# Part II

# SAMPLED DATA SINGLE-INPUT SINGLE-OUTPUT SYSTEMS

# Chapter 7

# FUNDAMENTALS OF SAMPLED-DATA SYSTEMS CONTROL

The chapters in this book dealing with the control of sampled-data systems are not self-contained. It is assumed that the reader has mastered the preceding chapters addressing the same topics for continuous-time systems. Indeed, some issues which are essentially identical (for example, the design of two-degree-of-freedom controllers) are completely omitted. In other cases only those features which distinguish sampled-data systems from continuous systems are emphasized. The equivalence of the classic feedback structure with the IMC structure was firmly established for continuous systems. Therefore, rather than deriving all stability and performance conditions first for the classic feedback structure (Chap. 2) and then translating them to the IMC structure (Chap. 3) we will proceed directly with the IMC structure after some general definitions and results for sampled-data systems control.

Out treatment of sampled-data systems is different from that in many other books in that we define performance in terms of the *continuous* plant output – i.e., we pay close attention to the intersample behavior.

## 7.1 Sampled-Data Feedback Structure

The block diagram of a typical computer-controlled system is shown in Fig. 7.1-1A. Thick lines are used to represent the paths along which the signals are continuous (analog). The sampling switch is used to describe the A/D converter which is modelled as an impulse modulator. When a signal $a(t)$ is fed to a switch with a sampling time $T$, it yields as an output the impulse sequence $a^*(t)$

$$a^*(t) = \sum_{k=0}^{\infty} a(kT)\delta(t-kT) \qquad (7.1-1)$$

The Laplace transform of $a^*(t)$ is

$$\mathcal{L}\{a^*(t)\} = a^*(e^{sT}) = \sum_{k=0}^{\infty} a(kT)e^{-skT} \quad (7.1-2)$$

Alternatively we can represent the impulse sequence by its Fourier series

$$a^*(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} a(t)e^{ik\omega_s t} \quad (7.1-3)$$

where $\omega_s$ is the sampling frequency

$$\omega_s = \frac{2\pi}{T} \quad (7.1-4)$$

From (7.1–3) we obtain a different representation of $\mathcal{L}\{a^*(t)\}$

$$\mathcal{L}\{a^*(t)\} = a^*(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} a(s+ik\omega_s) \quad (7.1-5)$$

The transformation

$$z = e^{sT} \quad (7.1-6)$$

will be used throughout the book. Then $a^*(z)$ is the $z$-transform of the signal $a(t)$. The following notation describes (7.1–2) and (7.1–5).

$$a^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{a(s)\} \quad (7.1-7)$$

It is clear from (7.1–5) that $a^*(e^{i\omega T})$ is periodic in $\omega$ with period $\omega_s$. It is also important to note that for a rational function $a^*(z)$ we have $a^*(z)^H = a^*(z^H)$, where the superscript $^H$ indicates complex conjugate, and therefore for $\pi/T < \omega < 2\pi/T$ we have:

$$a^*(e^{i\omega T})^H = a^*(e^{-i\omega T}) = a^*(e^{i(\omega_s-\omega)})^T \quad (7.1-8)$$

Hence, in addition to periodicity, a rational $z$-transform $a^*(z)$ has the property that its values for frequencies larger than $\pi/T$ are uniquely determined by those for $0 \leq \omega \leq \pi/T$.

For the signals in Fig. 7.1-1A we have

$$r^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{r(s)\} \quad (7.1-9)$$

$$d^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{d(s)\} \quad (7.1-10)$$

$$y^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{y(s)\} \quad (7.1-11)$$

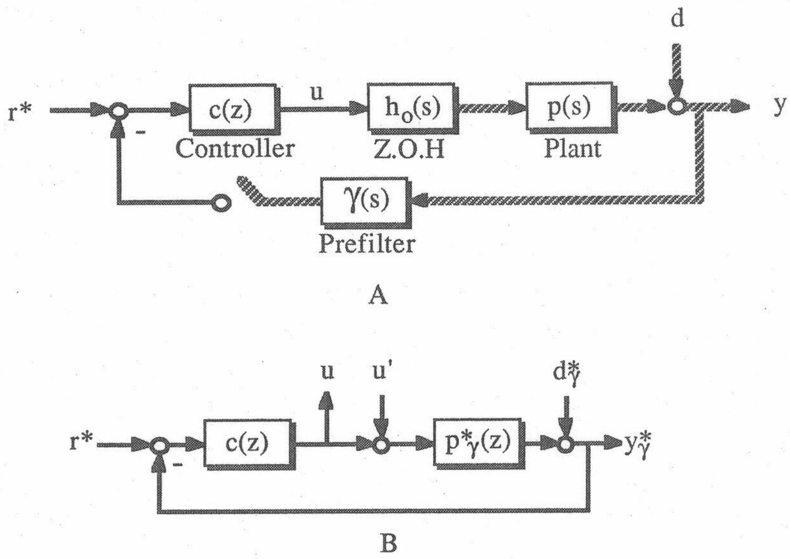



*Figure 7.1-1.* Block diagram of computer controlled system. A: Sampled-data structure with thick lines indicating analog signals. B: Discrete structure with all signals discrete.

The controller $z$-transfer function $c(z)$ represents a difference equation, which models the computer program. The zero-order hold $h_0(s)$ models the D/A converter, which constructs the piecewise-constant input to the plant from the impulse sequence described by $u(z)$. We have

$$h_0(s) = \frac{1 - e^{-sT}}{s} \qquad (7.1-12)$$

The block $\gamma(s)$ represents an analog anti-aliasing prefilter. Briefly one can understand the problem of aliasing from (7.1–5). After substitution of $s$ with $i\omega$, it follows that the value of $a^*$ at a frequency $\omega$ is the sum of the values of the continuous signal $a$ at the frequencies $\omega + k\omega_s$ divided by $T$. The result is that after sampling, a high-frequency disturbance or measurement noise cannot be distinguished from an equivalent low frequency one. The objective of the prefilter is to cut off high frequency components from the analog signals before sampling, when that is necessary. Its transfer function is stable.

Note that no measurement device block is included in Fig. 7.1-1. When the dynamics of the measurement device function are significant they can be included in the prefilter $\gamma(s)$.

When the continuous output $y$ is not observed directly but after the prefilter and only at the sampling intervals, then Fig. 7.1-1A can be simplified to Fig.

7.1-1B.

$$d_\gamma^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{\gamma(s)d(s)\} \tag{7.1-13}$$

$$y_\gamma^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{\gamma(s)y(s)\} \tag{7.1-14}$$

Here all signals are impulse sequences. The block $p_\gamma^*(s)$ is the pulse transfer function representing the zero-order hold equivalent of $p(s)\gamma(s)$. We define

$$p_\gamma^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)p(s)\gamma(s)\} \tag{7.1-15}$$

and similarly

$$p^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)p(s)\} \tag{7.1-16}$$

Pulse transfer functions are always rational in $z$, although the continuous transfer functions may include time delays. Time delays appear as poles at $z = 0$. It should also be noted that in the case of pulse transfer functions, the definitions of *proper* and *causal* in the spirit of Sec. 2.1 coincide.

**Definition 7.1-1.** *A system $g(z)$ is proper or causal if $\lim_{z\to\infty} g(z)$ is finite. A proper system is strictly proper if $\lim_{z\to\infty} g(z) = 0$ and semiproper if $\lim_{z\to\infty} |g(z)| > 0$. All systems which are not proper are called improper or noncausal.*

A system $g(z)$ is improper if the order of the numerator polynomial exceeds the order of the denominator polynomial and proper otherwise. An improper system is not physically realizable because it requires prediction.

It is useful to understand the relationship between the poles and the zeros of a continuous-time system and of the corresponding discrete-time system. Poles are mapped in a simple manner: if $\pi_i$ is a pole of the continuous system then $e^{\pi_i T}$ is a pole of the corresponding discrete system (zero order hold included). It is not possible to give a simple formula for the mapping of the zeros. The zeros of the discrete-time system depend on the sampling period. In particular, it is possible for a discrete-time system to have zeros outside the unit circle (UC) even when the corresponding continuous system is MP. The converse can also happen.

It is well known that poles of a continuous system can become unobservable by sampling. We will assume throughout the book that the sampling rate has been chosen such that all unstable poles of the continuous system $p(s)$ appear in the pulse transfer function $p_\gamma^*(z)$. With this assumption the internal stability of the system in Fig. 7.1-1A can be assessed in terms of the system in Fig. 7.1-1B.

**Theorem 7.1-1.** *The sampled-data system in Fig. 7.1-1A is internally stable if and only if the transfer matrix in (7.1-17)*

$$\begin{pmatrix} y_\gamma^* \\ u \end{pmatrix} = \begin{pmatrix} \frac{p_\gamma^* c}{1+p_\gamma^* c} & \frac{p_\gamma^*}{1+p_\gamma^* c} \\ \frac{c}{1+p_\gamma^* c} & \frac{-p_\gamma^* c}{1+p_\gamma^* c} \end{pmatrix} \begin{pmatrix} r^* \\ u' \end{pmatrix} \tag{7.1-17}$$

*is stable — i.e., if and only if all poles of the four pulse-transfer functions are strictly inside the unit circle (UC).*

## 7.2 IMC Structure

The block diagram of the sampled-data IMC loop is shown in Fig. 7.2-1A. The block $\tilde{p}^*_\gamma(z)$ is the pulse transfer function representing the zero order hold equivalent of $\tilde{p}(s)\gamma(s)$, where $\tilde{p}(s)$ is the continuous plant model. We define

$$\tilde{p}^*_\gamma(z) = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)\tilde{p}(s)\gamma(s)\} \qquad (7.2-1)$$

and similarly

$$\tilde{p}^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)\tilde{p}(s)\} \qquad (7.2-2)$$

The same block manipulations as in the continuous case can be used here to derive the relations between the feedback controller $c(z)$ and the IMC controller $q(z)$:

$$c = \frac{q}{1 - \tilde{p}^*_\gamma q} \qquad (7.2-3)$$

$$q = \frac{c}{1 + \tilde{p}^*_\gamma c} \qquad (7.2-4)$$

When $c$ and $q$ are related through (7.2–3) or (7.2–4), $u(z)$ and $y(s)$ react to inputs $r^*(z)$ and $d(s)$ in exactly the same way for both the classic feedback and the IMC structure.

In Fig. 7.2-1B a different configuration is drawn for the sampled-data IMC structure. This configuration is equivalent to that of Fig. 7.2-1A, but is not suitable for computer implementation because of the presence of the continuous model $\tilde{p}(s)$. However Fig. 7.2-1B demonstrates the properties of the IMC structure, that were discussed in Sec. 3.1, in a clearer way.

If only the sampled signals are of interest, then Fig. 7.2-1A and B are equivalent to Fig. 7.2-1C where all signals are digital.

Finally, it should be noted that the implicit assumption has been made throughout this section that an exact model is available for the anti-aliasing prefilter $\gamma(s)$. The simplicity of this control-loop element makes this assumption valid and allows us to avoid unnecessary complications.

## 7.3 Formulation of Control Problem

For the design of a discrete controller the same items have to be specified as in the continuous case:

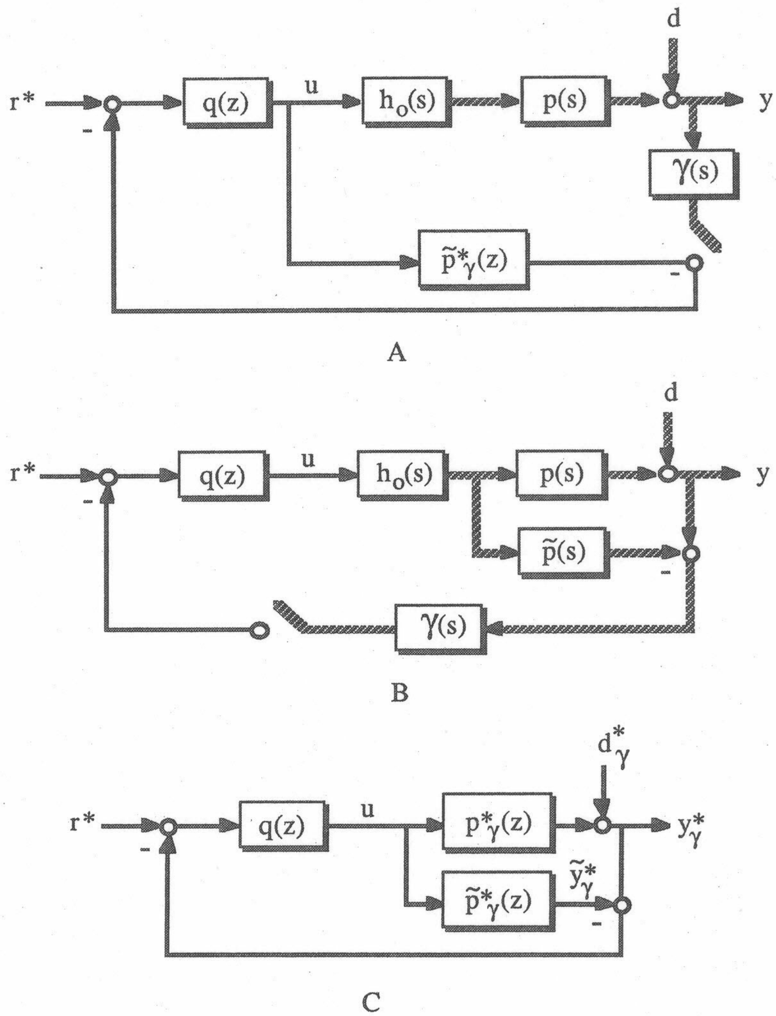



*Figure 7.2-1.* IMC structure. A: Sampled-data structure; B: Structure equivalent to (A) but not implementable; C: Discrete structure (all signals discrete).

- process model
- model uncertainty bounds
- type of inputs
- performance objectives

The process model can be continuous or discrete. There are advantages to starting with a continuous model (see Sec. 7.3.1). The inputs of interest, in particular the disturbances, are continuous in nature. Therefore the same input specifications (specific inputs, sets of inputs) as discussed in Sec. 2.2.3 are relevant here. In terms of performance, one is usually interested in the behavior of the *continuous* rather than the *sampled* output. The fact that only the sampled output is available to the controller leads to some complications in the specification of a meaningful design objective which will be addressed in Sec. 7.5.

### 7.3.1 Process Model

Most popular identification schemes generate pulse transfer function models. Such models are sufficient for control system design but do not allow the analysis of the intersample behavior which can be significantly worse than the behavior at the sampling points, as we will show later in this chapter. Furthermore, model uncertainty is more naturally described in terms of the continuous system. Thus, it is desirable that a continuous system model be available. The system itself will be assumed to be linear and time invariant but not necessarily finite dimensional. Systems with time delays do not cause any problems for the design of discrete controllers.

### 7.3.2 Model Uncertainty Description

In Sec. 2.2.2 the additive and multiplicative uncertainty descriptions were presented, which assume that for each frequency $\omega$, the actual plant $p(i\omega)$ lies in a disk-shaped region of known radius around the model $\tilde{p}(i\omega)$. For sampled data systems we also need to know how far $p^*_\gamma(e^{sT})$ lies from the known $\tilde{p}^*_\gamma(e^{sT})$. This information can be obtained from the information on $p(s)$. Let $p(s)$ belong to the family $\Pi$ of plants defined by

$$\Pi = \{p : |p(i\omega) - \tilde{p}(i\omega)| \le \bar{\ell}_a(\omega)\} \qquad (7.3-1)$$

or equivalently

$$p(i\omega) = \tilde{p}(i\omega) + \ell_a(i\omega) \qquad (7.3-2)$$

$$|\ell_a(i\omega)| \le \bar{\ell}_a(\omega) \qquad \forall p \in \Pi \qquad (7.3-3)$$

From the definitions (7.1–15) and (7.2–1) we find

$$p_\gamma^*(e^{sT}) - \tilde{p}_\gamma^*(e^{sT}) = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)\gamma(s)(p(s) - \tilde{p}(s))\} = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)\gamma(s)\ell_a(s)\} \tag{7.3 - 4}$$

and by using (7.1–5)

$$p_\gamma^*(e^{i\omega T}) - \tilde{p}_\gamma^*(e^{i\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} h_0\gamma\ell_a(i\omega + ik\omega_s) \tag{7.3 - 5}$$

With (7.3–3) we obtain the following bound

$$|p_\gamma^*(e^{i\omega T}) - \tilde{p}_\gamma^*(e^{i\omega T})| \le \frac{1}{T}\sum_{k=-\infty}^{\infty} |h_0\gamma(i\omega + ik\omega_s)|\bar{\ell}_a(\omega + k\omega_s) \triangleq \bar{\ell}_a^*(\omega) \tag{7.3 - 6}$$

The above sum converges because $|h_0\gamma(i\omega)|\bar{\ell}_a(\omega) \to 0$ faster than $1/\omega$ as $\omega \to \infty$. This happens because $|h_0\gamma(i\omega)| \to 0$ at least as fast as $1/\omega$ as $\omega \to \infty$, even if $\gamma(s) = 1$. Also a bound $\bar{\ell}_a(\omega)$ such that $\bar{\ell}_a(\omega) \to 0$ as $\omega \to \infty$ can always be found since any physical system $p(s)$ and its model $\tilde{p}(s)$ are strictly proper and therefore $\ell_a(\omega) \to 0$ as $\omega \to \infty$. Note that if a prefilter $\gamma(s)$ is used, the property $\ell_a(\omega) \to 0$ as $\omega \to \infty$ is not needed for convergence.

Let us now define the family $\Pi^*$ of plants $p(s)$ as follows

$$\Pi^* = \{p(s) : |p_\gamma^*(e^{i\omega T}) - \tilde{p}_\gamma^*(e^{i\omega T})| \le \bar{\ell}_a^*(\omega)\} \tag{7.3 - 7}$$

Clearly $\Pi^*$ depends on the choice of $T$ and $\gamma(s)$. However, the steps to arrive at (7.3–6) imply that if a plant $p(s)$ belongs to $\Pi$, then it also belongs to $\Pi^*$.

Because of the step from (7.3–5) to (7.3–6) the description (7.3–7) is conservative but not much so. The reason is that the sum in (7.3–5) and (7.3–6) has only a few dominant terms: $|\gamma(i\omega)|$ is designed to be small for $\omega > \pi/T$ in order to cut off high frequency components. Also $h_0(i\omega)/T$ is small for $\omega > \pi/T$. Therefore, the only dominant term in (7.3–5) and (7.3–6) is the one for which $-\pi/T \le \omega + k\omega_s \le \pi/T$. Hence for $0 \le \omega \le \pi/T$, the dominant term corresponds to $k = 0$. Computationally it is rare that more than two or three terms are significant.

## 7.4 Internal Stability

Assuming that the sampling time $T$ has been chosen to avoid unobservable unstable poles in $p_\gamma^*$ we only need to study the internal stability of the system in Fig. 7.2-1C where all signals are digital. The internal stability conditions can be stated in terms of pulse transfer functions and the arguments of Section 3.2.1 carry over directly.

**Theorem 7.4-1.** *Assume that the model is perfect ($p(s) = \tilde{p}(s)$); then the IMC system in Fig. 7.2.1A is internally stable if and only if both the plant $p(s)$ and the controller $q(z)$ are stable.*

## 7.5 Nominal Performance

The objective is to keep the *continuous* error $e$ between the plant output $y$ and the reference $r$ small when the overall system is affected by external signals $r$ and $d$. Contrary to the continuous case, there is no transfer function between $d$ and $e$ but the relationship is time varying. We will explain the problem in Sec. 7.5.1 and suggest meaningful approximations.

### 7.5.1 Sensitivity and Complementary Sensitivity Function

From the IMC structure of Fig. 7.2-1A or B we can easily obtain for $p = \tilde{p}$

$$y(s) = h_0(s)\tilde{p}(s)q(e^{sT})(r^*(e^{sT}) - d_\gamma^*(e^{sT})) + d(s) \qquad (7.5-1)$$

We are interested in finding transfer functions relating the external inputs $r(s)$ and $d(s)$ to the error

$$e(s) = y(s) - r(s) \qquad (7.5-2)$$

where $r(s)$ is the Laplace transform of the continuous time function we wish the plant output to follow. The signal $r(s)$ is related to $r^*(e^{sT})$ through (7.1–9) but it does not appear in the block diagrams since no hardware (A/D converter modelled by the sampling switch) is actually used to obtain $r^*(e^{sT})$.

Simple inspection of (7.5–1) indicates that it is not possible to obtain transfer functions relating $r(s)$ and $d(s)$ to $e(s)$. Let us first consider the relation between $r(s)$ and $e(s)$. Equations (7.5–1) and (7.5–2) yield

$$e(s) = h_0(s)\tilde{p}(s)q(e^{sT})r^*(e^{sT}) - r(s) \qquad (7.5-3)$$

Clearly there is no transfer function relating $r(s)$ to $e(s)$. The reason is that the relation is time-varying — i.e., the response of $e(s)$ to $r(s)$ depends on the time relative to the sampling instant at which the signal $r(s)$ is applied. A transfer function can be obtained in the special case when $\mathcal{L}^{-1}\{r(s)\}$ remains constant between sampling instants. In this case we have $r(s) = h_0(s)r^*(e^{sT})$ and then (7.5–3) yields

$$\frac{-e(s)}{r(s)} = \frac{-e(s)}{h_0(s)r^*(e^{sT})} = 1 - \tilde{p}(s)q(e^{sT}) \triangleq \tilde{\epsilon}_r(s) \qquad (7.5-4)$$

The complementary sensitivity function $\tilde{\eta}_r(s)$ relating $y(s)$ to $r(s)$ can be obtained by subtracting the sensitivity function $\tilde{\epsilon}_r(s)$ from unity.

$$\frac{y(s)}{h_0(s)r^*(e^{sT})} = \tilde{p}(s)q(e^{sT}) \triangleq \tilde{\eta}_r(s) \tag{7.5-5}$$

Let us now consider the relation between $d(s)$ and $e(s)$ or equivalently $d(s)$ and $y(s)$. From (7.5–1) we have

$$y(s) = d(s) - h_0(s)\tilde{p}(s)q(e^{sT})d^*_\gamma(e^{sT}) \tag{7.5-6}$$

Again the relation is time varying and there is no transfer function connecting $d(s)$ to $y(s)$. If, of course, $\gamma(s) = 1$ and $\mathcal{L}^{-1}\{d(s)\}$ remained constant between the sampling instants, then we could proceed in a manner similar to that for $r(s)$ and obtain the same expression for the sensitivity function as in (7.5–4). The assumption, however, that $\mathcal{L}^{-1}\{d(s)\}$ is constant between the sampling instants is not realistic.

There are three possible approaches to deal with this problem:

1. The time varying sensitivity operator can be bounded by a "conic sector."
2. The bandwidth of the disturbance signal $d(s)$ can be assumed to be limited and an approximate sensitivity function can be defined.
3. The plant output can be studied at the sampling instants only and an appropriate pulse-transfer function can be derived.

We will discuss the latter two approaches in the following.

*Approximate sensitivity function for bandlimited disturbance signal.* We will assume the disturbance to be approximately limited to the frequency band up to $\pi/T$.

From (7.1–5) we find

$$d^*_\gamma(e^{sT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} d(s + ik\omega_s)\gamma(s + ik\omega_s) \tag{7.5-7}$$

Because $d$ is band limited and because $\gamma$ is designed to attenuate signals at frequencies larger than $\pi/T$, (7.5–7) can be approximated by

$$d^*_\gamma(e^{i\omega T}) \cong \frac{1}{T}d(i\omega)\gamma(i\omega) \qquad 0 \le \omega \le \frac{\pi}{T} \tag{7.5-8}$$

With this approximation (7.5–6) becomes

$$y(i\omega) \cong \left[1 - \frac{1}{T}h_0(i\omega)\tilde{p}(i\omega)q(e^{i\omega T})\gamma(i\omega)\right] d(i\omega) \qquad 0 \le \omega \le \frac{\pi}{T} \tag{7.5-9}$$

Defining the new "controller"

$$\hat{q}(s) = \frac{1}{T} h_0(s) q(e^{sT}) \gamma(s) \qquad (7.5-10)$$

(7.5–9) can be rewritten as

$$y(i\omega) \cong (1 - \tilde{p}(i\omega)\hat{q}(i\omega)) d(i\omega) \triangleq \tilde{\epsilon}_d(i\omega) d(i\omega) \qquad 0 \leq \omega \leq \frac{\pi}{T} \qquad (7.5-11)$$

which is identical in *structure* with what can be obtained for the continuous system. For continuous systems the expression is exact, while for sampled-data systems it represents an approximation of a time-varying relationship.

*Sensitivity pulse-transfer function.* Sampling of (7.5–1) yields

$$y^*(z) = \tilde{p}^*(z) q(z)(r^*(z) - d^*_\gamma(z)) + d^*(z) \qquad (7.5-12)$$

Then by assuming $\gamma(s) = 1$, we can obtain sensitivity and complementary sensitivity pulse-transfer functions, connecting $e^*(z)$ to $r^*(z)$ and $d^*(z)$, where

$$e^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{e(s)\} \qquad (7.5-13)$$

$$\tilde{\epsilon}^*(z) \triangleq 1 - \tilde{p}^*(z) q(z) \qquad (7.5-14)$$

$$\tilde{\eta}^*(z) \triangleq \tilde{p}^*(z) q(z) \qquad (7.5-15)$$

However, disregarding the intersample behavior of the plant output may lead to serious problems as will be illustrated in Sec. 7.5.3.

### 7.5.2 Asymptotic Properties of Closed-Loop Response

"System types" were defined in Sec. 2.4.3 to classify the asymptotic closed-loop behavior. A "Type m" system, where $m$ is a non-negative integer is defined as a system which tracks perfectly, as time $\rightarrow \infty$, inputs $r(s)$ and $d(s)$ with all the poles in the LHP except $m$ or less poles at $s = 0$. The conditions that have to be satisfied in order for this to happen impose certain requirements on the controller $q(z)$ and the anti–aliasing prefilter $\gamma(s)$, described by the following theorem (see for comparison Sec. 3.3.3).

**Theorem 7.5-1.** *Provided that the closed-loop system is stable, the necessary and sufficient conditions for the system to be "Type m" ($m > 0$) are the following:*

$$\lim_{z \to 1} \frac{d^k}{dz^k}(1 - \tilde{p}^*(z) q(z)) = 0, \quad 0 \leq k < m \qquad (7.5-16)$$

$$\lim_{s \to 0} \frac{d^k}{ds^k}(1 - \gamma(s)) = 0, \quad 0 \leq k < m \qquad (7.5-17)$$

*Proof.* The disturbance $d(s)$ goes through $\gamma(s)$ before it is sampled and therefore we clearly need

$$\lim_{\text{time}\to\infty} \mathcal{L}^{-1}\{d(s) - \gamma(s)d(s)\} = \lim_{s\to 0}(s(1-\gamma(s))d(s)) = 0 \qquad (7.5-18)$$

Since (7.5–18) must be satisfied for all $d(s)$ with $m$ or less poles at $s = 0, (1-\gamma(s))$ has to have $m$ zeros at $s = 0$, which will be the case if and only if (7.5–17) holds.

We obtain from Fig. 7.2-1A or B

$$e^*(z) = \frac{p^*(z)q(z)}{1 + q(z)(p^*_\gamma(z) - \tilde{p}^*_\gamma(z))}(r^*(z) - d^*_\gamma(z)) + (d^*(z) - r^*(z)) \qquad (7.5-19)$$

Condition (7.5–17) implies that

$$\lim_{\text{time}\to\infty} \mathcal{Z}^{-1}\{d^*(z) - d^*_\gamma(z)\} = \lim_{z\to 1}(1 - z^{-1})(d^*(z) - d^*_\gamma(z)) = 0 \qquad (7.5-20)$$

Hence for tracking considerations, $d^*_\gamma(z)$ can be replaced by $d^*(z)$ in (7.5–19)

$$e^*(z) = \frac{1 + q(p^*_\gamma - \tilde{p}^*_\gamma) - p^*q}{1 + q(p^*_\gamma - \tilde{p}^*_\gamma)}v^* \qquad (7.5-21)$$

where $v^* = d^* - r^*$. (7.5–17) also implies that

$$\lim_{\text{time}\to\infty} \mathcal{Z}^{-1}\{(p^*_\gamma - p^*)v^*\} = 0 \qquad (7.5-22)$$

Thus, for tracking considerations $p^*_\gamma$ can be replaced by $p^*$ and similarly $\tilde{p}^*_\gamma$ by $\tilde{p}^*$. Then (7.5–21) becomes

$$e^*(z) = \frac{1 - \tilde{p}^*q}{1 + q(p^* - \tilde{p}^*)}v^* \qquad (7.5-23)$$

Assume that $v^*$ has at most $m$ poles at $z = 1$ and apply the final value theorem to (7.5–23). Condition (7.5–16) follows directly. □

The implications of (7.5–16) for the design of $q(z)$ will be considered in Chap. 8. Let us discuss briefly the design of the prefilter $\gamma(s)$, whose objective is to cut off high-frequency components. Most digital control books discuss different types of anti-aliasing prefilters, which satisfy (7.5–16) for $m = 1$, like Butterworth and Bessel filters. For the case of $m > 1$ a simple modification can be used. Let us write

$$\gamma(s) = \gamma_1(s)\gamma_m(s) \qquad (7.5-24)$$

where

$$\gamma_m(s) = \frac{c_{m-1}s^{m-1} + \ldots + c_1 s + 1}{(\tau s + 1)^{m-1}} \qquad (7.5-25)$$

and $\gamma_1(s)$ is an appropriate prefilter for $m = 1$. Then for a specified $\tau$, (7.5–17) can be used to compute the coefficients $c_1, \ldots, c_{m-1}$. Qualitatively it is clear that the use of $\gamma_m(s)$ to satisfy (7.5–17) should not change the behavior of $\gamma_1(s)$ significantly. Condition (7.5–17) simply adds some properties at $\omega = 0$ and this can be done without affecting the high-frequency properties of $\gamma_1(s)$. A large $\tau$ should be used to push the effect of $\gamma_m(s)$ toward $\omega = 0$. Indeed for a usual second-order filter $\gamma_1(s) = \omega_0^2/(s^2 + 2\omega_0\zeta s + \omega_0^2)$ and for $m = 2$ (ramp inputs), (7.5–17) yields $c_1 = \tau + 2\zeta/\omega_0$ and therefore for a sufficiently large $\tau, \gamma_m(s)$ does not affect the high-frequency performance of $\gamma(s)$ significantly.

### 7.5.3 Limitations on Achievable Performance

In Sec. 3.3.4 the concept of "perfect control" was discussed and three sources of limitations on the achievable closed-loop performance were given, namely the NMP characteristics of the plant, constraints on the inputs and model uncertainty. In this section some additional sources, particular to sampled-data control systems will be discussed.

(i) *Intersample rippling*

To demonstrate the problem we shall assume that $p(s) = \tilde{p}(s)$ and $d(s) = 0$. Let us consider the system

$$p(s) = \frac{2}{(s^2 + 1.2s + 1)(s + 2)} \tag{7.5 – 26}$$

and choose a sampling time $T = 1.8$. Then

$$p^*(z) = 0.483\frac{z^2 + 1.01z + 0.0597}{z^3 - 0.116z^2 + 0.118z - 0.00315} \tag{7.5 – 27}$$

The behavior of two control algorithms will be examined:

$$q_1(z) = (zp^*(z))^{-1} \tag{7.5 – 28}$$

$$q_2(z) = 1.001\frac{z^3 - 0.116z^2 + 0.118z - 0.00315}{z^3} \tag{7.5 – 29}$$

The response to a step change in the setpoint $r(s)$ is shown for both algorithms in Fig. 7.5-1. Clearly $q_1(z)$ produces an unacceptable response. However if one concentrated only at the sampling instants, which is equivalent to using (7.5–12) instead of (7.5–1), then it would seem that $q_1(z)$ produces a perfect response which reaches the setpoint in one sampling interval and remains there. On the other hand, although $q_2(z)$ produces an excellent response, if one looked only at the sample points it would seem inferior to that of $q_1(z)$ since it takes three sampling intervals to reach the setpoint.

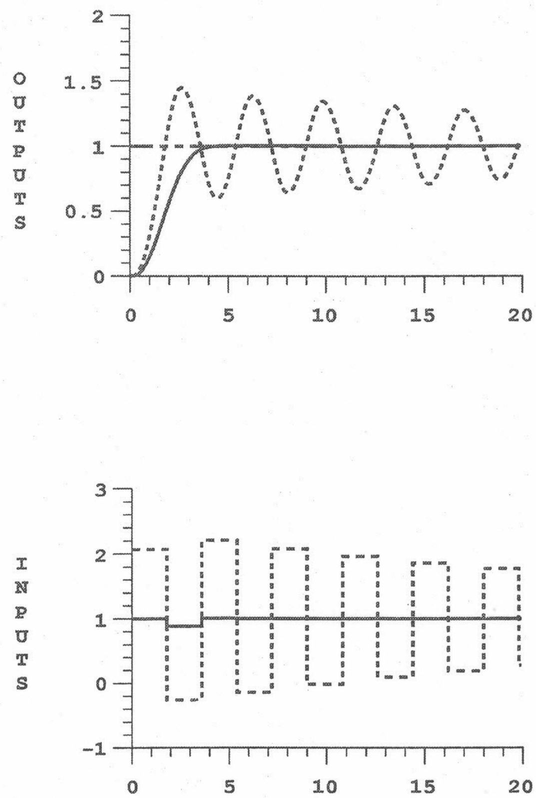



*Figure 7.5-1.* Demonstration of intersample rippling. Dash: $q_1$; Solid: $q_2$.

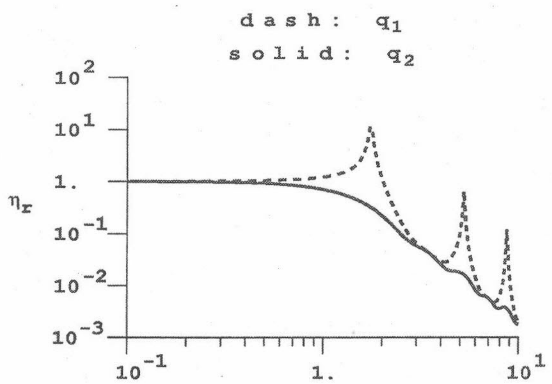



*Figure 7.5-2.* Bode plot of complementary sensitivity $\eta_r$.

The cause of the problem is the pole of $q_1(z)$ at $z = -0.94$. From (7.5–1), and (7.5–12) we obtain for $p(s) = \tilde{p}(s)$:

$$y(s) = p(s)q(e^{sT})h_0(s)r^*(e^{sT}) \tag{7.5 - 30}$$

$$y^*(z) = p^*(z)q(z)r^*(z) \tag{7.5 - 31}$$

In (7.5–31) this pole cancels with the zero of $p^*(z)$ and its bad effect does not show up in $y^*(z)$. This does not happen in (7.5–30), however, as is shown on the Bode plots of $\eta_r(s)$ in Fig. 7.5-2, where the pole of $q_1(z)$ at $z = -0.94$ causes a peak in $|\eta_r(i\omega)|$. Figure 7.5-1 clearly indicates that the problem appears because $q_1(z)$ produces an oscillatory output $u(z)$ with a period that matches the sampling period and whose effect does not show up in the sampled output $y^*(z)$. This is a characteristic of poles near (-1,0) on the $z$-plane. Hence, to avoid such hidden oscillations (intersample rippling) one should use an IMC controller $q(z)$ which has no poles near (-1,0) or in general no poles with negative real part. A controller $q(z)$ which inverts the model $\tilde{p}(z)$ cannot be used when $\tilde{p}(z)$ has zeros close to (-1,0).

(ii) *Effect of sampling on performance*

From a qualitative point of view, sampling clearly puts a limitation on the achievable performance since one can obtain information on the system output and change the control action only at every sampling point. We can demonstrate this fact quantitatively by looking at $\eta_r(s)$, given by (7.5–5) for $p(s) = \tilde{p}(s)$. In Fig. 7.5-3 a typical Bode plot of $p(s)$ is shown. For perfect performance $\eta_r(s) = 1$ – i.e., $q(e^{sT})$ should be equal to the inverse of $p(s)$. However, as shown by (7.1–8),

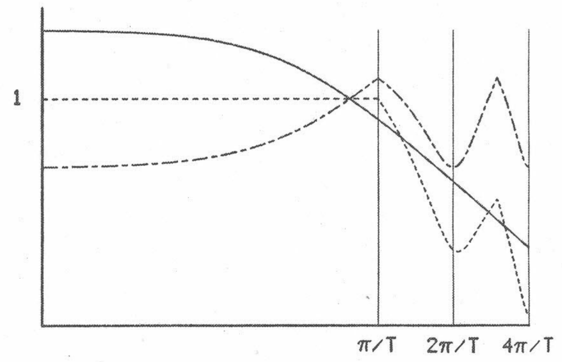



*Figure 7.5-3.* Effect of sampling on performance (logarithmic plot). Solid line: $|\tilde{p}(i\omega)|$. Dash and dot line: $|q(e^{i\omega T})|$. Dashed line: $|\tilde{p}(i\omega)q(e^{i\omega T})|$. (Reprinted with permission from Int. J. Control, 44, 716(1986), Taylor & Francis Ltd.)

$q(e^{i\omega T})$ is periodic in $\omega$ with period $\omega_s$ and its values for frequencies larger than $\pi/T$ are uniquely determined by those for $\omega \leq \pi/T$. In Fig. 7.5-3 an ideal $q$ is plotted which inverts $p(s)$ for $\omega$ up to $\pi/T$. In order for this to be accomplished, $q$ has to be of infinite order. Even for this $q$, it is clear from Fig. 7.5-3 that the closed-loop transfer function $p(s)q(e^{sT})$ cannot have a bandwidth larger than $\pi/T$.

### 7.5.4 Discrete Linear Quadratic ($H_2^*$-) Optimal Control

In the continuous case, the objective of $H_2$-optimal control theory is to minimize the integral of the squared error — i.e., the $H_2$ norm of the error — for a particular input. The $H_2^*$ norm for a discrete signal $e^*(z)$ is given by

$$\|e^*\|_2^2 = \sum_{k=0}^{\infty} e_k^2 \qquad (7.5-32)$$

where the sequence $\{e_k\}$ is defined from

$$\{e_k\} = \mathcal{Z}^{-1}\{e^*(z)\} \qquad (7.5-33)$$

The objective of the $H_2^*$-optimal controller $\tilde{q}_H$ is to minimize (7.5–32) resulting from a particular reference and/or disturbance change. Recall that a discrete controller is not effective in rejecting disturbances in the frequency range $\omega > \pi/T$. Thus for the computation of the $H_2^*$-optimal control law it is not meaningful to specify disturbances with large high-frequency components. Therefore $\tilde{q}_H^*$ should be designed for the filtered disturbance $d_\gamma^*$ rather than $d^*$. Then we find from Sec. 7.5.1

$$e^*(z) = \epsilon^*(z)(d_\gamma^*(z) - r^*(z)) \qquad (7.5-34)$$

Let us define the combined inputs

$$v^*(z) = d_\gamma^*(z) - r^*(s) \qquad (7.5-35)$$

With the help of Parseval's theorem we can rewrite the objective (7.5–32)

$$\|e^*\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^*(e^{i\theta})|^2 d\theta \qquad (7.5-36)$$

and upon substitution of (7.5–34) and (7.5–35)

$$\|e^*\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{\epsilon}^*(e^{i\theta})v^*(e^{i\theta})|^2 d\theta \qquad (7.5-37)$$

Thus, the $H_2^*$-optimal control problem becomes

$$\min_{q(z)} \|\tilde{\epsilon}^* v^*\|_2 = \min_{q(z)} \|(1-\tilde{p}^* q)v^*\|_2 \qquad (7.5-38)$$

Note that in this formulation no attention is paid to intersample behavior. Experience has shown that the $H_2^*$-optimal controller can lead to unacceptable intersample rippling.

### 7.5.5 $H_\infty$ Performance Objective

In Sec. 2.4.5 we introduced the $H_\infty$ performance objective

$$\|\epsilon w\|_\infty < 1 \qquad (2.4-20)$$

We found it particularly relevant for disturbance rejection because rather than restricting the disturbance to a specific function it assumes the disturbance to belong to a set. Usually this is more realistic. Because of the time varying nature of the sampling operation there exists no sensitivity function for sampled-data systems and thus (2.4–20) cannot be defined. We can, however, state an objective similar to (2.4–20) for the approximate relation (7.5–11).

We assume the disturbance to be approximately limited to the frequency band up to $\pi/T$. This implies for the weight $w$

$$|w(\omega)| << 1, \quad \omega > \frac{\pi}{T} \qquad (7.5-39)$$

Therefore (2.4–20) can be approximated by

$$|\epsilon w| < 1, \quad 0 \le \omega \le \frac{\pi}{T} \qquad (7.5-40)$$

Using (7.5–11), (7.5–40) can be expressed as

$$|(1-\tilde{p}(i\omega)\hat{q}(i\omega))w(\omega)| < 1, \qquad 0 \le \omega \le \frac{\pi}{T} \qquad (7.5-41)$$

Note that (7.5–41) is exactly equal to (2.4–20) if the disturbance $d$ has no components at frequencies higher than $\pi/T$.

Let us assume that the performance specification for disturbance rejection has been stated in the form of (2.4–20) in terms of the continuous input and output signals. Then (7.5–41) can be used to assess if these specifications can be met with the digital controller $q(z)$.

## 7.6 Robust Stability

We wish to derive a condition that guarantees stability of the control loop for all plants in the family $\Pi^*$ defined by (7.3–7). The Nyquist stability criterion as applied to discrete systems can be used to obtain such a condition in exactly the same way as for continuous systems. Hence in the same way as in Sec. 2.5 we can derive the following theorem, where for consistency with the continuous case we define

$$\bar{\ell}_m^*(\omega) = \bar{\ell}_a^*(\omega)/|\tilde{p}^*(e^{i\omega T})| \qquad (7.6-1)$$

**Theorem 7.6-1 (Robust Stability).** *Assume that all plants $p(s)$ in the family*

$\Pi^*$

$$\Pi^* = \{p(s) : |p_\gamma^*(e^{i\omega T}) - \tilde{p}_\gamma^*(e^{i\omega T})| \le \bar{\ell}_a^*(\omega)\} \qquad (7.3-7)$$

*have the same number of RHP poles and that these poles do not become unobservable after sampling. Let $c(z)$ be a controller that stabilizes the system in Fig. 7.1-1A for the nominal plant $\tilde{p}(s)$. Then the system is robustly stable with the controller $c$ if and only if the complementary sensitivity function $\tilde{\eta}^*(z)$ for $\tilde{p}(s)$ satisfies*

$$|\tilde{\eta}^*(e^{i\omega T})|\bar{\ell}_m^*(\omega) < 1, \quad 0 \le \omega \le \pi/T \qquad (7.6-2)$$

(Note that the periodicity and (7.1–8) imply that (7.6–2) holds for all $\omega$ if it holds for $0 \le \omega \le \pi/T$.)

The IMC structure can be used for control system implementation only when the plant is stable. Then the robust stability condition is described by the following theorem.

**Theorem 7.6-2 (Robust Stability).** *Assume that all plants $p(s)$ in the family $\Pi^*$ are stable, that $q(z)$ is stable, and that $c(z)$ is related to $q(z)$ through (7.2–3). Then the systems in Figs. 7.1-1A and 7.2-1A are robustly stable if and only if*

$$|\tilde{\eta}^*(e^{i\omega T})|\bar{\ell}_m^*(\omega) < 1, \quad 0 \le \omega \le \pi/T \qquad (7.6-3)$$

## 7.7 Robust Performance

In a similar manner as in Sec. 7.5.5 we will develop an approximate sensitivity function on the basis of which we will assess robust performance. It follows from Figs. 7.2-1A or B that $y(s)$ and $d(s)$ are related by the time-varying expression

$$y(s) = d(s) - \frac{h_0(s)p(s)q(e^{sT})}{1 + q(e^{sT})(p_\gamma^*(e^{sT}) - \tilde{p}_\gamma^*(e^{sT}))} d_\gamma^*(e^{sT}) \qquad (7.7-1)$$

When $d$ is bandlimited we can use the approximation

$$d_\gamma^*(e^{i\omega T}) \cong \frac{1}{T} d(i\omega)\gamma(i\omega) \qquad 0 \le \omega \le \frac{\pi}{T} \qquad (7.5-8)$$

derived in Sec. 7.5.1. We can use the arguments of Sec. 7.3.2 to justify

$$p_\gamma^*(e^{i\omega T}) - \tilde{p}_\gamma^*(e^{i\omega T}) \cong \frac{1}{T} h_0(i\omega)\gamma(i\omega)\ell_a(i\omega) \quad 0 \le \omega \le \frac{\pi}{T} \qquad (7.7-2)$$

With (7.5–8) and (7.7–2), (7.7–1) becomes

$$y(i\omega) \cong \frac{1 - h_0(i\omega)\tilde{p}(i\omega)q(e^{i\omega T})\gamma(i\omega)/T}{1 + \ell_a(i\omega)q(e^{i\omega T})h_0(i\omega)\gamma(i\omega)/T} d(i\omega) \quad 0 \le \omega \le \frac{\pi}{T} \qquad (7.7-3)$$

With the "controller"

$$\hat{q}(s) = \frac{1}{T} q(e^{sT}) h_0(s)\gamma(s) \qquad (7.5-10)$$

equation (7.7–3) can be rewritten as

$$y(i\omega) \cong \frac{1 - \tilde{p}(i\omega)\hat{q}(i\omega)}{1 + \ell_a(i\omega)\hat{q}(i\omega)} d(i\omega) \quad 0 \le \omega \le \frac{\pi}{T} \qquad (7.7-4)$$

Equation (7.7–4) is identical in structure with what can be obtained for the continuous system. For continuous systems the expression is exact, while for sampled-data systems it represents an approximation of the time-varying relationship between $y$ and $d$. We can take advantage of this structural similarity and restate approximate conditions for sampled-data systems which were derived for continuous systems in Sec. 2.6.

### 7.7.1 $H_2$ Performance Objective

To estimate the worst error that can occur when a specific controller is used for a family $\Pi$ of plants we can use the expression derived from (7.7–4)

$$\max_{p \in \Pi} \|e\|_2^2 \cong \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left| \frac{(1 - \tilde{p}(i\omega)\hat{q}(i\omega))d(i\omega)}{1 - \bar{\ell}_a(\omega)|\hat{q}(i\omega)|} \right|^2 d\omega \qquad (7.7-5)$$

Because of the robust stability condition (7.6–2)

$$\bar{\ell}_a(\omega)|\hat{q}(i\omega)| < \bar{\ell}_a^*(\omega)|q^*(e^{i\omega T})| < 1 \qquad (7.7-6)$$

and the integrand in (7.7–5) is always bounded. Because of the approximations made to arrive at (7.7–4), (7.7–5) is valid only when the bandwidth of $d$ is limited to $\pi/T$. Furthermore, it is optimistic — i.e., the error bound is underestimated because (7.7–2) underestimates the uncertainty.

### 7.7.2 $H_\infty$ Performance Objective

Based on the approximation (7.7–4), the $H_\infty$ objective (2.4–20) for robust performance can be stated

$$\left|\frac{1-\tilde{p}(i\omega)\hat{q}(i\omega)}{1+\ell_a(i\omega)\hat{q}(i\omega)}\right| < \frac{1}{w(\omega)}, \quad \forall \ell_a \ni |\ell_a(i\omega)| \le \bar{\ell}_a(\omega), \quad 0 \le \omega \le \pi/T \qquad (7.7-7)$$

where the weight $w(\omega)$ is designer specified. Note however that $w(\omega)$ cannot be chosen arbitrarily large because even for $\ell_a = 0$, the left hand side of (7.7–7) may be nonzero. The selection of $w(\omega)$ will be discussed in Sec. 8.4.1.

The following conditions are completely equivalent to (7.7–7)

$$\frac{|1-\tilde{p}(i\omega)\hat{q}(i\omega)|w(\omega)}{1-\bar{\ell}_a(\omega)|\hat{q}(i\omega)|} < 1, \quad 0 \le \omega \le \pi/T \qquad (7.7-8)$$

$$|\hat{q}(i\omega)|\bar{\ell}_a(\omega) + |1-\tilde{p}(i\omega)\hat{q}(i\omega)|w(\omega) < 1, \quad 0 \le \omega \le \pi/T \qquad (7.7-9)$$

Condition (7.7–9) is identical in structure with the result for continuous systems. While it is exact for continuous systems, (7.7–9) is generally optimistic because of the approximation (7.5-8) and (7.7–4).

## 7.8 Summary

The basic IMC concepts carry over to the discrete case without major modifications. Because of the sampling operation an anti-aliasing prefilter $\gamma(s)$ has to be included in the control system (Fig. 7.2-1) and the process model $\tilde{p}_\gamma^*(z)$ (7.2–1) has to be defined accordingly. When the classic feedback controller $c(z)$ and the IMC controller $q(z)$ are related through

$$c = \frac{q}{1-\tilde{p}_\gamma^* q} \qquad (7.2-3)$$

$$q = \frac{c}{1+\tilde{p}_\gamma^* c} \qquad (7.2-4)$$

then the input-output behavior of the IMC structure and the classic feedback structure is the same. The IMC structure can be used for implementation only if $\tilde{p}$ and $q$ are stable. The classic feedback system is internally stable for $p = \tilde{p}$ if and only if $q$ defined by (7.2–4) is stable (Thm. 7.4-1).

For the design of a discrete controller the following has to be specified:

- process model
- model uncertainty bounds
- type of inputs
- performance objectives

In order to account for the intersample performance the availability of a continuous plant model is essential. Model uncertainty bounds for the discrete model can be obtained from the bounds $\bar{\ell}_a$ for the continuous model:

$$|p_\gamma^*(e^{i\omega T}) - \tilde{p}_\gamma^*(e^{i\omega T})| \leq \frac{1}{T} \sum_{k=-\infty}^{\infty} |h_0\gamma(i\omega + ik\omega_s)| \bar{\ell}_a(\omega + k\omega_s) \triangleq \bar{\ell}_a^*(\omega) \qquad (7.3-6)$$

If the reference trajectory $r$ is assumed to be constant between samples ($r(s) = h_0(s)r^*(e^{sT})$) then the sensitivity and complementary sensitivity can be defined in the usual manner:

$$\tilde{\epsilon}_r(s) = 1 - \tilde{p}(s)q(e^{sT}) \qquad (7.5-4)$$

$$\tilde{\eta}_r(s) = \tilde{p}(s)q(e^{sT}) \qquad (7.5-5)$$

The disturbance $d$ is usually *not* constant between samples. Then the relationship between $d$ and $y$ is time varying and a transfer function cannot be defined. However, if $d$ is band limited up to $\pi/T$ then approximately

$$y(i\omega) \cong (1 - \tilde{p}(i\omega)\hat{q}(i\omega))d(i\omega) \qquad 0 \leq \omega \leq \frac{\pi}{T} \qquad (7.5-11)$$

where

$$\hat{q}(s) = \frac{1}{T}h_0(s)q(e^{sT})\gamma(s) \qquad (7.5-10)$$

For asymptotically error-free response to polynomial inputs (i.e., Type m behavior) the controller $q$ and anti-aliasing filter $\gamma$ must have the following properties

$$\lim_{z\to 1} \frac{d^k}{dz^k}(1 - \tilde{p}^*(z)q(z)) = 0, \quad 0 \leq k < m \qquad (7.5-16)$$

$$\lim_{s\to 0} \frac{d^k}{ds^k}(1 - \gamma(s)) = 0, \qquad 0 \leq k < m \qquad (7.5-17)$$

Apart from the factors which limit the closed loop performance of continuous systems (NMP characteristics, constraints and model uncertainty) two more limitations arise for discrete systems: intersample rippling caused by poles of the controller $q$ close to (-1,0) and a limitation of the effective closed loop bandwidth to $\pi/T$ caused by the sampling operation.

The two performance objectives discussed in this book for discrete systems are the sum of the squared errors

$$H_2^*: \quad \|e^*\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{\epsilon}^*(e^{i\theta})v^*(e^{i\theta})|^2 d\theta \qquad (7.5-37)$$

and a bound on the sensitivity utilizing the approximate relationship (7.5-11):

$$H_\infty: \quad |(1-\tilde{p}(i\omega)\hat{q}(i\omega))w(\omega)| < 1, \qquad 0 \le \omega \le \frac{\pi}{T} \qquad (7.5-41)$$

The robust stability condition for discrete systems is formally similar to that for continuous systems

$$|\tilde{\eta}^*(e^{i\omega T})|\bar{\ell}_m^*(\omega) < 1, \qquad 0 \le \omega \le \frac{\pi}{T} \qquad (7.6-3)$$

An approximate (somewhat optimistic) condition for robust performance in the $H_\infty$ sense can be derived for (7.5–11):

$$|\hat{q}(i\omega)|\bar{\ell}_a(\omega) + |1-\tilde{p}(i\omega)\hat{q}(i\omega)|w(\omega) < 1, \quad 0 \le \omega \le \frac{\pi}{T} \qquad (7.7-9)$$

## 7.9 References

7.1. A detailed discussion of $z$-transforms can be found in any book on digital control. Two very good such books are Åström and Wittenmark (1984) and Kuo (1980). The effect of sampling time on the location of the zeros of discrete systems was analyzed by Åström, Hagander and Sternby (1984).

7.3.1. Åström and Wittenmark (1984, Chap. 12) provide a concise discussion of identification techniques for discrete systems and further references.

7.5. Another well known class of digital control algorithms, besides the $H_2$ and $H_\infty$ types, are the so-called deadbeat controllers. A brief but good discussion on these algorithms can be found in Kuo (1980), Chap. 10. For more details see Isermann (1981). For a comparison of the deadbeat and $H_2$ controllers to the Dahlin and Vogel-Edgar controllers see Zafiriou and Morari (1985).

7.5.1. Dailey (1987) and Thompson (1982) propose design approaches that consider the continuous plant output by bounding appropriate time varying operators, like the sensitivity operator or the sampling switch, with "conic sectors."

7.5.2. For a list of anti-aliasing prefilters for "Type 1" systems, see Åström and Wittenmark (1984, p. 28).

# Chapter 8

# SISO IMC DESIGN FOR STABLE SAMPLED-DATA SYSTEMS

As in the continuous case the IMC design procedure consists of two steps.

*STEP 1: Nominal Performance*

The controller $\tilde{q}(z)$ is selected to yield a "good" system response for the input(s) of interest, without regard for constraints and model uncertainty.

*STEP 2: Robust Stability and Performance*

The controller $\tilde{q}(z)$ is augmented by a lowpass filter $f(z)$ ($q(z) = \tilde{q}(z)f(z)$) to achieve robust stability and robust performance.

## 8.1 Nominal Performance

In the continuous case $\tilde{q}$ is designed so that it minimizes the integral of the squared error for a particular input. The analogous approach in the discrete case would be to design the controller to minimize the sum of the squared errors for some external setpoint or disturbance input. Although such a controller may suffer from the problem of intersample rippling as exhibited in Sec. 7.5.3, it can be used as a starting point for the design of $\tilde{q}(z)$. In Sec. 8.1.1 the design of the discrete linear quadratic optimal controller will be discussed and in Sec. 8.1.2 an appropriate simple modification of this controller will be introduced to avoid intersample rippling.

### 8.1.1 $H_2^*$-Optimal Control

The $H_2^*$-optimal controller $\tilde{q}_H(z)$ is designed by solving the following minimization problem

$$\min_{\tilde{q}_H(z)} \|e^*\|_2 = \min_{\tilde{q}_H(z)} \|(1 - \tilde{p}^*(z)\tilde{q}_H(z))v^*(z)\|_2 \qquad (7.5-38)$$

subject to the constraint that $\tilde{q}_H(z)$ be stable and causal.

The following theorem which provides the solution of (7.5–38) will be proven in Chapter 9 for the general case of unstable plants.

**Theorem 8.1-1.** *Assume that $\tilde{p}$ is stable. Factor the model $\tilde{p}^*(z)$ into an allpass part $\tilde{p}_A^*(z)$ and $\tilde{p}_M^*(z)$*

$$\tilde{p}^*(z) = \tilde{p}_A^*(z)\tilde{p}_M^*(z) \qquad (8.1-1)$$

*where*

$$\tilde{p}_A^*(z) = z^{-N}\prod_{j=1}^{h}\frac{(1-(\zeta_j^H)^{-1})(z-\zeta_j)}{(1-\zeta_j)(z-(\zeta_j^H)^{-1})} \qquad (8.1-2)$$

*and $\zeta_j$, $j = 1,\ldots,h$ are the zeros of $\tilde{p}^*(z)$ which are outside the UC. The positive integer $N$ is chosen such that $\tilde{p}_M^*(z)$ is semi-proper — i.e., its numerator and denominator have the same degree, which is equivalent to saying that $N$ is such that $z^N\tilde{p}^*(z)$ is semi-proper.*

*Factor the input $v^*(z)$ similarly — i.e.,*

$$v^*(z) = v_A^*(z)v_M^*(z) \qquad (8.1-3)$$

$$v_A^*(z) = z^{-N_v}\prod_{j=1}^{h_v}\frac{(1-(\zeta_{vj}^H)^{-1})(z-\zeta_{vj})}{(1-\zeta_{vj})(z-(\zeta_{vj}^H)^{-1})} \qquad (8.1-4)$$

*where $\zeta_{vj}$, $j = 1,\ldots,h_v$ are the zeros of $v^*(z)$ outside the UC and $N_v$ is such that $z^{N_v}v^*(z)$ is semi-proper. The $H_2^*$- optimal controller $\tilde{q}_H(z)$ is given by*

$$\tilde{q}_H(z) = z(\tilde{p}_M^* v_M^*)^{-1}\left\{z^{-1}\tilde{p}_A^{*-1}v_M^*\right\}_* \qquad (8.1-5)$$

*where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand only the strictly proper and stable (including poles at $z = 1$) terms are retained.*

Note that $\tilde{q}_H(z)$ is stable and causal. Also note that in order for the system to be Type m when $\tilde{q}_H(z)$ is used as the controller, the input $v(s)$ for which $\tilde{q}_H(z)$ is designed must have $m$ poles at $s = 0$.

The evaluation of (8.1–5) for specific inputs $v^*$ yields the results shown in Table 8.1-1. As an illustration, let us compute the $H_2^*$-optimal controller for two different inputs.

**Example 8.1-1.**

$$v^* = v_M^* = z(z-1)^{-1} \qquad \text{(Step)}$$

$$\{z^{-1}\tilde{p}_A^{*-1}v_M^*\}_* = \{\tilde{p}_A^{*-1}(z-1)^{-1}\}_* = (z-1)^{-1}$$

$$\tilde{q}_H(z) = z(\tilde{p}_M^* v_M^*)^{-1}(z-1)^{-1} = \tilde{p}_M^{*-1}$$

□

*Table 8.1-1.* $H_2^*$-optimal controller for some typical input forms.

| Input $v(s)$ | Input $v(z)$ | Controller $\tilde{q}_H(z)$ |
|---|---|---|
| $\frac{1}{s}$ | $\frac{z}{z-1}$ | $(\tilde{p}_M^*(z))^{-1}$ |
| $\frac{1}{\tau s+1}$ | $\frac{z/\tau}{z-e^{-T/\tau}}$ | $(\tilde{p}_M^*(z))^{-1}(\tilde{p}_A^*(e^{-T/\tau}))^{-1}$ |
| $\frac{1}{s(\tau s+1)}$ | $\frac{z(1-e^{-T/\tau})}{(z-1)(z-e^{-T/\tau})}$ | $(\tilde{p}_M^*(z))^{-1}\frac{(1-\tilde{p}_A^{*-1}(e^{-T/\tau})e^{-T/\tau})z+(\tilde{p}_A^{*-1}(e^{-T/\tau})-1)e^{-T/\tau}}{(1-e^{-T/\tau})z}$ |
| $\frac{1}{s^2}$ | $\frac{Tz}{(z-1)^2}$ | $(\tilde{p}_M^*(z))^{-1}\frac{(N+\Xi+1)z-N-\Xi}{z}$ <br> where $\Xi \triangleq \frac{d}{dz}(\tilde{p}_A^{*-1}(z)z^{-N})\|_{z=1}$ <br> $= \sum_{j=1}^{h}\frac{(\zeta_j^H)^{-1}-\zeta_j}{(1-\zeta_j)(1-(\zeta_j^H)^{-1})}$ |

**Example 8.1-2.**

$$v^* = v_M^* = \frac{z/\tau}{z - e^{-T/\tau}}$$

$$\{z^{-1}\tilde{p}_A^{*-1}v_M^*\}_* = \left\{\tilde{p}_A^{*-1}\frac{1/\tau}{z - e^{-T/\tau}}\right\}_* = (\tilde{p}_A^*(e^{-T/\tau}))^{-1}\frac{1/\tau}{z - e^{-T/\tau}}$$

$$\tilde{q}_H(z) = z(\tilde{p}_M^* v_M^*)^{-1}(\tilde{p}_A^*(e^{-T/\tau}))^{-1}\frac{1/\tau}{z - e^{-T/t}} = \tilde{p}_M^{*-1}(\tilde{p}_A^*(e^{-T/\tau}))^{-1}$$

□

The derivation of $\tilde{q}_H$ for the other inputs listed in Table 8.1-1 is left as an exercise.

In the case of setpoint following, one sometimes has available and supplies to the controller future values of the setpoint, which the system output is to follow after $N_p$ time steps. By doing so, better servo-behavior is accomplished. In this case $\tilde{q}_H(z)$ can be obtained from

$$\tilde{q}_H(z) = z(\tilde{p}_M^* v_M^*)^{-1}\{z^{-N_p-1}\tilde{p}_A^{*-1}v_M^*\}_* \qquad (8.1-6)$$

### 8.1.2 Design of the IMC Controller $\tilde{q}(z)$

The $H_2^*$- optimal controller $\tilde{q}_H(z)$ obtained in Sec. 8.1.1 may exhibit intersample rippling caused by poles of $\tilde{q}_H(z)$ close to $(-1,0)$ as explained in Sec. 7.5.3. Hence a modification is necessary to obtain $\tilde{q}(z)$ from $\tilde{q}_H(z)$. We can write

$$\tilde{q}(z) = \tilde{q}_H(z)\tilde{q}_-(z)B(z) \qquad (8.1-7)$$

where $\tilde{q}_-(z)$ cancels all the poles of $\tilde{q}_H(z)$ with negative real part and substitutes them with poles at the origin. $B(z)$ is selected to preserve the system type. The introduction of poles at the origin aims at incorporating into the design some of the advantages of a deadbeat-type response while at the same time avoiding known problems of deadbeat controllers like overshoot or undershoot.

Let $\kappa_i$, $i = 1, \ldots, \rho$ be the poles of $\tilde{q}_H(z)$ with negative real part. Then we can write

$$\tilde{q}_-(z) = z^{-\rho}\prod_{j=1}^{\rho}\frac{z - \kappa_j}{1 - \kappa_j} \qquad (8.1-8)$$

$$B(z) = \sum_{j=0}^{m-1} b_j z^{-j} \qquad (8.1-9)$$

where $m$ is the system type and the coefficients $b_j$, $j = 0, \ldots, m-1$ are chosen such that $\tilde{q}(z)$ satisfies (7.5–16). By construction $\tilde{q}_H(z)$ satisfies (7.5–16). Then it follows that $\tilde{q}(z)$ satisfies (7.5–16) if and only if

$$\lim_{z\to 1}\frac{d^k}{dz^k}(1 - \tilde{q}_-(z)B(z)) = 0, \qquad k = 0, 1, \ldots, m-1 \qquad (8.1-10)$$

For the important special cases of $m = 1$ and 2 we find

$$\text{Type 1}: \ B(z) = 1 \qquad (8.1-11)$$

$$\text{Type 2}: \ B(z) = b_0 + b_1 z^{-1} \qquad (8.1-12)$$

with

$$b_0 = 1 - b_1 \qquad (8.1-13)$$

$$b_1 = \sum_{j=1}^{\rho} \frac{\kappa_j}{1-\kappa_j} \qquad (8.1-14)$$

In general, use of the transformation $z = \lambda^{-1}$ in (8.1–10) leads to a system of linear equations which can be solved easily by successive substitution.

The proposed "correction scheme" might seem somewhat *ad hoc* but at least for step inputs it can be shown to lead to controllers which combine the advantages of the algorithm that minimizes the sum of squared errors and of deadbeat-type algorithms. For step inputs Table 8.1-1 shows the $H_2^*$-optimal controller to be $\tilde{q}_H(z) = (\tilde{p}_M^*(z))^{-1}$. In order for the system to be Type 1, $B(z) = 1$. Application of (8.1–7) leads to a controller $\tilde{q}(z)$ with the following properties:

- In the case where all the unstable zeros of $\tilde{p}^*(z)$ have negative real part, the controller is of the deadbeat type and drives the discrete output of the system to the setpoint in a finite number of time steps.
- When $\tilde{p}^*(z)$ has unstable zeros with positive real part, the controller drives the output to the setpoint asymptotically in order to avoid large overshoot or undershoot.
- When all the zeros, stable or unstable, have positive real part, the controller minimizes the sum of the squared errors of the output.

Similar desirable properties are maintained for other input types when the minimum number of coefficients $b_i$ necessary to satisfy (8.1–10) is used. Unfortunately, unlike for the continuous case, it is impossible to state general formulas for the IMC controller $\tilde{q}(z)$ for commonly occurring process models. The reason is that the factor $\tilde{q}_-(z)$ depends on both the MP and the NMP zeros of the plant $p^*(z)$ which in turn depend on the zeros and poles of the continuous system and the sampling time. Thus, we will simply illustrate the benefits of the IMC controller with an example.

**Example 8.1-3.** Consider the system given by (7.5–26) in Sec. 7.5.3. Its zero-order-hold discrete equivalent for $T = 1.8$ has two zeros, both inside the UC, at $z = -0.95$ and $z = -0.06$. Hence $p_M^* = zp^*$. For a step input $v$, the expression for the $H_2$-optimal controller given by (8.1–5) simplifies to that in Ex. 8.1-1 — i.e.,

$$q_1 \triangleq \tilde{q}_H = (\tilde{p}_M^*)^{-1}$$

This controller has a pole at $z = -0.95$ which is close enough to (-1,0) to produce the unacceptable input ringing and output intersample rippling shown in Fig. 7.5-1.

Application of (8.1–7) yields the controller $q_2(z)$ given by (7.5–29), which produces the excellent response shown also in Fig. 7.5-1. Note that the pole at $z = -0.06$ is so close to the origin, that it does not really make a difference whether it is substituted with a pole at the origin or not. □

**Example 8.1-4.** Consider the system

$$p(s) = \frac{1}{(10s+1)(25s+1)}$$

For $T = 3$ we get

$$p^*(z) = \frac{0.0157(z+0.869)}{(z-0.887)(z-0.741)}$$

The discrete system has a zero at $z = -0.869$, which is close enough to (-1,0) to produce the intersample rippling shown in Fig. 8.1-1 when it appears as a pole of the $H_2^*$-optimal controller $q_H$. Again, application of (8.1–7) eliminates the problem and results in a deadbeat type response for this particular example. □

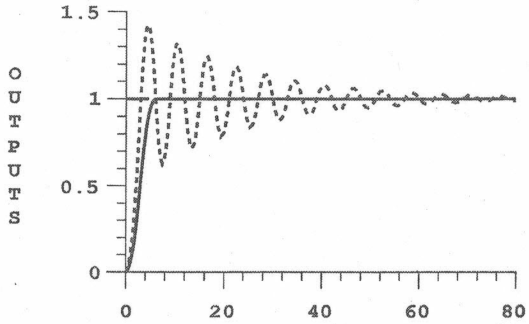
## 8.2 The Discrete IMC Filter

Similar to the continuous case, $\tilde{q}(z)$ is augmented by a low-pass filter $f(z)$ ($q = \tilde{q}f$), whose structure and parameters should be determined such that an optimal compromise between performance and robustness is reached. To simplify the design task the filter structure is fixed and only a few adjustable parameters are included. The simplest form is a first order one-parameter filter:

$$f_1(z) = \frac{(1-\alpha)z}{z-\alpha} \qquad (8.2-1)$$

The filter should preserve the asymptotic properties of the closed-loop system — i.e., (7.5–16) should be satisfied. The design procedure in Sec. 8.1.2 assures that (7.5–16) is satisfied for $q(z) = \tilde{q}(z)$. Therefore, for the system to be Type m, the filter $f(z)$ has to satisfy

$$\text{Type m}: \quad \frac{d^k}{dz^k}(1-f(z))\Big|_{z=1} = 0, \quad 0 \le k < m \qquad (8.2-2)$$



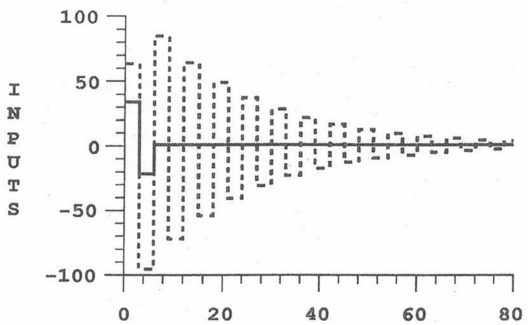




*Figure 8.1-1.* Dashed line: $q_1 = \tilde{q}_H$. Solid line: $q_2 = \tilde{q}_H \tilde{q}_- B$.

For a Type 1 system only $f(1) = 1$ is required and the filter given by (8.2–1) clearly meets that requirement. For $m \geq 2$ however, the filter (8.2–1) is not sufficient. In this case we postulate

$$f(z) = (\beta_0 + \beta_1 z^{-1} + \ldots + \beta_w z^{-w})\frac{(1-\alpha)z}{z-\alpha} \qquad (8.2-3)$$

where the coefficients $\beta_0, \ldots, \beta_w$ are to be chosen such that $f(z)$ satisfies (8.2–2) for some specified $\alpha$.

**Theorem 8.2-1.** *For a Type $m$ system the coefficients $\beta_i$ of the filter (8.2–3) have to satisfy*

$$\beta_0 = 1 - (\beta_1 + \ldots \beta_w) \qquad (8.2-4)$$

*and for $m \geq 2$, $w \geq m-1$*

$$N_w \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_w \end{bmatrix} = \begin{bmatrix} -\alpha/(1-\alpha) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad (8.2-5)$$

*where the elements $\nu_{ij}$ of the $(m-1) \times w$ matrix $N_w$ are defined by*

$$\nu_{ij} = \begin{cases} 0 & \text{for } i > j \\ \frac{j!}{(j-i)!} & \text{for } i \leq j \end{cases} \qquad (8.2-6)$$

For the proof the following lemma will be used.

**Lemma 8.2-1.** *Let $h(\lambda) = \frac{1-\alpha}{1-\alpha\lambda}$. Then*

$$h^{(k)}(\lambda) = (1-\alpha)k!\ \alpha^k(1-\alpha\lambda)^{-(k+1)} \qquad (8.2-7)$$

*where the superscript $(k)$ denotes $k^{th}$ derivative.*

*Proof.* By induction.
$k = 1$.

$$\frac{d}{d\lambda}h(\lambda) = (1-\alpha)\alpha(1-\alpha\lambda)^{-2}$$

$k = n$. Assume

$$h^{(n)}(\lambda) = (1-\alpha)n!\ \alpha^n(1-\alpha\lambda)^{-(n+1)} \qquad (8.2-8)$$

$k = n+1$. From (8.2–8) we get

$$h^{(n+1)}(\lambda) = (1-\alpha)n!\ \alpha^n\frac{d}{d\lambda}(1-\alpha\lambda)^{-(n+1)} =$$

$$= (1-\alpha)(n+1)!\ \alpha^{n+1}(1-\alpha\lambda)^{-(n+2)}$$

□

*Proof of Theorem 8.2-1.* Equation (8.2–4) follows directly from (8.2–2) for $k = 0$. For proving (8.2–5) we define

$$\Gamma(\lambda) \triangleq \beta_0 + \beta_1\lambda + \ldots + \beta_w\lambda^w \qquad (8.2-10)$$

$$h(\lambda) \triangleq \frac{1-\alpha}{1-\alpha\lambda}$$

and express the filter (8.2–3) as

$$f(\lambda^{-1}) = \Gamma(\lambda)h(\lambda) \qquad (8.2-11)$$

Thus for $k \geq 1$ we can rewrite (8.2–2) as

$$\frac{d^k}{d\lambda^k}f(\lambda^{-1})\Big|_{\lambda=1} = 0, \qquad k = 1, \ldots m-1 \qquad (8.2-12)$$

For $k = 1$, (8.2–12) yields

$$\Gamma^{(1)}(1)h(1) + \Gamma(1)h^{(1)}(1) = 0 \qquad (8.2-13)$$

From Lemma 8.2-1 we find

$$h^{(k)}(1) = k!\alpha^k(1-\alpha)^{-k} \qquad (8.2-14)$$

Substituting (8.2–14) into (8.2–13) yields

$$\Gamma^{(1)}(1) = -h^{(1)}(1) = -\alpha(1-\alpha)^{-1} \qquad (8.2-15)$$

We will show next that for $k \geq 2$, (8.2–12) requires $\Gamma^{(k)}(1) = 0$. The proof will be by induction.

$k = 2$. Condition (8.2–12) becomes

$$\Gamma^{(2)}(1)h(1) + 2\Gamma^{(1)}(1)h^{(1)}(1) + \Gamma(1)h^{(2)}(1) = 0 \qquad (8.2-16)$$

Using (8.2–14) and (8.2–15), (8.2–16) yields

$$\Gamma^{(2)}(1) = 0 \qquad (8.2-17)$$

$2 \leq k \leq n < m-1$. Assume

$$\Gamma^{(k)}(1) = 0 \qquad (8.2-18)$$

$k = n+1$. Because of (8.2–18), (8.2–12) becomes

$$\Gamma^{(n+1)}(1)h(1) + (n+1)\Gamma^{(1)}(1)h^{(n)}(1) + \Gamma(1)h^{(n+1)}(1) = 0 \qquad (8.2-19)$$

or by using (8.2–14) and (8.2–15)

$$\Gamma^{(n+1)}(1) = 0 \tag{8.2-20}$$

Hence by induction

$$\Gamma^{(k)}(1) = 0, \qquad k = 2, \ldots, m-1 \tag{8.2-21}$$

But one can easily see that

$$\begin{bmatrix} \Gamma^{(1)}(1) \\ \cdot \\ \cdot \\ \cdot \\ \Gamma^{(m-1)}(1) \end{bmatrix} = N_w \begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_w \end{bmatrix} \tag{8.2-22}$$

and (8.2–5) follows from (8.2–21), (8.2–22). □

For $w > m - 1$, there are several solutions to (8.2–5) and one can obtain $\beta_1, \ldots, \beta_w$ as the minimum norm solution. It can be shown that as $w \to \infty$ the norm of this solution goes to zero and from (8.2–3), (8.2–4) it follows that the properties of $f(z)$ are not significantly different from those of $f_1(z)$. Finally note that for $m = 2$, one should choose $w \geq 2$ in order to avoid the trivial solution $f(z) = 1$. Then the minimum norm solution for $m = 2$, $w \geq 2$, is found to be

$$\beta_k = \frac{-6k\alpha}{(1-\alpha)w(w+1)(2w+1)}, \qquad k = 1, \ldots, w \tag{8.2-23}$$

Note that $\lim_{w\to\infty} \beta_0 = 1$.

## 8.3 Robust Stability

### 8.3.1 Filter Design

The robust stability condition derived in Sec. 7.6 can be stated in terms of the IMC controller $q(z)$ $(= \tilde{q}(z)f(z))$.

**Corollary 8.3-1 (Robust Stability).** *Assume that all plants $p(s)$ in the family $\Pi^*$ are stable, that $q(z)$ is stable and that $c(z)$ is related to $q(z)$ through (7.2–3). Then the systems in Figs. 7.1-1A and 7.2-1A are robustly stable if and only if*

$$|f(e^{i\omega T})| < \left[|\tilde{p}^*\tilde{q}(e^{i\omega T})|\bar{\ell}^*_m(\omega)\right]^{-1} \qquad 0 \leq \omega \leq \frac{\pi}{T} \tag{8.3-1}$$

Clearly an $f(z)$ can always be found such that (8.3–1) is satisfied. However, a small $|f|$ implies a small $|\tilde{\eta}^*|$ and thus poor performance. Hence, if performance requirements have to be met the uncertainty has to be limited. A simple performance specification is to require the closed-loop system to be Type 1 – i.e., $\tilde{p}^*(1)\tilde{q}(1) = f(1) = \gamma(0) = 1$. Then from (8.3–1) we can obtain the following corollary.

**Corollary 8.3-2.** *Assume that $\bar{\ell}^*_m(\omega)$ is continuous. Then there exists a filter $f(z)$ such that the closed-loop system is Type 1 and robustly stable for the family $\Pi^*$ if and only if $\bar{\ell}_m(0) < 1$, where $\bar{\ell}_m(0)$ is the multiplicative steady-state error bound for the continuous system.*

*Proof.* All that is needed is to show that (8.3–1) is satisfied for $\omega = 0$, where $f(1) = \tilde{p}^*(1)\tilde{q}(1) = 1$. Hence we need $\bar{\ell}^*_m(0) < 1$. The steady-state gain of the zero-order hold equivalent of $\tilde{p}(s)$ is the same as the steady-state gain of $\tilde{p}(s)$ — i.e., $\tilde{p}^*(1) = \tilde{p}(0)$. Also from (7.3–6) we get $\bar{\ell}^*_a(0) = \bar{\ell}_a(0)$ since $h_0(i2\pi k/T) = 0$ for $k = \pm 1, \pm 2, \ldots$ and $h_0(0)/T = \gamma(0) = 1$. Thus $\bar{\ell}^*_m(0) = \bar{\ell}^*_a(0)/\tilde{p}^*(1) = \bar{\ell}_a(0)/\tilde{p}(0) = \bar{\ell}_m(0)$. □

Note that Cor. 8.3-2 requires simply that the error between the steady-state gain of the plant and that of the model is not more than 100% of the model gain. This condition can always be satisfied by appropriate selection of the model if all the possible plants have steady-state gains with the same sign.

Note that the condition $\bar{\ell}_m(0) < 1$ is the same as the one we found for the continuous system (Cor. 4.3-2). This makes sense because a steady-state requirement should not be affected by the sampling operation.

A simple way to design the IMC filter is to use an $f(z)$ of the structure in (8.2–3) and to vary the parameter $\alpha$ so that (8.3–1) is satisfied. Equation (8.3–1) places a lower bound $\alpha^*$ on $\alpha$. It can be obtained from a Bode plot of $(|\tilde{p}^*\tilde{q}(e^{i\omega T})|\bar{\ell}^*_m(\omega))^{-1}$. If this quantity is never less than 1, then $\alpha^* = 0$. If it obtains values less than 1, then $\alpha^*$ can be found from a Bode plot of $f(z)$, which is practically the same as that of the first-order filter $f_1(z)$ in (8.2–1) provided that the number of coefficients $w$ in (8.2–3) is sufficiently large. For example, if $(|\tilde{p}^*\tilde{q}(e^{i\omega T})|\bar{\ell}^*_m(\omega))^{-1}$ decreases like a first-order system and reaches a value of 0.7 at $\omega = \omega_\ell$ then

$$\alpha^* \cong e^{-T\omega_\ell} \qquad (8.3-2)$$

Note that for an open-loop stable sampled-data system a *first-order* filter $f_1(z)$ can always be designed to satisfy the robust stability condition regardless of the magnitude of the model uncertainty. For continuous systems, depending on the uncertainty a higher order filter might be required. The reason is that for sampled-data systems the frequency range over which the condition has to be met is bounded.

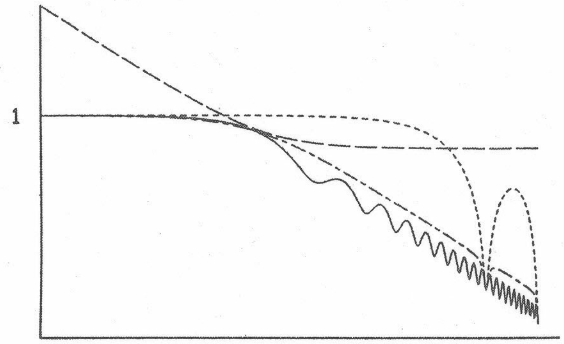



*Figure 8.3-1.* Effect of sampling on robust stability (logarithmic plot). Long dash: $1/\bar{\ell}_m(\omega)$. Solid: $|\tilde{p}(i\omega)\tilde{q}(e^{i\omega T})|, T = T_1$. Short dash: $|\tilde{p}(i\omega)\tilde{q}(e^{i\omega T})|, T = T_2 < T_1$. Dash and dot: $|\tilde{p}(i\omega)\tilde{q}(e^{i\omega T})f_1(e^{i\omega T})|, T = T_2$. (Reprinted with permission from Int. J. Control, **44**, 721(1986), Taylor & Francis Ltd.)

### 8.3.2 Effect of Sampling

As explained in Sec. 8.3-1, condition (8.3–1) can be satisfied by simply increasing the time constant of the filter, provided that $\bar{\ell}_m(0) < 1$. The increase of the filter time constant reduces the closed-loop bandwidth of the nominal system. In Sec. 7.5.3 we saw that a larger sampling time $T$ also reduces the bandwidth. This becomes clearer if we write (8.3–1) as

$$|\tilde{p}(i\omega)\tilde{q}(e^{i\omega T})f(e^{i\omega T})| < |\tilde{p}(i\omega)|/\bar{\ell}_a^*(\omega) \qquad (8.3-3)$$

One can see that the bandwidth of the left hand side term can be reduced by either increasing $\alpha$ in $f(z)$ or leaving $f(z) = 1$ and increasing $T$. A graphical illustration of this discussion is given in Fig. 8.3-1. Note that in Fig. 8.3-1 the right-hand-side term of (8.3–3) is assumed independent of $T$ by using the approximation $\bar{\ell}_a^*(\omega) \cong \bar{\ell}_a(\omega)$. For illustrative purposes this is a reasonable approximation for $0 \le \omega \le \pi/T$ but it should not be used to check (8.3–1); $\bar{\ell}_a^*(\omega)$ should be computed from (7.3–6).

## 8.4 Robust Performance

In Sec. 7.7.2 we derived that for robust performance the controller has to be designed such that

$$M(\omega) \triangleq |\tilde{q}(i\omega)|\bar{\ell}_a(\omega) + |1 - \tilde{p}(i\omega)\tilde{q}(i\omega)|w(\omega) < 1, \quad 0 \le \omega \le \frac{\pi}{T} \qquad (8.4-1)$$

is satisfied.

### 8.4.1 Filter Design

The simplest approach is to specify the structure of the filter as that in (8.2–3) and to try to satisfy (8.4–1) by varying the parameter $\alpha$. Increasing $\alpha$ will tend to decrease the first term in $M(\omega)$ and increase the second term. Hence, depending on $\bar{\ell}_a$ and $w$ there might be no value of $\alpha$ for which (8.4–1) is satisfied. Let us assume that $\tilde{q}(z)$ and $f(z)$ are selected such that the system is Type 1 or higher ($\tilde{p}^*\tilde{q}f(1) = 1$) and that $\bar{\ell}_m(0) < 1$. Then robust performance at $\omega = 0$ can be achieved for any $w$.

**Corollary 8.4-1.** *There exists a filter $f(z)$ such that (8.4–1) is satisfied at $\omega = 0$ for any weight $w$ if and only if $\bar{\ell}_m(0) < 1$.*

Corollary 8.4-1 is similar to Cor. 4.4-2 for continuous systems and it simply states that if the system is robustly stable for a controller $c(z)$ with integral action (Type 1), then the steady-state performance is perfect even when there is modelling error.

*Selection of the weight $w(\omega)$.* The choice of $w(\omega)$ depends on the performance requirements set by the designer. It is consistent with the overall design philosophy to assume that an $H_2$-optimal controller $\tilde{q}(s)$ designed for the continuous model $\tilde{p}(s)$ would achieve the ideal performance. Hence it is reasonable to use the ideal sensitivity function $\tilde{\eta}(s) = \tilde{p}(s)\tilde{q}(s)$ as a guide for the choice of the weight:

$$w(\omega)^{-1} \geq |1 - \tilde{p}(i\omega)\tilde{q}(i\omega)| \qquad (8.4-2)$$

This sensitivity function, however, is achieved only by a non-proper controller. The properness requirement adds to (8.4–2) the condition that $1/w(\infty) \geq 1$. Also note that though for a Type m system ($m \geq 1$), (8.4–2) becomes $w(0)^{-1} \geq 0$ for $\omega = 0$, there is no need to choose $w(0) = \infty$, since $\tilde{q}(z)$ and $f(z)$ have been designed so that conditions (7.5–16) and (8.2–2) are satisfied. These conditions guarantee no steady-state offset under modelling error, provided that stability is maintained.

*Computation of $\alpha$.* The filter parameter $\alpha$ has to be adjusted in an effort to satisfy (8.3–1) and (8.4–1). It was shown in Sec. 8.3.1 that (8.3–1) puts a lower bound $\alpha^*$ on the values of $\alpha$ that are allowed. Hence, to find $\alpha$ one must solve the following optimization problem:

$$\min_{\alpha^* \leq \alpha < 1} \max_{0 \leq \omega \leq \pi/T} M(\omega) \triangleq \psi(T) \qquad (8.4-3)$$

where $M(\omega)$ is defined in (8.4–1) and the argument $T$ has been used in $\psi$ to indicate that the optimum value of the objective function depends on the sampling time $T$.

The above minimization can be carried out by computing $M(\omega)$ for a number of values for $\alpha$. The computational effort is very small. It is advisable to write $\alpha = e^{-T/\tau}$ where $\tau$ is in $[\tau^*, \infty)$ with $\alpha^* = e^{-T/\tau^*}$ and minimize over $\tau$.

### 8.4.2 Sampling Time Selection

A short sampling time improves the nominal performance as we have discussed in Sec. 7.5.3. However, high-frequency sampling puts a large load on the computer and for robustness nominal performance generally has to be sacrificed anyway. Thus a longer sampling time might be acceptable for robust stability and robust performance. On the other hand if the sampling time is too long, it might be impossible to meet the robust performance requirements.

As a rule, $\pi/T$ should be selected larger than the bandwidth over which good performance is desired. If for a certain sampling time $T^*$ it is found that the robust performance requirements are exceeded ($\psi(T^*) < 1$), then the specifications could be met even with a larger $T$. If $\psi(T^*) > 1$ then for the assumed model uncertainty and controller structure the specifications are too tight for the specific $T^*$ and have to be relaxed.

### 8.4.3 Example

Let us consider the system

$$\tilde{p}(s) = \frac{3}{(s+1)(s+3)} \tag{8.4-4}$$

A delay-type uncertainty is assumed:

$$p(s) = \tilde{p}(s)e^{-\theta s} \tag{8.4-5}$$

where

$$0 \le \theta \le 0.05 \tag{8.4-6}$$

Then from (2.2–2), (2.2–4)

$$\ell_m(s) = e^{-\theta s} - 1 \tag{8.4-7}$$

from which one can easily obtain the bound $\bar{\ell}_m$ (2.2–8)

$$\bar{\ell}_m(\omega) = \begin{cases} |e^{-0.05i\omega} - 1| & 0 \le \omega \le 20\pi \\ 2 & \omega \ge 20\pi \end{cases} \tag{8.4-8}$$

Let us examine two sampling times, different by an order of magnitude, $T_1 = 0.1$ and $T_2 = 0.01$. For the robust performance design the following weight is selected:

$$w(s)^{-1} = 0.4\frac{0.5s+1}{0.1s+1} \tag{8.4-9}$$

This selection was based on the observation that at $\omega = 2$, $|\tilde{p}(i\omega)|$ is small enough ($\simeq 0.35$) to justify a relaxation of the performance requirement. Also $1/w(\infty) = 2 > 1$. It should be noted that the above choice is a rather strict performance requirement, but it is justified because the system is not inherently difficult to control and the uncertainty is small. Also note that in this simple case where $\tilde{p}(s)$ is minimum phase, the right-hand side of (8.4–2) is zero and this leaves us the freedom to select $w(\omega)$ as above.

The next step is to compute $\tilde{q}$ for the two sampling times according to the procedure of Sec. 8.1. We obtain

$$q_1(z) = \frac{40.55(z^2 - 1.64566z + 0.67032)}{z^2} \qquad (8.4-10a)$$

$$q_2(z) = \frac{3400(z^2 - 1.960495z + 0.960789)}{z^2} \qquad (8.4-10b)$$

for $T_1$ and $T_2$, respectively.

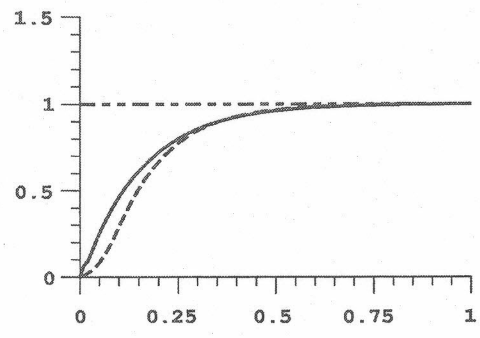
Then the quantity $\psi(T)$, which measures robust performance must be computed. For the two sampling times the solution of (8.4–3) yields

$$\psi(T_1) = 1.22 \qquad (8.4-11a)$$

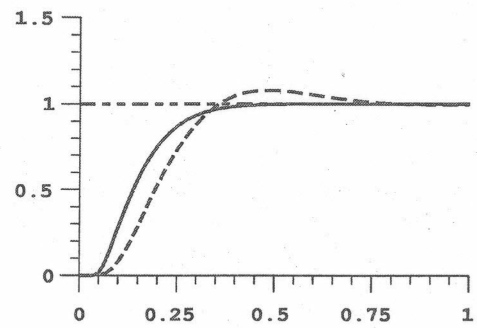
$$\psi(T_2) = 0.90 \qquad (8.4-11b)$$

The corresponding optimal $\alpha$'s are $\alpha_1 = 0.4625$ and $\alpha_2 = 0.9363$. The optima (8.4–11) imply that for the sampling time $T_2$ it is possible to satisfy the tight robust performance specification set through (8.4–9), while this cannot be done for the larger $T_1$. Equation (8.4–11b) indicates that the specification can be met even when $T$ is somewhat larger than $T_2$. Further search shows that $\psi(0.032) = 0.98 < 1$.

Let us now compare the time responses for the two controllers designed for $T_1$ and $T_2$ to see how (8.4–11) translates into the time-domain. Figure 8.4-1A shows the responses to a unit step setpoint change for the case when there is no model-plant mismatch. As expected, the controller with the smaller sampling time is somewhat better. Note that when the procedure described in this chapter is used for controller design, the use of a smaller sampling time cannot harm the nominal behavior, contrary to what could happen for some other digital algorithms, like deadbeat-type controllers. Figure 8.4-1B shows the response when the plant is $p(s) = \tilde{p}(s)e^{-0.05s}$. Again the response for $T_2$ is clearly better. Note that because of the robust design the faster nominal response ($T_2$) does not imply increased sensitivity to model uncertainty; the response for $T_2$ remains superior even in the presence of plant/model mismatch.




A




B

*Figure 8.4-1.* Step response for controllers with sampling time $T_1$ (dash) and $T_2$ (solid). A: No model error ($p = \tilde{p}$); B: Model error $p = \tilde{p}e^{-0.05s}$.

Roughly speaking, both controllers produce acceptable responses. This is not surprising since the $\psi$'s for the two controllers are similar. This simple example demonstrates, however, that the frequency domain based quantity $\psi(T)$ captures the time domain behavior in an excellent manner and even small differences in $\psi(T)$ translate into noticeable differences in the time responses.

## 8.5 Summary

In the *first step* of the IMC design procedure the controller is designed to yield a "good" system response for the input(s) of interest without regard for constraints or uncertainty. The starting point is the $H_2^*$-optimal controller which is calculated from

$$\tilde{q}_H(z) = z(\tilde{p}_M^* v_M^*)^{-1}\left\{z^{-1}\tilde{p}_A^{*-1}v_M^*\right\}_* \qquad (8.1-5)$$

Here the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand only the strictly proper and stable (including poles at $z=1$) terms are retained. The allpass and MP portions of the model are denoted by $\tilde{p}_A^*$ and $\tilde{p}_M^*$ respectively (8.1–1 and 8.1–2); $v_M^*$ is defined similarly (8.1–3 and 8.1–4). Table 8.1-1 lists formulas for $\tilde{q}_H(z)$ for some typical inputs $v_M^*$.

Because the $H_2^*$-optimal controller can lead to undesirable intersample rippling it is modified to

$$\tilde{q}(z) = \tilde{q}_H(z)\tilde{q}_-(z)B(z) \qquad (8.1-7)$$

where

$$\tilde{q}_-(z) = z^{-\rho}\prod_{j=1}^{\rho}\frac{z-\kappa_j}{1-\kappa_j} \qquad (8.1-8)$$

$$B(z) = \sum_{j=0}^{m-1} b_j z^{-j} \qquad (8.1-9)$$

Here $\kappa_i$, $i = 1, \ldots \rho$ are the poles of $\tilde{q}_H(z)$ with negative real part, $m$ is the system type and $b_j$ are coefficients to be chosen to satisfy the type requirements.

In the *second step* of the IMC design procedure the controller $\tilde{q}(z)$ is augmented by a filter $f(z)$ for robustness

$$q(z) = \tilde{q}(z)f(z)$$

Recommended one-parameter filters are

$$\text{Type 1: } f_1(z) = \frac{(1-\alpha)z}{z-\alpha} \qquad (8.2-1)$$

$$\text{Type 2: } f_2(z) = (\beta_0 + \beta_1 z^{-1} + \ldots + \beta_w z^{-w})\frac{(1-\alpha)z}{z-\alpha} \qquad (8.2-3)$$

where

$$\beta_k = \frac{-6k\alpha}{(1-\alpha)w(w+1)(2w+1)}, \quad k = 1, \ldots, w; \quad w \geq 2 \qquad (8.2-23)$$

For robust stability the filter parameter $\alpha$ is increased until

$$|f(e^{i\omega T})| < [|\tilde{p}^* \tilde{q}(e^{i\omega T})| \bar{\ell}_m^*(\omega)]^{-1} \qquad 0 \leq \omega \leq \frac{\pi}{T} \qquad (8.3-1)$$

For robust performance

$$\min_{\alpha^* \leq \alpha < 1} \max_{0 \leq \omega \leq \pi/T} M(\omega) \qquad (8.4-3)$$

where

$$M(\omega) \triangleq |\hat{q}(i\omega)| \bar{\ell}_a(\omega) + |1 - \tilde{p}(i\omega)\hat{q}(i\omega)| w(\omega) < 1, \quad 0 \leq \omega \leq \frac{\pi}{T} \qquad (8.4-1)$$

and $\alpha^*$ is the minimum filter parameter needed to assure robust stability. Instead of increasing the filter parameter $\alpha$, the sampling time $T$ can be increased with a similar effect on robustness.

## 8.6 References

8.1.1. For a discussion of the state-space approach to discrete Linear Quadratic Control see Kwakernaak and Sivan (1972). For a brief discussion, which also includes deadbeat controllers, see Kucera (1972).

8.1.2. The reasons behind the "correction scheme" were presented by Zafiriou and Morari (1985).

8.3. The original formulation of the discrete IMC controller and filter and some early results on robust stability can be found in Garcia and Morari (1982, 1985).

8.4.2. For a more detailed discussion of the procedure for sampling time selection see Zafiriou and Morari (1986a). Additional material on signal sampling and reconstruction is available from Åström and Wittenmark (1984, Chap. 2).

# Chapter 9

# SISO DESIGN FOR UNSTABLE SAMPLED-DATA SYSTEMS

In Chap. 5 it was pointed out that the IMC structure is unsuitable for implementation when the plant is open-loop unstable. However, the IMC controller parametrization remains a valuable tool that simplifies the controller design and greatly clarifies the robustness problems of open-loop unstable plants.

## 9.1 Parametrization of All Stabilizing Controllers

### 9.1.1 Internal Stability

For open-loop unstable systems the classic feedback structure shown in Fig. 7.1-1 has to be used for implementation. For internal stability the system described by (7.1–17) has to be stable. We can express (7.1–17) in terms of the IMC controller $q(z)$ by substituting (7.2–3) into (7.1–17). We obtain (for $p = \tilde{p}$)

$$\begin{pmatrix} y_\gamma^* \\ u \end{pmatrix} = \begin{pmatrix} p_\gamma^* q & (1 - p_\gamma^* q) p_\gamma^* \\ q & -p_\gamma^* q \end{pmatrix} \begin{pmatrix} r^* \\ u' \end{pmatrix} \qquad (9.1-1)$$

All four transfer functions in (9.1–1) have to be stable. Note that since the prefilter $\gamma(s)$ is stable, the only unstable poles of $p_\gamma^*(z)$ are the unstable poles of $p^*(z)$. By using the same arguments as in Sec. 5.1.1, we can derive the following theorem.

**Theorem 9.1-1**. *Assume that the model is perfect ($p = \tilde{p}$) and that $p^*(z)$ has $k$ unstable poles at $\pi_1, \ldots, \pi_k$, and that $p(s)$ has also $k$ unstable poles (i.e., that none of the unstable poles of $p(s)$ become unobservable after sampling). Then the feedback system in Fig. 7.1-1 with the controller $c = q(1 - p_\gamma^* q)^{-1}$ is internally stable if and only if*

*(i) $q(z)$ is stable.*

*(ii)* $(1 - p_\gamma^* q)$ *has zeros at* $\pi_1, \ldots, \pi_k$.

Theorem 9.1-1 reduces to Thm. 7.4-1 when $p$ is stable.

### 9.1.2 Controller Parametrization

A parametrization of *all* $q$'s that satisfy the conditions of Thm. 9.1-1 will be found in this section. Define the allpass $z$-transfer function

$$b_p^*(z) = \prod_{j=1}^{k} \frac{(1 - (\pi_j^H)^{-1})(z - \pi_j)}{(1 - \pi_j)(z - (\pi_j^H)^{-1})} \tag{9.1 - 2}$$

where $\pi_j$, $j = 1, \ldots, k$ are the poles of $p^*(z)$ strictly outside the UC.

**Theorem 9.1-2.** *Assume that* $p^*(z) = \tilde{p}^*(z)$ *has* $k$ *poles* $\pi_1, \ldots, \pi_k$ *strictly outside the UC and* $\ell$ *poles at* $z = 1$, *and that* $p(s)$ *has no unstable poles that become unobservable after sampling. Also assume that there exists a causal* $q_0(z)$ *such that* $c = q_0(1 - p_\gamma^* q_0)^{-1}$ *stabilizes the system in Fig. 7.1-1. Then all causal controllers that stabilize the system are parametrized by*

$$c = q(1 - p_\gamma^* q)^{-1} \tag{7.2 - 3}$$

$$q = q_0 + (b_p^*)^2 (1 - z^{-1})^{2\ell} q_1 \tag{9.1 - 3}$$

*where* $q_1(z)$ *is any arbitrary causal stable* $z$*-transfer function.*

*Proof.* The proof is similar to that of Thm. 5.1-2, and uses the fact that $p^*(z)$ and $p_\gamma^*(z)$ have the same unstable poles. □

Note that Thm. 9.1-2 assumes the existence of a stabilizing $q_0(z)$. The construction of the $H_2^*$-optimal controller in Sec. 9.2.1 will serve as proof that such a controller always exists. Also note that for an open-loop stable system we have $b_p = 1$, $\ell = 0$ and by choosing the stabilizing $q_0 = 0$, we obtain $q = q_1$, which is the IMC parametrization for stable systems.

## 9.2 Nominal Performance

The design procedure for unstable systems is the same as for stable ones. First the $H_2^*$-optimal controller $\tilde{q}_H(z)$ is designed and then a modification is introduced to avoid the problem of intersample rippling. Subsequently $\tilde{q}(z)$ is augmented by a low-pass filter to achieve robust stability and performance. In this section we shall derive the formulas for the design of $\tilde{q}(z)$.

### 9.2.1 $H_2^*$-Optimal Controller

As explained in Sec. 8.1.1, the $H_2^*$-optimal controller solves the problem defined by (7.5–38), subject to the constraint that $\tilde{q}$ is a stabilizing controller. The external system input $v(s)$ can be either a setpoint ($v = r,\ d = 0$) or a disturbance ($r = 0,\ v = d$ or $v = \gamma d$). Then the following theorem holds:

**Theorem 9.2-1.** *Let $p^*(z) = \tilde{p}^*(z)$ have $k$ poles at $\pi_1, \ldots, \pi_k$ strictly outside the UC and a pole of multiplicity $\ell$ at $z = 1$. Define*

$$b_p^* = \prod_{j=1}^{k} \frac{(1-(\pi_j^H)^{-1})(z-\pi_j)}{(1-\pi_j)(z-(\pi_j^H)^{-1})} \qquad (9.1-2)$$

*and factor the plant into an allpass portion $p_A^*(z)$ and a semi-proper MP portion $p_M^*(z)$.*

$$p^*(z) = p_A^*(z)p_M^*(z) \qquad (9.2-1)$$

*Factor the input $v$ similarly*

$$v^*(z) = v_A^*(z)v_M^*(z) \qquad (9.2-2)$$

*Assume without loss of generality that the unstable poles of $v^*(z)$ strictly outside the UC are the first $k'$ poles $\pi_i$ of the plant*[1] *and define accordingly*

$$b_v^* = \prod_{j=1}^{k'} \frac{(1-(\pi_j^H)^{-1})(z-\pi_j)}{(1-\pi_j)(z-(\pi_j^H)^{-1})} \qquad (9.2-3)$$

*Assume further that $v^*(z)$ has at least $\ell$ poles at $z = 1$.*[2] *Then the $H_2^*$-optimal controller $\tilde{q}_H(z)$ is given by*

$$\tilde{q}_H = z b_p^* (p_M^* b_v^* v_M^*)^{-1} \{(z b_p^* p_A^*)^{-1} b_v^* v_M^*\}_* \qquad (9.2-4)$$

*where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion only the strictly proper terms are retained except for those corresponding to poles of $(p_A^*)^{-1}$.*

*Proof.* Some preliminary definitions and facts are necessary. Let $L_2^*$ be the Hilbert space of complex-valued functions defined on the unit circle ($UC = \{e^{i\theta} : -\pi \leq \theta < \pi\}$) and square integrable with respect to $\theta$. The inner product on $L_2^*$ is

$$< f, g > = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\theta})^H d\theta \qquad (9.2-5)$$

[1] If this assumption were not made the problem would not be meaningful. Unbounded controller action would be necessary for the error to vanish as $t \to \infty$. See also the discussion at the end of Sec. 2.2.3.

[2] This assumption is made so that disturbances occurring at the plant *input* can be rejected with vanishing error as $t \to \infty$.

The closed subspace of $L_2^*$ of functions having analytic continuations inside the UC is defined as $(H_2^*)^\perp$; its orthogonal complement is denoted by $H_2^*$.[3] Note that with the above definitions a constant function is in $(H_2^*)^\perp$. $(H_2^*)^\perp$ also includes all rational $z$-transfer functions that are strictly unstable — i.e., which have all their poles strictly outside the UC [including poles at $z = \infty$ (improper transfer functions)]. All strictly proper, stable rational $z$-transfer functions are in $H_2^*$. Any rational $f(z)$ with no poles on the UC, can be uniquely decomposed into a strictly proper, stable part $\{f\}_+$ in $H_2^*$ and a strictly unstable part $\{f\}_-$ in $(H_2^*)^\perp$:

$$f = \{f\}_- + \{f\}_+ \tag{9.2 – 6}$$

For a rational $z$-transfer function, such a decomposition can be obtained by simply taking a partial fraction expansion. Note that any constant and improper terms belong in $\{f\}_-$.

If $f_1(z) \in H_2^*$ and $f_2(z) \in (H_2^*)^\perp$, then

$$< f_1, f_2 >= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(e^{i\theta}) f_2(e^{i\theta})^H d\theta = 0 \tag{9.2 – 7}$$

Hence the $L_2^*$-norm ($H_2^*$-norm) of $f$, defined by the inner product (9.2–5) can be computed as

$$\|f\|_2^2 =< f, f >=< \{f_-\}, \{f_-\} > + < \{f_+\}, \{f_+\} >= \|\{f\}_-\|_2^2 + \|\{f\}_+\|_2^2 \tag{9.2 – 8}$$

According to (7.5–38) we define the objective function $\phi$:

$$\phi \triangleq \|(1 - p^*(z)\tilde{q}(z))v^*(z)\|_2^2 \tag{9.2 – 9}$$

Since multiplication of a function by an allpass does not change its $L_2^*$-norm, we get:

$$\phi = \|(zb_p^* p_A^* v_A^*)^{-1} b_v^*(1 - p^*\tilde{q})v^*\|_2^2 \tag{9.2 – 10}$$

Use of (9.1–3), (9.2–1), (9.2–2) yields:

$$\phi = \|(zb_p^* p_A^*)^{-1}(1 - \tilde{p}^* q_0) b_v^* v_M^* - z^{-1} p_M^* b_p^* (1 - z^{-1})^{2\ell} b_v^* v_M^* q_1\|_2^2$$

[3] This definition of $H_2^*$ and $(H_2^*)^\perp$ is exactly the opposite of the one encountered in the mathematics literature, where $H_2$ corresponds to the $L_2$-functions with analytic continuations inside the UC. Our definitions have been chosen to be consistent with the common definitions of $H_2, H_2^\perp$ for Laplace transfer functions (Chap. 5) in the control literature. The transformation $\lambda = z^{-1}$ could have been employed to introduce consistency with the mathematics literature but this would unnecessarily complicate the notation.

$$\triangleq \|f\|_2^2 \triangleq \|f_1 - f_2 q_1\|_2^2 \qquad (9.2-11)$$

It will be assumed that in addition to being a stabilizing controller, $q_0$ satisfies (7.5–16). Not every stabilizing controller has this property, since $v^*$ may have more poles at $z = 1$ than $p^*$. The final construction of the $H_2^*$-optimal $q$ serves as proof of the existence of a $q_0$ with such properties.

Inspection of (9.2–11) shows that $f_2$ has no poles on or outside the UC except possibly for poles at $z = 1$ in the case where $v^*(z)$ has more than $\ell$ poles at $z = 1$. However, $f_1$ has no poles at $z = 1$ because $q_0$ satisfies (7.5–16). Hence the optimal $q_1$ *must* have the required number of zeros at $z = 1$ to produce an $f_2 q_1$ without any poles at $z = 1$ so that $\phi$ is finite. Also $f_2 q_1$ is strictly proper since $q_1$ is proper. Therefore $f_2 q_1$ is in $H_2^*$. Hence

$$\{f\}_+ = \{f_1\}_+ - f_2 q_1 \qquad (9.2-12)$$

$$\{f\}_- = \{f_1\}_- \qquad (9.2-13)$$

Then (9.2–8, 11, 12, 13) imply

$$\phi = \|\{f_1\}_-\|_2^2 + \|\{f_1\}_+ - f_2 q_1\|_2^2 \qquad (9.2-14)$$

Since $\{f_1\}_-$ is independent of $q_1$, the obvious solution to the minimization of (9.2–14) is

$$q_1 = f_2^{-1}\{f_1\}_+ \qquad (9.2-15)$$

However this solution is optimal only if $q_1$ is proper, stable and $f_2 q_1$ has no poles at $z = 1$. Careful inspection of (9.2–15), (9.2–11) shows that $q_1$ is proper and stable. Also $f_2 q_1 = \{f_1\}_+$ is in $H_2^*$ and therefore it has no poles at $z = 1$. Hence (9.2–15) gives the optimal $q_1$.

Substitution of (9.2–15) into (9.1–3) yields the $H_2^*$-optimal controller $\tilde{q}_H(z)$:

$$\tilde{q}_H = q_0 + b_p^{*2}(1 - z^{-1})^{2\ell} f_2^{-1}\{f_1\}_+$$

$$= zb_p^*(p_M^* b_v^* v_M^*)^{-1}[(zb_p^*)^{-1} p_M^* b_v^* v_M^* q_0$$

$$+ \left\{(zb_p^* p_A^*)^{-1} b_v^* v_M^*\right\}_+ - \left\{(zb_p^* p_A^*)^{-1} p^* q_0 b_v^* v_M^*\right\}_+]$$

$$= zb_p^*(p_M^* b_v^* v_M^*)^{-1}[\left\{(zb_p^* p_A^*)^{-1} p^* b_v^* v_M^* q_0\right\}_{0-} + \left\{(zb_p^* p_A^*)^{-1} b_v^* v_M^*\right\}_+] \qquad (9.2-16)$$

where $\{\cdot\}_{0-}$ indicates that in the partial fraction expansion only the terms corresponding to poles on or outside the UC are retained. These are the poles of $(b_p^*)^{-1}b_v^*v_M^*$ on or outside the UC because $(p_A^*)^{-1}p^*q_0 = p_M q_0$ is stable since $q_0$ is a stabilizing controller. Also since $q_0$ satisfies (7.5–16), if $\pi$ is an unstable (on or outside the UC) pole of $(b_p^*)^{-1}b_v^*v_M^*$ of multiplicity $m$, then $(1-p^*q_0)$ has at least $m$ zeros at $z=\pi$ — i.e.,

$$p^*(\pi)q_0(\pi) = 1 \tag{9.2 – 17a}$$

$$\left.\frac{d^k}{dz^k}p^*(z)q_0(z)\right|_{z=\pi} = 0, \quad k = 1,\ldots,m-1 \tag{9.2 – 17b}$$

Thus (9.2–16) simplifies to

$$\tilde{q}_H = zb_p^*(p_M^*b_v^*v_M^*)^{-1}\left\{(zb_p^*p_A^*)^{-1}b_v^*v_M^*\right\}_*$$

□

In situations where future values of the setpoint, $r$ are supplied to the controller to be followed by the system output after $N_p$ time steps, $\tilde{q}_H(z)$ can be obtained from

$$\tilde{q}_H = zb_p^*(p_M^*b_M^*v_M^*)^{-1}\left\{(z^{N_p+1}b_p^*p_A^*)^{-1}b_v^*v_M^*\right\}_* \tag{9.2 – 18}$$

The proof follows that of Thm. 9.2-1 by changing the objective function to

$$\phi = \|(z^{-N_p} - \tilde{p}^*(z)\tilde{q}(z))r^*(z)\|_2^2 \tag{9.2 – 19}$$

### 9.2.2 Design of the IMC Controller $\tilde{q}(z)$

As explained in Sec. 7.5.3 the $H_2^*$-optimal controller $\tilde{q}_H(z)$ may exhibit intersample rippling caused by poles of $\tilde{q}_H(z)$ close to (-1,0). As in Sec. 8.1.2, $\tilde{q}(z)$ is obtained as

$$\tilde{q}(z) = \tilde{q}_H(z)\tilde{q}_-(z)B(z) \tag{9.2 – 20}$$

where $\tilde{q}_-(z)$ cancels all poles of $\tilde{q}_H(z)$ with negative real part and replaces them with poles at the origin. In this case, however, $B(z)$ is selected to preserve both the system type and the internal stability requirements described by Thm. 9.1-1 (ii). In this section we assume $\gamma(s) = 1$. Section 9.2.3 discusses the choice of $\gamma(s)$ further.

Similarly to Sec. 8.1.2 let $\kappa_i$, $i = 1,\ldots,\rho$ be the poles of $\tilde{q}_H(z)$ with negative real part. Then

$$\tilde{q}_-(z) = z^{-\rho} \prod_{j=1}^{\rho} \frac{z - \kappa_j}{1 - \kappa_j} \qquad (8.1 - 8)$$

Let $\pi_i$, $i = 1, \ldots, \xi$ be the unstable roots (including $z = 1$) of the least common denominator of $\tilde{p}^*(z), v^*(z)$ with multiplicity $m_i$. Recall that according to the assumption of Thm. 9.2-1, $v^*(z)$ has at least as many poles at $z = 1$ as $\tilde{p}^*(z)$ and each strictly unstable pole of $v^*(z)$ is also a pole of $\tilde{p}^*(z)$. The system type and the internal stability requirements can be unified as

$$\frac{d^k}{dz^k}(1 - \tilde{q}_-(z)B(z))\Big|_{z=\pi_i} = 0,\ k = 0, \ldots, m_i - 1;\ i = 1, \ldots, \xi \qquad (9.2 - 21)$$

We can write

$$B(z) = \sum_{j=0}^{M-1} b_j z^{-j} \qquad (9.2 - 22)$$

where

$$M = \sum_{i=1}^{\xi} m_i \qquad (9.2 - 23)$$

and compute the coefficients $b_j$, $j = 0, \ldots, M-1$ from (9.2–21). Note that since none of the $\pi_i$'s is 0 or $\infty$, (9.2–21) is equivalent to

$$\frac{d^k}{d\lambda^k}(1 - q_-(\lambda^{-1})B(\lambda^{-1}))\Big|_{\lambda=\pi_i^{-1}} = 0,\ k = 0, \ldots, m_i - 1,\ i = 1, \ldots, \xi \qquad (9.2 - 24)$$

Since both $\tilde{q}_-(\lambda^{-1})$ and $B(\lambda^{-1})$ are polynomials their derivatives with respect to $\lambda$ can be computed easily. Equation (9.2–24) yields a system of $M$ linear equations with $M$ unknowns $(b_0, b_1, \ldots, b_{M-1})$. The resulting controller $\tilde{q}(z)$ combines the desirable properties of the $H_2^*$-optimal controller and deadbeat type controllers, as explained in Sec. 8.1.2.

### 9.2.3 Anti-aliasing Prefilter

If the designer decides to add a prefilter $\gamma(s)$ in the block structure (Fig. 7.1-1A), it should be such that the system type (asymptotic properties) and the internal stability requirements are satisfied.

Section 7.5.2 discussed in detail the design of $\gamma(s)$ so that the system type is preserved. When the only unstable poles of $\tilde{p}^*(z)$ are at $z = 1$ (i.e., of $\tilde{p}(s)$ at $s = 0$), the assumption of Thm. 9.2-1 that $v^*(z)$ has at least as many poles at

$z = 1$ as $\tilde{p}^*(z)$, ensures that the internal stability conditions are satisfied for any prefilter which preserves system type.

When $\tilde{p}^*(z)$ has unstable poles in addition to those at $z = 1$, it is not a simple manner to design $\gamma$ such that condition (ii) of Thm. 9.1-1 is satisfied *after* $\tilde{q}(z)$ has been determined as outlined in the preceding two sections. The preferred approach is to design $\gamma(s)$ *first* according to Sec. 7.5.2. Then one computes $\tilde{p}^*_\gamma(z)$ and uses it instead of $\tilde{p}^*(z)$ in Thm. 9.2-1 in order to obtain $\tilde{q}_H(z)$ and subsequently $\tilde{q}(z)$. However, this means that the objective function which is minimized is not the one given by (9.2–9) but

$$\phi_\gamma = \|(1 - \tilde{p}^*_\gamma(z)\tilde{q}(z))v^*(z)\|_2^2 \qquad (9.2-25)$$

which does not correspond to the true physical problem. Usually (9.2–25) is a good approximation of (9.2–9).

### 9.2.4 Design for Common Input Forms

In this section we shall examine the $H_2^*$-optimal controller $\tilde{q}_H(z)$, given by (9.2–4), for specific systems and inputs.

(i) *MP System.*

When $p(s)$ is MP and also strictly proper (all physical systems are strictly proper), $p^*(z)$ will have a delay of one unit because of sampling. Hence $p_A^* = z^{-1}$, $p_M^* = zp^*$, and (9.2–4) yields

$$\begin{aligned}\tilde{q}_H &= (p^*)^{-1}((b_p^*)^{-1}b_v^*v_M^*)^{-1}((b_p^*)^{-1}b_v^*v_M^* - \kappa) \\ &= (p^*)^{-1}(1 - \kappa b_p^*(b_v^*v_M^*)^{-1}) \qquad (9.2-26)\end{aligned}$$

where $\kappa$ is the constant term in the partial fraction expansion of $(b_p^*)^{-1}b_v^*v_M^*$. Equivalently, since $b_p^*, b_v^*, v_M^*$ are semi-proper, $\kappa$ is the product of the constant terms of the PFE's of $b_p^{*-1}$, $b_v^*, v_M^*$. After some algebra we obtain

$$\kappa = v_0 \prod_{j=k'+1}^{k} \frac{1 - \pi_j}{1 - (\pi_j^H)^{-1}} \qquad (9.2-27)$$

where $k', k, \pi_j$ are defined in Thm. 9.2-1 and $v_0$ is the first non-zero coefficient obtained by long division of $v^*(z)$ (equal to the constant term in the PFE of $v_M^*(z)$).

(ii) *Stable System.* $b_p^* = b_v^* = 1$

$$\tilde{q}_H(z) = z(p_M^*v_M^*)^{-1}\{z^{-1}p_A^{*-1}v_M^*\}_* \qquad (8.1-5)$$

This formula was stated in Thm. 8.1-1.

(iii) *Integrator.* $p(s) = \frac{1}{s} \Rightarrow p^*(z) = \frac{T}{z-1}$. For $b_p^* = b_v^* = 1$,

$$\tilde{q}_H(z) = z(p_M^* v_M^*)^{-1}\{z^{-1}p_A^{*-1}v_M^*\}_* \tag{9.2-28}$$

The comments made in Sec. 5.2.2.iii for the continuous case, apply here as well.

(iv) *Type 1 design for system with one unstable pole.*
Consider the MP system

$$p(s) = \tilde{p}(s) = \frac{b}{-s+b}, \quad b > 0 \tag{9.2-29}$$

and assume that a step disturbance acts at the process input

$$v(s) = d(s) = \frac{b}{s(-s+b)} \tag{9.2-30}$$

Then for a sampling time T we have

$$p^*(z) = \frac{1-e^{bT}}{z-e^{bT}} \tag{9.2-31}$$

$$v^*(z) = \frac{(1-e^{bT})z}{(z-1)(z-e^{bT})} \tag{9.2-32}$$

$$v_M^*(z) = zv^*(z) \tag{9.2-33}$$

Note that $e^{bT} > 1$ since $b > 0$. The $H_2^*$-optimal controller can be obtained from (9.2–26). We have $b_p^* = b_v^*$ and so from (9.2–27)

$$\kappa = v_0 = 1 - e^{bT} \tag{9.2-34}$$

Substitution of (9.2–31 through 34) into (9.2–26) yields

$$\tilde{q}_H(z) = \frac{(z-e^{bT})((1+e^{bT})z - e^{bT})}{(1-e^{bT})z^2} \tag{9.2-35}$$

Since $\tilde{q}_H(z)$ has no poles with negative real parts,

$$\tilde{q}(z) = \tilde{q}_H(z) \tag{9.2-36}$$

### 9.2.5 Integral Squared Error (ISE) for Step Inputs to Stable Systems

The $H_2^*$-optimal controller $\tilde{q}_H(z)$ minimizes the sum of squared errors (SSE) for a particular input. To correct intersample rippling, the IMC controller $\tilde{q}(z)$ is obtained through the modification discussed in Sec. 8.1.2.

The ISE can be computed for the closed-loop system with $\tilde{q}(z)$ from (7.5–1) which describes the continuous plant output. For the specific case of a step setpoint or disturbance input ($v = -r$ or $d$), we have $h_0(s)v^*(e^{sT}) = v(s) = s^{-1}$ and then (7.5–1), (7.5–2) yield

$$e(s) = (1 - p(s)\tilde{q}(e^{sT}))s^{-1} \tag{9.2 – 37}$$

We have

$$ISE \triangleq \int_0^\infty e^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |e(i\omega)|^2 d\omega = \|e\|_2^2 \tag{9.2 – 38}$$

where $\|\cdot\|_2$ denotes the $H_2$-norm defined in Sec. 2.4.4.

For step inputs, we find from Table 8.1-1

$$\tilde{q}_H(z) = (p_M^*(z))^{-1} \tag{9.2 – 39}$$

From (8.1–11) we have $B(z) = 1$ and therefore

$$\tilde{q}(z) = \tilde{q}_H(z)\tilde{q}_-(z) \tag{9.2 – 40}$$

where $\tilde{q}_-(z)$ is defined in (8.1–8).

Hence we can write

$$ISE = \|(1 - p(s)(p_M^*(e^{sT}))^{-1}\tilde{q}_-(e^{sT}))s^{-1}\|_2^2 \tag{9.2 – 41}$$

By following the steps used in the proof of Thm. 4.1-3 we can break (9.2–41) into two parts:

$$ISE = \|(1 - p_A(s))s^{-1}\|_2^2 + \|(1 - p_M(s)(p_M^*(e^{sT}))^{-1}\tilde{q}_-(e^{sT}))s^{-1}\|_2^2 \tag{9.2 – 42}$$

where $p_A(s), p_M(s)$ are defined in (4.1–3 through 4.1–5).

Note that the first term in (9.2–45) is the minimum ISE for the continuous case. Hence the second term represents the additional ISE that is introduced because of the use of a discrete rather than a continuous controller (designed according to Secs. 8.1.1 and 8.1.2.)

## 9.3 The Discrete IMC Filter

The philosophy behind the IMC filter is the same as for stable systems (Sec. 8.2). The filter structure is fixed and only a few parameters are adjusted to meet the robustness objectives. The simplest filter form is

$$f_1(z) = \frac{(1-\alpha)z}{z-\alpha} \qquad (8.2-1)$$

### 9.3.1 Filter Form

The discrete IMC filter $f(z)$ has to satisfy the following requirements

(i) Asymptotic tracking of external system inputs (setpoints and/or disturbances) — i.e., $(1-\tilde{p}^*\tilde{q}f)v^*$ has to be stable.

(ii) Internal stability — i.e., $\tilde{q}f$ and $(1-\tilde{p}^*\tilde{q}f)\tilde{p}^*$ have to be stable.

Since $\tilde{q}(z)$ has been designed so that (i) and (ii) are satisfied for $f(z)=1$, $f(z)$ should satisfy

$$\frac{d^k}{dz^k}(1-f(z))\Big|_{z=\pi_i} = 0,\ k=0,\ldots,m_i-1, \quad i=1,\ldots,\xi \qquad (9.3-1)$$

where $\pi_i$, $m_i$ were defined in Sec. 9.2.2. Note that (9.3–1) implies for $k=0$:

$$f(z)=1 \quad \text{at} \quad z=\pi_1,\ldots,\pi_\xi \qquad (9.3-2)$$

One can now select a filter of the form

$$f(z)=\phi(z)f_1(z) \qquad (9.3-3)$$

where

$$\phi(z)=\sum_{j=0}^{w}\beta_j z^{-j} \qquad (9.3-4)$$

and choose the coefficients $\beta_0,\ldots,\beta_w$ so that (9.3–1) is satisfied for some specified $\alpha$. The parameter $\alpha$ can be used as a tuning parameter.

Note that for $\xi=1$, $\pi_1=1$, $m_1=1$, we only need $\phi(z)=1$. For the general case (9.3–1) can be transformed into a system of $M$ linear equations with $\beta_0,\ldots,\beta_w$ as unknowns, where $M$ is given by (9.2–23). Lemma 8.2-1 can help simplify the necessary algebra. One should select $w \geq M-1$ so that the system of linear equations has one or more solutions. When $w \geq M$ the system is underdetermined and $\beta_0, \ldots, \beta_w$ can be obtained as the minimum norm solution. Note that for $M = 2$ one should select $w \geq 2$ in order to avoid the trivial solution $f(z) = 1$.

The case $\xi = 1$, $\pi_1 = 1$ was examined in detail in Sec. 8.2. Let us now examine the common case where $\xi > 1$, but $m_i = 1$ for $i = 2, \ldots, \xi$. Then (9.3–1) is equivalent to:

$$\frac{d^k}{dz^k}(1 - f(z))\bigg|_{z=\pi_1=1} = 0, \quad k = 0, \ldots, m_1 - 1 \qquad (9.3-5a)$$

$$f(\pi_i) = 1, \qquad i = 2, \ldots, \xi \qquad (9.3-5b)$$

The following theorem holds:

**Theorem 9.3-1.** *For $\pi_1 = 1$, $\xi \geq 2$, $m_i = 1$ for $i = 2, \ldots, \xi$, the coefficients $\beta_0, \ldots, \beta_w$ must satisfy*

$$\beta_0 = 1 - \beta_1 - \ldots - \beta_w \qquad (9.3-6)$$

$$\begin{pmatrix} \Pi \\ N_w \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_w \end{pmatrix} = \begin{pmatrix} f_1(\pi_\xi)^{-1} - 1 \\ \vdots \\ f_1(\pi_2)^{-1} - 1 \\ -\alpha/(1-\alpha) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \xi - 1 \\ \triangleq \chi \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m_1 - 1 \end{matrix} \qquad (9.3-7)$$

*where*

$$\Pi = \begin{pmatrix} \pi_\xi^{-1} - 1 & \ldots & \pi_\xi^{-w} - 1 \\ \vdots & & \vdots \\ \pi_2^{-1} - 1 & \ldots & \pi_2^{-w} - 1 \end{pmatrix} \qquad (9.3-8)$$

*and the elements $\nu_{ij}$ of the $(m_1 - 1) \times w$ matrix $N_w$ are defined by (8.2–6)*

$$\nu_{ij} = \begin{cases} 0 & \text{for } i > j \\ \frac{j!}{(j-i)!} & \text{for } i \leq j \end{cases} \qquad (8.2-6)$$

*Proof.* Follows directly from Thm. 8.2-1, (9.3–5) and the fact that $f_1(1) = 1$. □

For $\xi = 1$, Thm. 9.3-1 reduces to Thm. 8.2-1. For $m_1 = 1$, the choice $w = \xi - 1$, reduces (9.3–7) to the Vandermonde form (5.3–4). In general one should select $w \geq M - 1 = \xi + m_1 - 2$ and obtain $\beta_1, \ldots, \beta_w$ as the minimum norm solution to (9.3–7):

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_w \end{pmatrix} = A^T(AA^T)^{-1}\chi \qquad (9.3-9)$$

where

$$A \triangleq \begin{pmatrix} \Pi \\ N_w \end{pmatrix} \qquad (9.3-10)$$

Note that from a numerical point of view it is preferable to compute the pseudo-inverse in (9.3–9) from a singular value decomposition of $\begin{pmatrix} \Pi \\ N_w \end{pmatrix}$.

### 9.3.2 Qualitative Interpretation of the Filter Function

The discussion in this section is in the spirit of that in Sec. 5.3.2 for continuous systems. With the filter $f(z)$ and the IMC controller $\tilde{q}(z)$ obtained in the first design step, the discrete complementary sensitivity function becomes for $p^* = \tilde{p}^*$:

$$\tilde{\eta}^*(z) = \tilde{p}^*(z)\tilde{q}(z)f(z) \qquad (9.3-11)$$

For open-loop stable systems, any stable filter that satisfies the system type requirements described by Thm. 7.5-1 is acceptable. For open-loop unstable systems however, the filter has to be unity at the unstable system poles, which limits the range of filter parameters $\alpha$ that can be chosen for reasonable performance, as we shall show next. For the effect of the unstable poles on $\tilde{\eta}^*$ to become negligible $f(z)$ has to approach $f_1(z)$ (8.2–1). Specifically, it follows from (9.3–3) that $\phi(z)$ has to approach unity. We will study the behavior of $\phi(z)$ for $w \to \infty$.

Consider the system studied in Sec. 9.2.4.iv. For internal stability and asymptotically error-free disturbance compensation we require

$$f(1) = f(e^{bT}) = 1 \qquad (9.3-12)$$

In the notation of Sec. 9.3.1, we have in this case $\xi = 2,\ \pi_1 = 1,\ \pi_2 = e^{bT}, m_1 = m_2 = 1$. Hence $f(z)$ is given by (9.3–3, 4, 6, 9) where

$$A = (e^{-bT} - 1 \quad \ldots \quad e^{-wbT} - 1) \qquad (9.3-13)$$

$$\chi = f_1(e^{bT})^{-1} - 1 = \frac{\alpha(1 - e^{-bT})}{1 - \alpha} \qquad (9.3-14)$$

From (9.3–9) it follows that

$$\sum_{j=1}^{w} \beta_j^2 = \chi^T(AA^T)^{-1}\chi = \frac{\alpha^2(1 - e^{-bT})^2}{(1 - \alpha)^2 S_1} \qquad (9.3-15)$$

where

$$S_1 \triangleq \sum_{j=1}^{w}(e^{-jbT}-1)^2$$
$$= \sum_{j=1}^{w} e^{-j2bT} - 2\sum_{j=1}^{w} e^{-jbT} + w$$
$$= \frac{1-e^{-2bTw}}{e^{2bT}-1} - 2\frac{1-e^{-bTw}}{e^{bT}-1} + w \qquad (9.3-16)$$

Since $|e^{-bT}| < 1$, it follows from (9.3–16) that $\lim_{w\to\infty} S_1 = \infty$ and $\lim_{w\to\infty} \sum_{j=1}^{w} \beta_j^2 = 0$. This fact, however, is not sufficient to produce an $f(z)$ that approximates the behavior of $f_1(z)$. For this to happen we need $\lim_{w\to\infty} \beta_0 = 1$. Let us compute this limit. From (9.3–9) we get

$$\beta_k = \frac{\alpha(1-e^{-bT})(e^{-kbT}-1)}{(1-\alpha)S_1}, \quad k = 1, \ldots, w \qquad (9.3-17)$$

(9.3–6), (9.3–17) yield

$$\beta_0 = 1 - \frac{\alpha(1-e^{-bT})S_2}{(1-\alpha)S_1} \qquad (9.3-18)$$

where

$$S_2 \triangleq \sum_{j=1}^{w}(e^{-jbT}-1) = \frac{1-e^{-bTw}}{e^{bT}-1} - w \qquad (9.3-19)$$

From (9.3–16), (9.3–19) it follows that $\lim_{w\to\infty} S_2/S_1 = -1$. Then (9.3–18) yields

$$\lim_{w\to\infty} \beta_0 = \frac{1-\alpha e^{-bT}}{1-\alpha} \qquad (9.3-20)$$

By writing $\alpha = e^{-T/\lambda}$ we get

$$\lim_{w\to\infty} \beta_0 = \frac{1-e^{-T(1/\lambda+b)}}{1-e^{-T/\lambda}} \qquad (9.3-21)$$

Hence in order for $\lim_{w\to\infty} \beta_0 \cong 1$ we need $1/\lambda \gg b$ or $\lambda b \ll 1$. In this case the behavior of $f(z)$ approaches that of $f_1(z)$ (compare to (5.3–25)) and if a $\lambda$ in that range is sufficient for robustness, the unstable pole $b$ produces no significant effect on the system behavior. If, however, one chooses a $\lambda$ for which $\lambda b \gg 1$, then the $\lim_{w\to\infty} \beta_0$ is very far from 1 and as a result problems similar to those discussed in Sec. 5.3.2 for the continuous case appear.

This is illustrated in Fig. 9.3-1, where amplitude plots of $f_1$ and $f$ are shown for different values of $\lambda$ and $w$. We see that as $w$ increases, $f$ tends towards $f_1$. For $\lambda b \ll 1$, the approximation is very good, while for $\lambda b \gg 1$, the closer we get to $f_1$, the higher the peak in $|f|$ becomes.

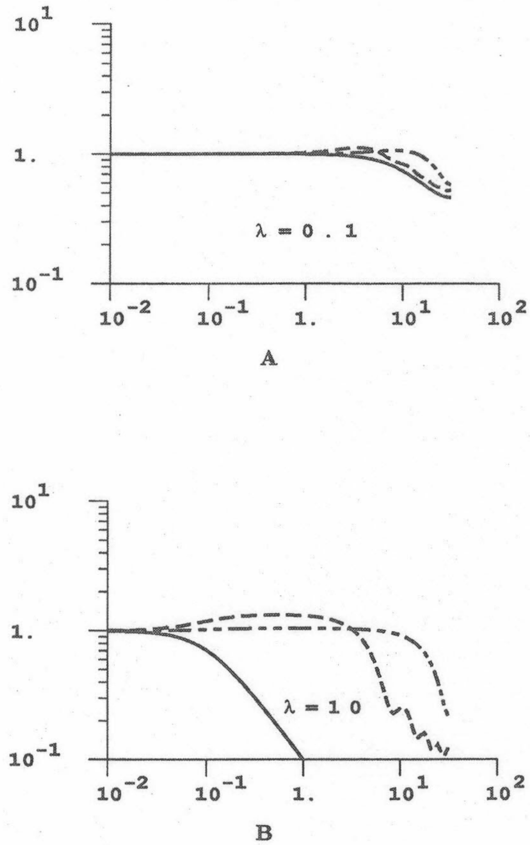



*Figure 9.3-1.* Effect of a RHP pole on the discrete IMC filter. $T = 0.1, b = 1$. Solid: $f_1$, Dash: $f_1\phi, w = 9$, Dot-Dash: $f_1\phi, w = 2$.

## 9.4 Robust Stability

For controllers designed via the IMC design procedure ($\tilde{\eta}^* = \tilde{p}^*\tilde{q}f$) Thm. 7.6-1 becomes Cor. 9.4-1.

**Corollary 9.4-1 (Robust Stability).** *Assume that all plants $p$ in the family $\Pi^*$*

$$\Pi^* = \left\{ p : \left| \frac{p_\gamma^*(i\omega) - \tilde{p}_\gamma^*(i\omega)}{\tilde{p}^*(i\omega)} \right| < \bar{\ell}_m^*(\omega) \right\} \qquad (9.4-1)$$

*have the same number of RHP poles and that these poles do not become unobservable after sampling. Then the system is robustly stable if and only if the IMC filter satisfies*

$$|f| < \frac{1}{|\tilde{p}^*\tilde{q}\bar{\ell}_m^*|}, \; 0 \le \omega \le \pi/T \qquad (9.4-2)$$

*where $\tilde{q}$ is a stabilizing controller for the nominal plant $\tilde{p}$.*

For stable systems $f$ is arbitrary. Therefore, there always exists a filter $f$ which satisfies (9.4–2) regardless of the magnitude of the uncertainty $\bar{\ell}_m^*$. For unstable systems $f$ is constrained to be unity at the poles of $\tilde{p}^*$ outside the UC. Thus, depending on $\bar{\ell}_m^*$, there might not exist any filter parameter $\alpha$ for which the constraint (9.4–2) is met. Indeed, there might not exist any filter — however complicated — which satisfies (9.4–2). A minimum amount of information is necessary or equivalently a maximum amount of uncertainty is allowed to stabilize an unstable system. The necessary information at $\omega = 0$ can be characterized easily.

**Corollary 9.4-2.** *Assume that a filter $f$ is to be designed for a system or disturbance pole(s) at $s = 0$ — i.e., $f(1) = 1$. There exists an $f$ such that the closed loop system is robustly stable for the family $\Pi^*$ described by (9.4–1) only if $\bar{\ell}_m(0) < 1$.*

Note that contrary to Cor. 8.3-2, Cor. 9.4-2 is only necessary. For unstable systems, in general, the filter has to satisfy other constraints in addition to the one at $z = 1$.

## 9.5 Robust Performance

The results in Sec. 7.7.2 hold for unstable systems if it is assumed that all plants in the family $\Pi^*$ have the same number of RHP poles and if the controller $\tilde{q}$ and filter $f$ are stabilizing for the nominal plant $\tilde{p}^*$.

## 9.6 Summary of the IMC Design Procedure

The required information for the IMC design is the same as that for stable systems: process model, input type, performance specifications and uncertainty information. The input specification requires some care. If the physical disturbance enters at the plant input, the disturbance used in the design procedure has to include the unstable system poles for the resulting controller to yield offset free performance.

*Design Procedure*

*Step 1: Nominal Performance*
The stabilizing $H_2$-optimal controller $\tilde{q}$ is determined which minimizes

$$\|(1 - \tilde{p}^* \tilde{q}) v^*\|_2$$

for the specified input $v^*$. The optimal controller $\tilde{q}$ can be found explicitly from (9.2–4).

$$\tilde{q}_H = z b_p^* (p_M^* b_v^* v_M^*)^{-1} \left\{ (z b_p^* p_A^*)^{-1} b_v^* v_M^* \right\}_* \qquad (9.2-4)$$

If all the unstable plant/disturbance poles are at the origin, the entries in Table 8.1-1 can be used to find $\tilde{q}_H$ for typical inputs.

The controller $\tilde{q}_H$ is then modified as described in Sec. 9.2-2 to eliminate the problems of intersample rippling:

$$\tilde{q} = \tilde{q}_H \tilde{q}_{-B} \qquad (9.2-20)$$

*Step 2: Robust Stability and Robust Performance*
The controller $\tilde{q}$ is augmented by the IMC filter $f$

$$q = \tilde{q} f$$

In order for $q$ to be stabilizing $f$ has to be unity at all unstable system $\tilde{p}^*$ and input $v^*$ poles $\pi_1, \ldots, \pi_k$. When the poles outside the UC are distinct, the one-parameter filter is determined through Thm. 9.3-1. The filter poles are a subset of the closed-loop poles. If they are made much slower than the mirror images of the poles of $\tilde{p}^*$ outside the UC undesirable performance and robustness properties result.

*Robust Stability.* Increase $\alpha$ (or $\lambda$ in $\alpha = e^{-T/\lambda}$) until

$$|\tilde{p}^* \tilde{q} f \bar{\ell}_m^*| < 1, \qquad 0 \le \omega \le \pi/T \qquad (9.4-2)$$

is satisfied for $\alpha \ge \alpha^*$.

*Robust Performance.* Increase $\alpha$, starting from $\alpha^*$, until the following condition is met

$$|\hat{q}|\bar{\ell}_a + |1 - \tilde{p}\hat{q}|w \leq 1, \qquad 0 \leq \omega \leq \pi/T \qquad (8.4-1)$$

where

$$\hat{q}(s) = \tilde{q}(e^{sT})f(e^{sT})h_0(s)\gamma(s)/T \qquad (7.5-10)$$

## 9.7 Application: Distillation Column Base Level Control

This example was discussed in detail in Sec. 5.7.1. It is briefly considered here again, because some interesting issues arise when a digital controller is designed for the process.

The process model is

$$\tilde{p}(s) = \frac{1}{s}(1 - 2e^{-s\theta}) \qquad (5.7-1)$$

Let us select a sampling time $T = \theta/N$, where $N$ is an integer. Then the zero-order hold discrete equivalent is

$$\tilde{p}^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)\tilde{p}(s)\} = (1 - 2z^{-N})\mathcal{Z}\mathcal{L}^{-1}\left\{h_0(s)\frac{1}{s}\right\} = (1 - 2z^{-N})\frac{T}{z-1} \qquad (9.7-1)$$

In this very special case we find that if $\zeta$ is a finite zero of $\tilde{p}(s)$, then $e^{\zeta T}$ is a zero of $\tilde{p}^*(z)$ and therefore $\zeta$ is a zero of $\tilde{p}^*(e^{sT})$. This mapping does not hold for zeros in general, although it is always true for the poles of $\tilde{p}(s)$ and $\tilde{p}^*(z)$.

Because of this mapping the zeros of $\tilde{p}^*(z)$ that appear as poles of $\tilde{q}_H(z)$ are cancelled by zeros of $\tilde{p}(s)$ in (7.5–6). Therefore, any such zeros close to (-1,0) do not produce intersample rippling even if the modification described in Sec. 9.2.2 is not made. However, this does not mean that the behavior of the control system will deteriorate if the suggested modification is introduced. Indeed, the steps proposed in Sec. 9.2 will result in a well-performing controller, regardless of whether $\tilde{q}_H(z)$ suffers from rippling problems or not.

Let us proceed to illustrate this point by simulating the response for the two controllers for a ramp disturbance $d(s) = s^{-2}$. For the simulations we choose $\theta = 5$ and $T = 1$, which implies that $N = 5$. It follows from (9.7–1) that the zeros of $\tilde{p}^*(z)$ are located at $2^{1/5}e^{i2k\pi/5}$, $k = 0, 1, 2, 3, 4$, where $2^{1/5}$ indicates the real fifth root of 2. Two of these zeros (the ones that correspond to $k = 2, 3$) have

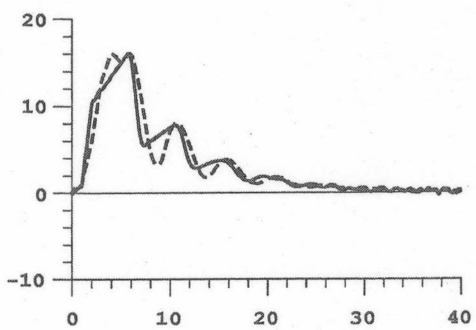



*Figure 9.7-1.* Distillation column base level control; response to $d(s) = s^{-2}$. Solid line: $\tilde{q}_H$. Dashed line: $\tilde{q}_H \tilde{q}_- B$.

negative real parts and will give rise to poles of $\tilde{q}_H(z)$ with negative real parts. The procedure of Sec. 9.2 yields

$$\tilde{q}_H(z) = \frac{z^3(17z - 16)(z - 1)}{(-2z^5 + 1)} \qquad (9.7-2)$$

$$\tilde{q}_-(z) = \frac{z^2 + 1.8586z + 1.3195}{4.1781z^2} \qquad (9.7-3)$$

$$B(z) = 2.0765 - 1.0765z^{-1} \qquad (9.7-4)$$

Figure 9.7-1 shows the responses to $d(s) = s^{-2}$ for both $q = \tilde{q}_H$ and $q = \tilde{q}_H \tilde{q}_- B$. Clearly, $\tilde{q}_H$ produces no intersample rippling. One can also see that when the modification of Sec. 9.2.2 is made anyway, the response is essentially unaffected.

Finally, note that because the open-loop system is unstable, the controller has to be implemented in the classic feedback structure. Its expression can be obtained from (7.2–3) and its implementation presents no problem.

## 9.8 References

9.1. The parametrization of all stabilizing controllers presented in this section is the discrete equivalent of that presented in Sec. 5.1. It was obtained by extending that proposed by Zames and Francis (1983) to include systems with integrators.

9.2.1. The $H^*$-optimal controller can also be obtained with state-space methods (e.g., Kwakernaak and Sivan, 1972).