performance and proposed the SSV as a tool to assess robust performance.

11.3.2, 11.3.3. The sufficient robust performance conditions in the presence of multiplicative input and output uncertainty were derived by Stein (1985).

11.3.4. The method in this section was proposed by Zafiriou and Morari (1986b).

11.3.5. The example is taken from the paper by Skogestad, Morari & Doyle (1988). The procedure described by Doyle (1985) was used to find the "μ-optimal" controller.

11.4. This section follows closely the paper by Skogestad & Morari (1988c). Postlethwaite & Foo (1985) also derive robustness conditions in terms of bounds on transfer matrices of interest. However, in particular for structured uncertainty, their bounds are not as tight as the bounds derived here.

11.4.1. Doyle (1984) showed that $N_{22} = 0$ when T is a *closed loop* transfer function.