Introduction to MVC

Definition---**Properness and strictly properness**

A system G(s) is proper if all its elements $\{g_{ij}(s)\}$ are proper, and strictly proper if all its elements are strictly proper.

Definition---**Causal**

A system G(s) is causal if all its elements are causal, and not causal if all its elements are noncausal.

Definition---**Poles**

The eigen values λ_i, i=1,---,n of the system G(s) are called the pole of the system.

The pole polynomial $\pi(s)$ is defined as: $\pi(s)=\prod_{i=1}^{n}(s-\lambda_i)$, where $\pi(s)$ is the least common denominator of all non-identical-zero minors of all order of G(s).

Example:

$$G(s)=\frac{1}{(s+1)(s+2)(s-1)}\begin{bmatrix}(s-1)(s+2) & 0 & (s-1)^2 \\ -(s+1)(s+2) & (s-1)(s+1) & (s-1)(s+1)\end{bmatrix}$$

The minor of order 2:

$$G_{1,2}^{1,2}=\frac{1}{(s+1)(s+2)};\ G_{1,3}^{1,2}=\frac{2}{(s+1)(s+2)};\ G_{2,3}^{1,2}=\frac{-(s-1)}{(s+1)(s+2)^2}$$

The least common denominator of the minors:

$$\pi(s)=(s+1)(s+2)^2(s-1)$$

Definition---**Zeros**

If the rank of G(z) is less than the normal rank, z is a zero of the system. The zero polynomial Z(s) is the greatest common divisor of the numerators of all order-r minors of G(s), where r is the norminal rank of G(s) provided that these minors have all been adjusted in such a way as to have the pole polynomial $\pi(s)$ as their denominator.

$$G_{1,2}^{1,2}(s)=\frac{(s-1)(s+2)}{\pi(s)};\ G_{1,3}^{1,2}(s)=\frac{2(s-1)(s+2)}{\pi(s)};\ G_{2,3}^{1,2}(s)=\frac{-(s-1)^2}{\pi(s)}$$

So, $Z(s)=(s-1)$

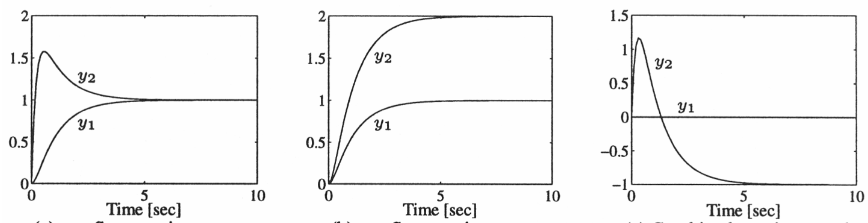
According to this definition, the zero polynomial of a square G(s) is:

$\det\{G(s)\}=0$

Notice that, in a MIMO system, there may be no inverse response to indicate the presence of RHP-zero. For example, in the MIMO system of the following:

$$G(s)=\frac{1}{(0.2s+1)(s+1)}\begin{bmatrix}1 & 1\\ 1+2s & 2\end{bmatrix};\quad G(0.5)=0;\quad \underline{\sigma}\{G(0.5)\}=0$$

$$G(0.5)=\frac{1}{1.65}\underbrace{\begin{bmatrix}0.45 & 0.89\\ 0.89 & -0.45\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}1.92 & 0\\ 0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}0.71 & -0.71\\ 0.71 & 0.71\end{bmatrix}^{H}}_{V^H}$$



(a) Step in u_1, $u=[1\quad 0]^T$ (b) Step in u_2, $u=[0\quad 1]^T$ (c) Combined step in u_1 and u_2, $u=[1\quad -1]^T$

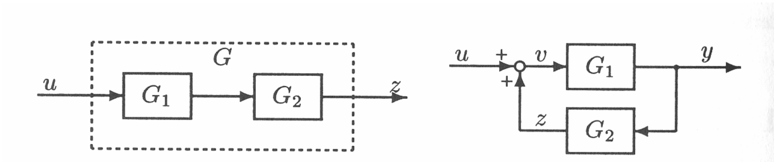
Transfer functions for Closed loop MIMO systems

1. **Cascade rule**. For the cascade interconnection of $G_1(s)$ and $G_2(s)$ in the following figure(a), the overall transfer function matrix is $G=G_1G_2$
2. **Feedback rule.** With reference to the positive feedback system in Figure (b):

$$v=(I-L)^{-1}u=(I-G_2G_1)^{-1}u$$

3. **Push-through rul**e. $G_1(I-G_2G_1)^{-1}=(I-G_1G_2)^{-1}G_1$

$$\begin{aligned}G_1-G_1G_2G_1&=G_1(I-G_2G_1)=(I-G_1G_2)G_1\\ &\Rightarrow (I-G_1G_2)^{-1}G_1(I-G_2G_1)=G_1\\ &\Rightarrow (I-G_1G_2)^{-1}G_1=G_1(I-G_2G_1)^{-1}\end{aligned}$$



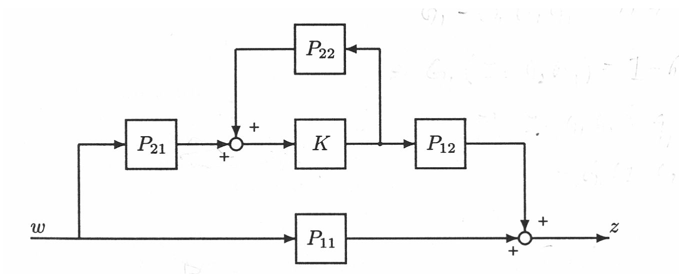
(a) Cascade system (b) Positive feedback system

4. **MIMO rule**.

(1). Start from the output, write down the blocks as moving backward by the most direct path toward the input

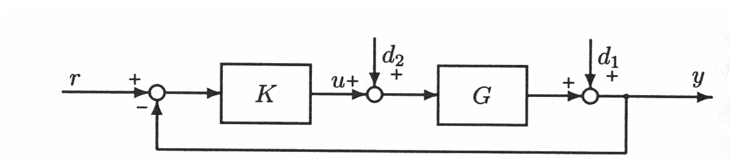
(2). When exit from a feedback loop, include a term $(I-L)^{-1}$ for positive feedback (or, $(I+L)^{-1}$ for negative feedback. Notice that L is the evaluated against signal flow starting at the point of exit from the loop.

Example: Consider the following block diagram:



$$z = \left[p_{11} + p_{12} K (1 - P_{22} K)^{-1} p_{21} \right] w$$

5. Consider the closed-loop system:



The following relationships are useful:

$$(I+L)^{-1} + L(I+L)^{-1} = S + T = 1$$
$$G(I+KG)^{-1} = (I+GK)^{-1} G$$
$$GK(I+GK)^{-1} = G(I+KG)^{-1} K = (I+GK)^{-1} GK$$
$$T = L(I+L)^{-1} = (I+L)^{-1} L = (I+L^{-1})^{-1}$$

Singular Values and Matrix Norms

1. Vector Norms

A real valued function $\|\bullet\|$ defined on a vector space X is said to be a norm on X, if it satisfies the following properties:

(1) $\|x\| \geq 0$;

(2) $\|x\| = 0$ only if $x = 0$;

(3) $\|\alpha x\| = |\alpha| \, \|x\|$, for any scalar α;

(4) $\|x + y\| \leq \|x\| + \|y\|$;

for any $x \in X$ and $y \in X$.

The vector p-norm of $x \in X$ is defined as: $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^{1/p} \right)$, for $1 \leq p \leq \infty$. In particular $p = 1, 2, \infty$ we have:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|; \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} (|x_i|)^2}; \qquad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| .$$

H$\ddot{o}$lder inequality:

$$|x^T y| \leq \|x\|_p \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Notice that:

(1) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$;

(2) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$;

(3) $\|x\|_\infty \leq \|x\|_1 \leq \sqrt{n} \|x\|_\infty$

2. Matrix norms:

A matrix norm is defined as a function valued that satisfies the following:

(1) $\|A\| \geq 0$ for all $A \in R^{m\times n}$ with equality only if A=0;

(2) $\|\alpha A\| = |\alpha| \, \|A\|$ for all $\alpha \in R,\ A \in R^{m\times n}$;

(3) $\|A+B\| \leq \|A\| + \|B\|$ for all $A, B \in R^{m\times n}$

Let $A = [a_{ij}] \in R^{m\times n}$, the vector induced p-norms is defined as:

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

In particular:

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(column sum)};$$

$$\|A\|_2 = \sqrt{\lambda_{\max}\left(A^* A\right)};$$

$$\|A\|_\infty = \max_{1\leq j\leq n} \sum_{j=1}^{n} |a_{ij}| \quad \text{(row sum)}$$

Another popular matrix norm is the Frobenius norm, i.e.:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} | a_{ij} |^2}$$

The p-norms have the following important property for every $A \in R^{m\times n}$:

(1) $\|AX\|_p \leq \|A\|_p \|x\|_p$; p=1,2, ∞;

(2) $\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2$;

(3) $\max\limits_{1\leq i\leq m;1\leq j\leq n} |a_{ij}| \leq \|A\|_2 \leq \sqrt{mn}\left\{\max\limits_{1\leq i\leq m;1\leq j\leq n} |a_{ij}|\right\}$;

(4) $\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$;

(5) $\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\,\|A\|_1$

Lemma 1: Let $x \in F^n$ and $y \in F^m$.

(1) Suppose $n \ge m$. $\|x\| = \|y\|$ iff $\exists U \in F^{n \times m}\ s.t.\ x = Uy,\ and\ U^*U = I$;

(2) Suppose $n = m$. $\left|x^* y\right| \le \|x\|\,\|y\|$. Moreover, equality holds iff $x = \alpha y$ for some $\alpha \in F\ or\ y = 0$;

(3) $\|x\| \le \|y\|\ iff\ \exists \Delta \in F^{n \times m}\ with \|\Delta\| \le 1\ s.t.\ x = \Delta y$. Furthermore,

$\|x\| \le \|y\|\ iff\ \|\Delta\| < 1$;

(4) $\|Ux\| = \|x\|$ for any appropriate dimensioned unitary matrix U.

Lemma 2: Let A and B be any appropriate dimensioned matrices.

(1) $\rho(A) \le \|A\|_p$, $(p = 1, 2, \infty, F)$;

(2) $\|AB\| \le \|A\|\,\|B\|$; $\|A^{-1}\| \ge \|A\|^{-1}$;

(3) $\|UAV\| = \|A\|,\ and\ \|UAV\|_F = \|A\|_F$, for any unitary matrices U and V;

(4) $\|AB\|_F \le \|A\|\,\|B\|_F,\ and\ \|AB\|_F \le \|B\|\,\|A\|_F$.

Lemma 3: Let A be a block partitioned matrix with:

$$A = \begin{bmatrix} A_{11} & A_{12} & & A_{1q} \\ A_{21} & A_{21} & & A_{2q} \\ & & & \\ A_{m1} & A_{m2} & & A_{mq} \end{bmatrix} = \left[A_{ij} \right]$$

Then, for any induced matrix p-norm,

$$\|A\|_p \le \left\| \begin{bmatrix} \|A_{11}\|_p & \|A_{12}\|_p & & \|A_{1q}\|_p \\ \|A_{21}\|_p & \|A_{22}\|_p & & \|A_{2q}\|_p \\ & & & \\ \|A_{m1}\|_p & \|A_{m2}\|_p & & \|A_{mq}\|_p \end{bmatrix} \right\|_p$$

Further, the equality holds if the F-norm is used.

Proof. It is obvious that if the F-norm is used, then the right hand side of inequality (2.2) equals the left hand side. Hence only the induced p-norm cases, $1 \le p \le \infty$, will be shown. Let a vector x be partitioned consistently with A as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix};$$

and note that

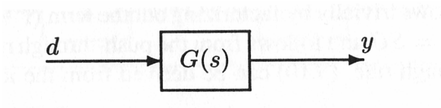
$$\|x\|_p = \left\| \begin{bmatrix} \|x_1\|_p \\ \|x_2\|_p \\ \vdots \\ \|x_q\|_p \end{bmatrix} \right\|_p .$$

Then

$$\begin{aligned}
\|[A_{ij}]\|_p &:= \sup_{\|x\|_p=1} \|[A_{ij}]\, x\|_p = \sup_{\|x\|_p=1} \left\| \begin{bmatrix} \sum_{j=1}^q A_{1j}x_j \\ \sum_{j=1}^q A_{2j}x_j \\ \vdots \\ \sum_{j=1}^q A_{mj}x_j \end{bmatrix} \right\|_p \\
&= \sup_{\|x\|_p=1} \left\| \begin{bmatrix} \left\|\sum_{j=1}^q A_{1j}x_j\right\|_p \\ \left\|\sum_{j=1}^q A_{2j}x_j\right\|_p \\ \vdots \\ \left\|\sum_{j=1}^q A_{mj}x_j\right\|_p \end{bmatrix} \right\|_p \le \sup_{\|x\|_p=1} \left\| \begin{bmatrix} \sum_{j=1}^q \|A_{1j}\|_p \|x_j\|_p \\ \sum_{j=1}^q \|A_{2j}\|_p \|x_j\|_p \\ \vdots \\ \sum_{j=1}^q \|A_{mj}\|_p \|x_j\|_p \end{bmatrix} \right\|_p \\
&= \sup_{\|x\|_p=1} \left\| \begin{bmatrix} \|A_{11}\|_p & \|A_{12}\|_p & \cdots & \|A_{1q}\|_p \\ \|A_{21}\|_p & \|A_{22}\|_p & \cdots & \|A_{2q}\|_p \\ \vdots & \vdots & & \vdots \\ \|A_{m1}\|_p & \|A_{m2}\|_p & \cdots & \|A_{1q}\|_p \end{bmatrix} \begin{bmatrix} \|x_1\|_p \\ \|x_2\|_p \\ \vdots \\ \|x_q\|_p \end{bmatrix} \right\|_p \\
&\le \sup_{\|x\|_p=1} \left\| \left[\|A_{ij}\|_p \right] \right\|_p \|x\|_p \\
&= \left\| \left[\|A_{ij}\|_p \right] \right\|_p .
\end{aligned}$$

Singular Value Decomposition

Consider a fixed frequency ω where $G(j\omega)$ is a $l \times m$ complex matrix.



Denote $G(j\omega)$ as G for simplicity. Any matrix G may be decomposed into $G = U\Sigma V^H$.

Where,

Σ is an $l \times m$ matrix with $k = \min\{l,\ m\}$ non-negative singular values, σ_i. arranged in descending order along its main diagonal; the other entries are zero. The singular values are the positive square roots of the eigenvalues of $G^H G$, and G^H is the conjugate transpose of G. That is:

$$\sigma_i(G) = \sqrt{\lambda_i(G^H G)}$$

$y = G \cdot d$;

Let u be one of the eigenvector of *G*.

$$d = v_i \Rightarrow \|v_i\|^2 = \alpha^2 \left\{ \sqrt{|v_{i,1}|^2 + |v_{i,2}|^2 + \cdots |v_{i,n}|^2} \right\}^2$$

$$y = G(v_i) = Gv_i = \lambda_i v_i$$

$$\Rightarrow \|y\|^2 = \alpha^2 \lambda_i^2 \|v_i\|^2$$

$$\therefore \frac{\|y\|_2}{\|d\|_2} = \frac{|\lambda_i| \|v_i\|}{\|v_i\|} = |\lambda_i|$$

On the other hand, it can be shown that the extreme values of $\|Gd\|_2 / \|d\|_2$ are $\lambda_i^{1/2}\left[G^H G\right]$, which is known as the singular value of G, in the directions of eigenvaectors of $G^H G$.

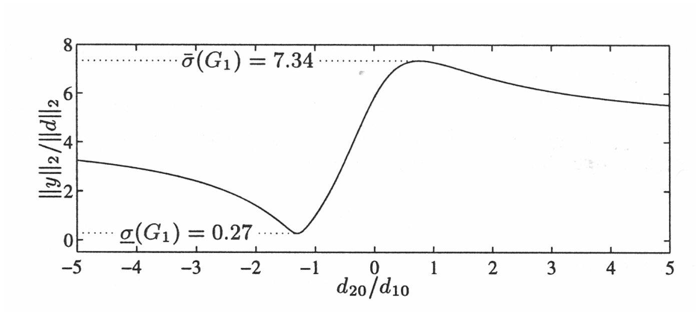
Since, $\underline{\sigma} = \min \frac{\|y\|_2}{\|d\|_2} \le \frac{\|y\|_2}{\|d\|_2} \le \max \frac{\|y\|_2}{\|d\|_2} = \bar{\sigma}$

As a result, $\underline{\sigma} \le |\lambda_i| \le \bar{\sigma}$

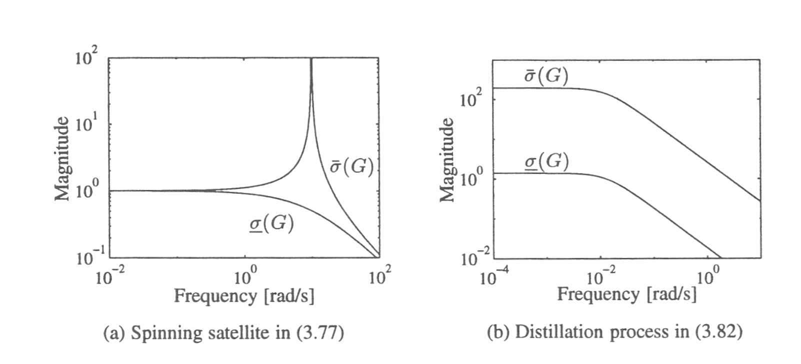
The singular values can be considered as the extreme gains of the MIMO system, which are local maximal values with respect to the direction of inputs. For example, consider the gain matrix at a specific frequency:

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

The gains with respect to the direction of d are given in the following figure:



Typical singular values with respect to frequency are as shown in the following figure:



Theorem 1: Let $A \in F^{m \times n}$. There exist unitary matrices:

$$U = [u_1, u_2, \cdots, u_m] \in F^{m \times m}; V = [v_1, v_2, \cdots, v_n] \in F^{n \times n}$$

such that: $A = U \Sigma V^*$, $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$

where,

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}; \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0; \quad p = \min\{m, n\}$$

[Proof]

Let $\sigma = \|A\|$, and assume $m \ge n$.

Then, from the definition of $\|A\|$, i.e.:

$$\|A\|_p = \sup \frac{\|Az\|_p}{\|z\|_p}; \quad (p = 1, 2, \infty) \Rightarrow \|Az\|_p = \|A\|_p \|z\|_p \text{ for some } z$$

In other words, there exists a vector $z \in F^n$ such that

$\|Az\| = \sigma \|z\| = \|\sigma z\|$

By the Lemma

($\|x\| = \|y\|$ iff there is a matrix $U \in F^{m\times n}$ such that $x = Uy$ and $U^*U = I$)

there is a matrix $\tilde{U} \in F^{m\times n}$ such that $\tilde{U}^*\tilde{U} = I$ and

$Az = \tilde{U}(\sigma z) = \sigma \tilde{U} z$.

Let: $x = \dfrac{z}{\|z\|} \in F^n$ and $y = \dfrac{\tilde{U}z}{\|\tilde{U}z\|} \in F^m$

We have: $Ax = \dfrac{Az}{\|z\|} = \dfrac{\sigma \tilde{U} z}{\left(\dfrac{\|Az\|}{\sigma}\right)} = \dfrac{\sigma^2 \tilde{U} z}{\|Az\|} = \dfrac{\sigma^2 \tilde{U} z}{\|\sigma \tilde{U} z\|} = \dfrac{\tilde{U}z}{\|\tilde{U}z\|} = y$

Let $U = [y, U_1] \in F^{m\times m}; V = [x, V_1] \in F^{n\times n}$ be unitary.
Thus,

$$\begin{aligned} A_1 &= U^* A V = [y, U_1]^* [Ax,\, AV_1] \\ &= \begin{bmatrix} y^*Ax & y^*AV_1 \\ U_1^*Ax & U_1^*AV_1 \end{bmatrix} = \begin{bmatrix} \sigma y^* y & y^*AV_1 \\ \sigma U_1^* y & U_1^*AV_1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma & w^* \\ 0 & B \end{bmatrix} \end{aligned}$$

Since,

$$\left\| A_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_2^2 = \sigma^2 + w^* w \quad \Rightarrow \quad \|A_1\|^2 \ge \sigma^2 + w^* w$$

and $\sigma = \|A\| = \|A_1\|$, we conclude that $w = 0$.

An obvious induction argument gives: $U^* AV = \Sigma$. The σ_i is the i-th

eigen value of A, and u_i and v_j are i-th left singular vector and j-th right

singular vectors, respectively. It is obvious to see:

$$Av_i = \sigma_i u_i \quad and \quad A^* u_i = \sigma_i v_i$$

or, $$A^* A v_i = \sigma_i^2 v_i \quad and \quad AA^* u_i = \sigma_i^2 u_i$$

Hence, σ_i^2 is an eigen value of AA^* *or* $A^* A$, u_i is an eigen vector of

AA^*, and is an eigen vector of $A^* A$. The following notations for singular values are often used:

$$\bar{\sigma}(A) = \sigma_{\max}(A) = \sigma_1 = \max_{\|x\|=1} \|Ax\|$$

$$\underline{\sigma}(A) = \sigma_{\min}(A) = \sigma_p = \min_{\|x\|=1} \|Ax\|$$

Lemma 1: Suppose A and D are square matrices.

(1) $\left|\underline{\sigma}(A+\Delta) - \underline{\sigma}(A)\right| \le \bar{\sigma}(A)$;

(2) $\underline{\sigma}(A\Delta) \ge \underline{\sigma}(A)\underline{\sigma}(\Delta)$;

(3) $\bar{\sigma}(A^{-1}) = \dfrac{1}{\underline{\sigma}(A)}$ if A is invertible;

Proof.

(i) By definition

$$\begin{aligned}
\underline{\sigma}(A+\Delta) &:= \min_{\|x\|=1} \|(A+\Delta)x\| \\
&\geq \min_{\|x\|=1} \{\|Ax\| - \|\Delta x\|\} \\
&\geq \min_{\|x\|=1} \|Ax\| - \max_{\|x\|=1} \|\Delta x\| \\
&= \underline{\sigma}(A) - \overline{\sigma}(\Delta).
\end{aligned}$$

Hence $-\overline{\sigma}(\Delta) \leq \underline{\sigma}(A+\Delta) - \underline{\sigma}(A)$. The other inequality $\underline{\sigma}(A+\Delta) - \underline{\sigma}(A) \leq \overline{\sigma}(\Delta)$ follows by replacing A by $A+\Delta$ and Δ by $-\Delta$ in the above proof.

(ii) This follows by noting that

$$\begin{aligned}
\underline{\sigma}(A\Delta) &:= \min_{\|x\|=1} \|A\Delta x\| \\
&= \sqrt{\min_{\|x\|=1} x^*\Delta^* A^* A\Delta x} \\
&\geq \underline{\sigma}(A) \min_{\|x\|=1} \|\Delta x\| = \underline{\sigma}(A)\underline{\sigma}(\Delta).
\end{aligned}$$

(iii) Let the singular value decomposition of A be $A = U\Sigma V^*$, then $A^{-1} = V\Sigma^{-1}U^*$. Hence $\overline{\sigma}(A^{-1}) = \overline{\sigma}(\Sigma^{-1}) = 1/\underline{\sigma}(\Sigma) = 1/\underline{\sigma}(A)$.

□

Lemma 2 *Let $A \in \mathbb{F}^{m\times n}$ and*

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = 0, \ \ r \leq min\{m, n\}.$$

Then

1. $rank(A) = r$;
2. $KerA = span\{v_{r+1}, \ldots, v_n\}$ *and* $(KerA)^{\perp} = span\{v_1, \ldots, v_r\}$;
3. $ImA = span\{u_1, \ldots, u_r\}$ *and* $(ImA)^{\perp} = span\{u_{r+1}, \ldots, u_m\}$;
4. $A \in \mathbb{F}^{m\times n}$ *has a dyadic expansion:*

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^* = U_r \Sigma_r V_r^*$$

where $U_r = [u_1, \ldots, u_r]$, $V_r = [v_1, \ldots, v_r]$, *and* $\Sigma_r = diag(\sigma_1, \ldots, \sigma_r)$;

5. $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2$;
6. $\|A\|_2 = \sigma_1$;
7. $\sigma_i(U_0 A V_0) = \sigma_i(A)$, $i = 1, \ldots, p$ *for any appropriately dimensioned unitary matrices* U_0 *and* V_0;
8. *Let* $k < r = rank(A)$ *and* $A_k := \sum_{i=1}^{k} \sigma_i u_i v_i^*$, *then*

$$\min_{rank(B)\leq k} \|A - B\| = \|A - A_k\| = \sigma_{k+1}.$$

Proof. We shall only give a proof for part 8. It is easy to see that $\text{rank}(A_k) \leq k$ and $\|A - A_k\| = \sigma_{k+1}$. Hence, we only need to show that $\min_{\text{rank}(B)\leq k} \|A - B\| \geq \sigma_{k+1}$. Let B be any matrix such that $\text{rank}(B) \leq k$. Then

$$\begin{aligned} \|A - B\| &= \|U\Sigma V^* - B\| = \|\Sigma - U^*BV\| \\ &\geq \left\| \begin{bmatrix} I_{k+1} & 0 \end{bmatrix} (\Sigma - U^*BV) \begin{bmatrix} I_{k+1} \\ 0 \end{bmatrix} \right\| = \left\| \Sigma_{k+1} - \hat{B} \right\| \end{aligned}$$

where $\hat{B} = \begin{bmatrix} I_{k+1} & 0 \end{bmatrix} U^*BV \begin{bmatrix} I_{k+1} \\ 0 \end{bmatrix} \in \mathbb{F}^{(k+1)\times(k+1)}$ and $\text{rank}(\hat{B}) \leq k$. Let $x \in \mathbb{F}^{k+1}$ be such that $\hat{B}x = 0$ and $\|x\| = 1$. Then

$$\|A - B\| \geq \left\|\Sigma_{k+1} - \hat{B}\right\| \geq \left\|(\Sigma_{k+1} - \hat{B})x\right\| = \|\Sigma_{k+1}x\| \geq \sigma_{k+1}.$$

Since B is arbitrary, the conclusion follows. □

Lemma 3:

(1) $\underline{\sigma}(A) \leq |\lambda| \leq \overline{\sigma}(A)$

(2) $|\lambda| \leq \|A\|_p$

(3) Let $\kappa(A) = \dfrac{\overline{\sigma}(A)}{\underline{\sigma}(A)}$ = condition number of A, and $Ax = b$. If

$A(x+\delta x)=(b+\delta b)$, then: $\dfrac{\|\delta x\|}{\|x\|}=\kappa(A)\times\left(\dfrac{\|\delta b\|}{\|b\|}\right)$

(4) Let $\tilde{A}\to A+\delta A$ and $\tilde{x}=x+\delta x$, where A and x satify $Ax=b$. Then:

$$\frac{\|\delta x\|}{\|x\|}\le\frac{\kappa(A)}{\left|1-\kappa(A)\dfrac{\|\delta A\|}{\|A\|}\right|}$$

Thus, when $\kappa(A)$ is large and matrix A is almost singular, a very small change of A will make the possible range of $\|\delta x\|$ large.

Applications of SVD

Physical Example

In order to develop a clearer picture of the physical significance of the decomposition, consider a very simple multivariable process. The process, as shown in Figure .1, is a simple piping arrangement in which hot and cold water are continuously mixed. The controlled variables are the mix temperature (T_m) and the

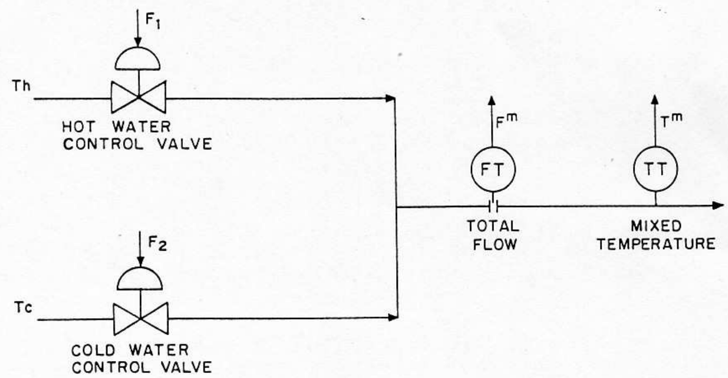


FIGURE .1. Simple Mixing Example

total flow (F_m), and the manipulated variables are the flow rates of the two inlet streams (F_1 and F_2). The linearized steady-state behavior of the system about some operating conditions can be described by the following equation:

$$\begin{matrix} \Delta T_m \\ \Delta F_m \end{matrix} = [\mathbf{K}] \begin{matrix} \Delta F_1 \\ \Delta F_2 \end{matrix} \qquad (\ 5)$$

where:

ΔT_m = steady-state change in the mixed temperature from base condition
ΔF_m = steady-state change in the total flow from the base condition
ΔF_1 = steady-state change in the hot water flow from the base condition
ΔF_2 = steady-state change in the cold water flow from the base condition

For a base case of $F_1 = 10$ gpm, $F_2 = 20$ gpm, $T_h = 100°F$, $T_c = 65°F$, let us assume that the gain matrix is as follows:

$$\mathbf{K} = \begin{bmatrix} 0.7778 & -0.3889 \\ 1.0000 & 1.0000 \end{bmatrix}$$

which decomposes to (see e.g., Reference 8 for mathematical details of how to carry out this decomposition):

$$\mathbf{U} = \begin{bmatrix} 0.2758 & -0.9612 \\ 0.9612 & 0.2758 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 0.8091 & -0.5877 \\ 0.5877 & 0.8091 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 1.4531 & 0 \\ 0 & 0.8029 \end{bmatrix}$$

Condition Number, CN = 1.7.

At this point these singular values and vectors are merely numbers; however, consider the relationship between these values and an experimental procedure that could be applied to measure the steady-state process characteristics. Suppose the mix temperature and total flow rate are measured for combinations of inlet flows defined by $\Delta F_1**2 + \Delta F_2**2 = 1$. The response of the system to such an experiment would be a direct indication of the sensitivity of the process to all possible combinations of inputs and would be useful in designing a control system. Under the present operating conditions, this simple process would respond as indicated in Figure 2(a).

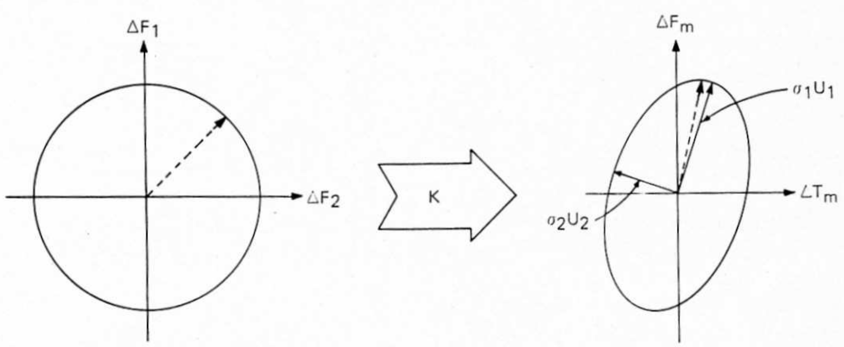


FIGURE 2(a). Mixer Output Ellipse—Case I

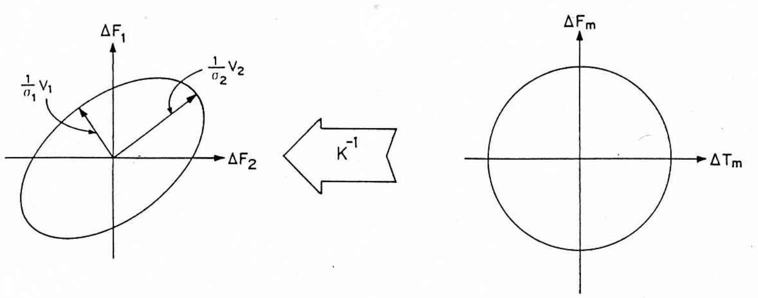


FIGURE 2(b). Mixer Input Ellipse—Case 1

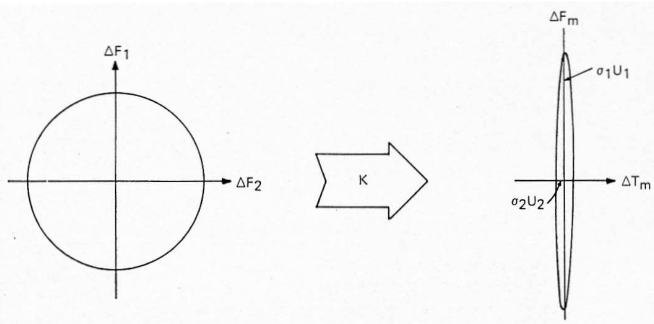


FIGURE (c). Mixer Output Ellipse—Case 2

Note that the response is presented in terms of the locus of two vectors. One vector represents the manipulated variables and is expressed in terms of the deviation of each from the base case position. The second vector represents the controlled variables and is also expressed in terms of the deviation of each from the base case. (The dotted line in Figure 2(a) is an example input and resulting output vector). The set of manipulated variables considered for this experiment forms a unit circle that maps over into the controlled variables as an ellipse. The ellipse has a major axis and a minor axis that define the relative strengths and weaknesses of the process response. Note that the mix temperature is slightly more responsive than is the total flow, but both are quite responsive to the two inlet flows.

It can be mathematically shown that the same information can be indirectly obtained from the SVD analysis (15). Note in this example that the column vectors of **U** describe the orientation of the major and minor axes of the ellipse and that the singular values describe the magnitude of each axis. The first column vector of **U** (U_1) describes the orientation of the major axis and the first singular value (σ_1) describes the magnitude of that axis. The second column vector of **U** (U_2) and the second singular value (σ_2) describe the direction and magnitude of the minor axis, respectively. In other words, the column vectors of **U** describe the rotations necessary for a sensor coordinate system that is aligned with the relative strengths and weaknesses of the system.

Also, the right singular vectors (columns of **V**) can be used to show the relative strengths and weaknesses of the manipulated variables. The first column of **V** (V_1) indicates the combinations of manipulated variables that have the greatest effect on the system. The second column of **V** (V_2) indicates the combination of manipulated variables that have the least effect on the system. This can also be visualized as an ellipse in the manipulated variable coordinate system with a major and a minor axis defined by **V** and the reciprocal of the singular values, as shown in Figure 2(b).

The physical significance of the condition number can also be seen in this simple example if we compare the operation of the system at the conditions above with the following conditions:

$$F_1 = 100 \text{ gpm}$$
$$F_2 = 150 \text{ gpm}$$
$$T_h = 100°\text{F}$$
$$T_c = 65°\text{F}$$

where the gain matrix and SVD analysis are the following:

$$\mathbf{K} = \begin{bmatrix} 0.084 & -0.056 \\ 1.000 & 1.000 \end{bmatrix}$$

which decomposes to the following:

$$\mathbf{U} = \begin{bmatrix} 0.014 & -0.999 \\ 0.999 & 0.014 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 0.708 & -0.706 \\ 0.706 & 0.708 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 1.414 & 0 \\ 0 & 0.0990 \end{bmatrix}$$

$$\text{CN} = 14.28$$

In the first case the condition number, which is the ratio of the largest to the smallest singular value, is 1.70. This indicates that the system is almost twice as responsive in the strong coordinate direction as it is in the weak coordinate direction. In the second case, the condition number is 14.28, indicating that the system is an order of magnitude more responsive in the strong coordinate direction than it is in the weak coordinate direction.

The differences in the condition number of the two cases can be easily seen by looking at the operating ellipses of the two cases (Figures 2a and 2c). In the first case, the operating ellipse is broad, indicating that the system has two clear degrees of freedom. In the second case, the operating ellipse is narrow, indicating that, while two degrees of freedom do exist, the area of operation will necessarily be along the major axis. Depending on the degree of difficulty of the dynamic problem, it may or may not be practical to try to control the system in the direction of the minor axis.

Problem of small singular values:

Very small singular values in a multivariable system are analogus to very small gains in a conventional siso system. It requires very large controller gains and results in excessively large controller actions. The typical presence of constraints in the manipulated variable and noise in the sensor makes it difficult even for siso system. In the context of mimo system, the additional complications presented by hidden loops, interactions make the problem even more severe.

A general rule of thumb to measure the small singular value is the magnitude of the noise in the signal. Singular values are equal or less than the magnitude of sensor noise should be assumed degenerate.

Problem of large singular values:

Large singular values are not as serious a problem as small values. It requires very small controller gains. This results in small controller outputs, which can be easily become lost in the resolution of the manipulator. The symptomatic behaviors are cyclic responses, which never settling down to a reasonable steady state.

A general rule of thumb concerning large singular values is that all singular values that are equal to or greater than the reciprocal of the valve resolution should be avoided.

Determining Good Sensor Locations

Consider, for example, the ethonal-water distillation column as shown in Figure 3.Assume that the first level objective is to control two column temperatures by manipulating D and Q. The basic concern is to determine which combination out of 1225 possible combinations of sensor locations from the point of view of column control.

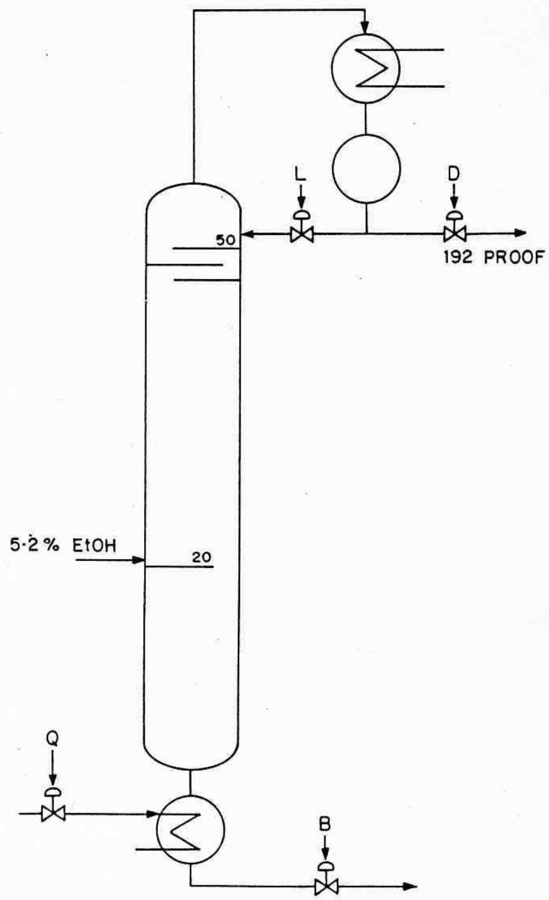


FIGURE 3 Ethanol-Water Distillation Example

The 50×2 gain matrix is as shown in figure 4.

	$\frac{\partial T_i}{\partial D}$	$\frac{\partial T_i}{\partial Q}$	i	
	9.90127E-04	−9.61801E-05	50	TOP TRAY
	1.22779E-03	−1.19262E-04	49	
	1.50924E-03	−1.45496E-04	48	
	1.84241E-03	−1.75168E-04	47	
	2.23707E-03	−2.08612E-04	46	
	2.70533E-03	−2.46212E-04	45	
	3.26229E-03	−2.88420E-04	44	
	3.92703E-03	−3.35763E-04	43	
	4.72377E-03	−3.88869E-04	42	
	5.68366E-03	−4.48478E-04	41	
	6.84724E-03	−5.15480E-04	40	
	8.26786E-03	−5.90948E-04	39	
	1.00168E-02	−6.76182E-04	38	
	1.21907E-02	−7.72778E-04	37	
	1.49226E-02	−8.82708E-04	36	
	1.83990E-02	−1.00843E-03	35	
	2.28854E-02	−1.15304E-03	34	
	2.87670E-02	−1.32047E-03	33	
	3.66118E-02	−1.51577E-03	32	
	4.72722E-02	−1.74546E-03	31	
	6.20473E-02	−2.01806E-03	30	
	8.29484E-02	−2.34481E-03	29	
	0.11313	−2.74058E-03	28	
	0.15762	−3.22536E-03	27	
	0.22452	−3.82624E-03	26	
K =	0.32704	−4.58073E-03	25	
	0.48706	−5.54272E-03	24	
	0.74146	−6.79568E-03	23	
	1.1541	−8.49150E-03	22	
	1.8420	−1.10013E-02	21	
	3.0319	−1.55514E-02	20	
	4.8898	−2.43362E-02	19	
	5.2408	−2.09634E-02	18	
	2.4720	1.07653E-02	17	
	0.40753	2.65409E-02	16	
	0.39395	4.17207E-02	15	
	0.35813	5.06657E-02	14	
	0.30683	5.30754E-02	13	
	0.24922	5.01423E-02	12	
	0.19346	4.38410E-02	11	
	0.14479	3.61175E-02	10	
	0.10533	2.84070E-02	9	
	7.49757E-02	2.15354E-02	8	
	5.24851E-02	1.58419E-02	7	
	3.62503E-02	1.13596E-02	6	
	2.47382E-02	7.96420E-03	5	
	1.66683E-02	5.47209E-03	4	
	1.10472E-02	3.69304E-03	3	
	7.13714E-03	2.45428E-03	2	
	2.87042E-03	1.11071E-03	1	REBOILER

FIGURE 4 GAIN MATRIX-ETHANOL-WATER COLUMN

U =

−1.15845E-04	−1.43652E-04
−7.27120E-04	−9.01615E-04
−1.77177E-04	−1.09961E-03
−2.16279E-04	−1.32333E-03
−2.62589E-04	−1.57508E-03
−3.17525E-04	−1.85761E-03
−3.82859E-04	−2.17410E-03
−4.60820E-04	−2.52826E-03
−5.54245E-04	−2.92444E-03
−6.66783E-04	−3.36777E-03
−8.03176E-04	−3.86431E-03
−9.69670E-04	−4.4?126E-03
−1.17461E-03	−5.04726E-03
−1.42930E-03	−5.75268E-03
−1.74932E-03	−6.55005E-03
−2.15648E-03	−7.45457E-03
−2.68186E-03	−8.48474E-03
−3.37052E-03	−9.66303E-03
−4.28891E-03	−1.10167E-02
−5.53677E-03	−1.25787E-02
−7.26608E-03	−1.43879E-02
−9.71211E-03	−1.64894E-02
−1.32442E-02	−1.89325E-02
−1.84501E-02	−2.17670E-02
−2.62769E-02	−2.50335E-02
−3.82708E-02	−2.87459E-02
−5.69912E-02	−3.28632E-02
−8.67503E-02	−3.72601E-02
−0.13502	−4.17745E-02
−0.21549	−4.68344E-02
−0.35468	−5.72102E-02
−0.57201	−8.64782E-02
−0.61305	−5.28765E-02
−0.28907	0.13565
−4.75439E-02	0.21494
−4.58861E-02	0.33268
−4.16561E-02	0.40148
−3.56458E-02	0.41914
−2.89204E-02	0.39513
−2.24278E-02	0.34496
−1.67708E-02	0.28388
−1.21903E-02	0.22310
−8.67139E-03	0.16903
−6.06674E-03	0.12428
−4.18827E-03	8.90898E-02
−2.85722E-03	6.24467E-02
−1.92469E-03	4.28993E-02
−1.27531E-03	2.89479E-02
−8.23620E-04	1.92336E-02
−3.30684E-04	8.69678E-03

V =

−1.0000	2.69431E-03
2.69541E-03	1.0000

$\sigma_1 = 8.5492$

$\sigma_2 = 0.12861$

CONDITION NUMBER = 66.47593

FIGURE 5 SVD ANALYSIS—ETHANOL-WATER COLUMN

From the SVD result in Figure 5 and is plotted on Figure 6. The largest elements in U_1 and U_2 occurs on stage 18 and 13. On the other hand, if we use abs(U_1)-abs(U_2) as criterion, as shown on Figure 7, the largest differences suggests that stage 18 and stage 13 are good choices.

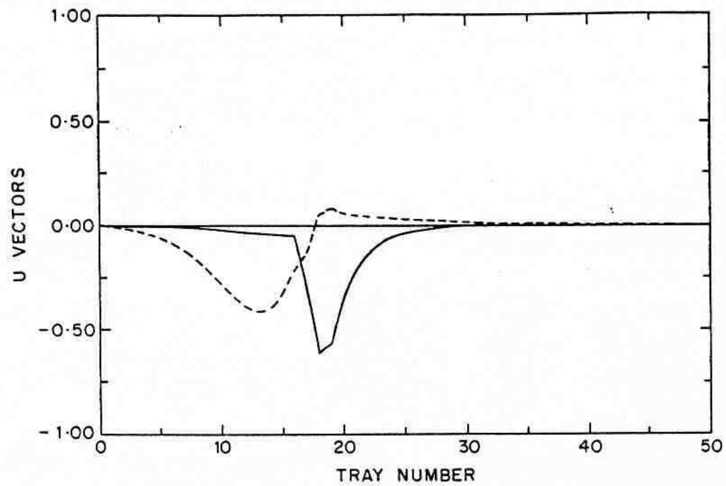


FIGURE 6 **U**-vector Plots—Principle Component Analysis

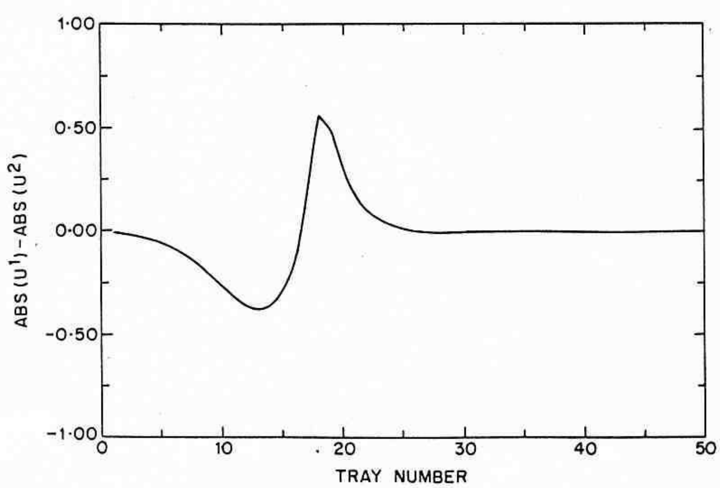


FIGURE 7 Modified Principles Component Analysis

Selection of Proper Manipulated variables

For example, in the design of control of a control for a distillation column, four manipulated variables are typically be considered:

D	distillate flow rate
L	reflux flow rate
B	bottoms flow rate
Q	steam rate to the reboiler

Two out of the four have to control levels (i.e. accumulator and column base). Thus, only the remaining two can be manipulated for the compositions. To choose two out of the four, SVD provides a straightforward method to compare the steady state behavior of various first level control strategies.

TABLE 1
Condition Numbers of Distillation Control Schemes

First-Level Scheme	*Overall SVD Analysis*		
	CN	*1*	*2*
LQ	1321.7	0.910	0.00068
LB	97.2	0.081	0.00084
DQ	66.4	0.081	0.00122
QR	179.5	0.124	0.00069
BR	96.8	0.081	0.00084

In this example, DQ is the best first-level control scheme. It has a much better condition number than the others. The second singular value of this DQ scheme is much stronger than for the other schemes.

Application to Feedback Properties

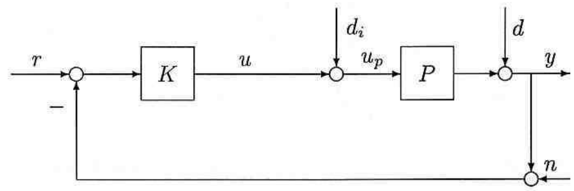


Figure   Standard Feedback Configuration

Consider again the feedback system shown in Figure 5.1. For convenience, the system diagram is shown again in Figure 5.3. For further discussion, it is convenient to define the *input loop transfer matrix*, L_i, and *output loop transfer matrix*, L_o, as

$$L_i = KP, \quad L_o = PK,$$

respectively, where L_i is obtained from breaking the loop at the input (u) of the plant while L_o is obtained from breaking the loop at the output (y) of the plant. The *input sensitivity* matrix is defined as the transfer matrix from d_i to u_p:

$$S_i = (I + L_i)^{-1}, \qquad u_p = S_i d_i.$$

And the *output sensitivity* matrix is defined as the transfer matrix from d to y:

$$S_o = (I + L_o)^{-1}, \qquad y = S_o d.$$

The *input* and *output complementary sensitivity* matrices are defined as

$$T_i = I - S_i = L_i(I + L_i)^{-1}$$

$$T_o = I - S_o = L_o(I + L_o)^{-1},$$

respectively. (The word *complementary* is used to signify the fact that T is the complement of S, $T = I - S$.) The matrix $I + L_i$ is called *input return difference matrix* and $I + L_o$ is called *output return difference matrix*.

It is easy to see that the closed-loop system, if it is internally stable, satisfies the following equations:

$$\begin{aligned}
y &= T_o(r - n) + S_o P d_i + S_o d \\
r - y &= S_o(r - d) + T_o n - S_o P d_i \\
u &= K S_o(r - n) - K S_o d - T_i d_i \\
u_p &= K S_o(r - n) - K S_o d + S_i d_i.
\end{aligned}$$

Hence, good disturbance rejection at the plant output (y) would require that

$$\begin{aligned}
\overline{\sigma}(S_o) &= \overline{\sigma}\left((I+PK)^{-1}\right) = \frac{1}{\underline{\sigma}(I+PK)}, \quad \text{(for disturbance at plant output, } d) \\
\overline{\sigma}(S_oP) &= \overline{\sigma}\left((I+PK)^{-1}P\right) = \overline{\sigma}(PS_i), \quad \text{(for disturbance at plant input, } d_i)
\end{aligned}$$

be made small and good disturbance rejection at the plant input (u_p) would require that

$$\begin{aligned}
\overline{\sigma}(S_i) &= \overline{\sigma}\left((I+KP)^{-1}\right) = \frac{1}{\underline{\sigma}(I+KP)}, \quad \text{(for disturbance at plant input, } d_i) \\
\overline{\sigma}(S_iK) &= \overline{\sigma}\left(K(I+PK)^{-1}\right) = \overline{\sigma}(KS_o), \quad \text{(for disturbance at plant output, } d)
\end{aligned}$$

be made small, particularly in the low frequency range where d and d_i are usually significant.

Note that

$$\begin{aligned}
\underline{\sigma}(PK) - 1 &\leq \underline{\sigma}(I+PK) \leq \underline{\sigma}(PK) + 1 \\
\underline{\sigma}(KP) - 1 &\leq \underline{\sigma}(I+KP) \leq \underline{\sigma}(KP) + 1
\end{aligned}$$

then

$$\begin{aligned}
\frac{1}{\underline{\sigma}(PK)+1} \leq \overline{\sigma}(S_o) \leq \frac{1}{\underline{\sigma}(PK)-1}, &\quad \text{if } \underline{\sigma}(PK) > 1 \\
\frac{1}{\underline{\sigma}(KP)+1} \leq \overline{\sigma}(S_i) \leq \frac{1}{\underline{\sigma}(KP)-1}, &\quad \text{if } \underline{\sigma}(KP) > 1.
\end{aligned}$$

These equations imply that

$$\begin{aligned}
\overline{\sigma}(S_o) \ll 1 &\iff \underline{\sigma}(PK) \gg 1 \\
\overline{\sigma}(S_i) \ll 1 &\iff \underline{\sigma}(KP) \gg 1.
\end{aligned}$$

Now suppose P and K are invertible, then

$$\begin{aligned}
\underline{\sigma}(PK) \gg 1 \text{ or } \underline{\sigma}(KP) \gg 1 &\iff \overline{\sigma}(S_oP) = \overline{\sigma}\left((I+PK)^{-1}P\right) \approx \overline{\sigma}(K^{-1}) = \frac{1}{\underline{\sigma}(K)} \\
\underline{\sigma}(PK) \gg 1 \text{ or } \underline{\sigma}(KP) \gg 1 &\iff \overline{\sigma}(KS_o) = \overline{\sigma}\left(K(I+PK)^{-1}\right) \approx \overline{\sigma}(P^{-1}) = \frac{1}{\underline{\sigma}(P)}.
\end{aligned}$$

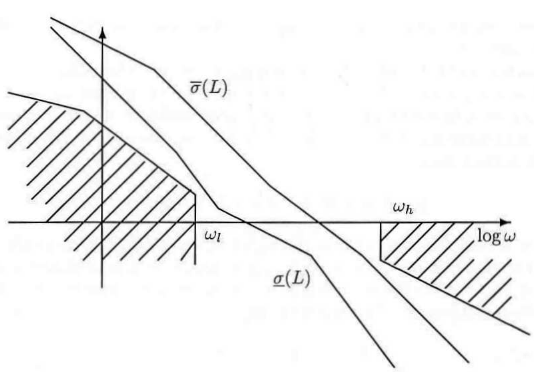


Figure 5.4: Desired Loop Gain

Use of the minimum singular value of the plant: The minimum singular value of the plant evaluated as a function of frequency is a useful measure for evaluating the feasibility of achieving acceptable control. In general, we want $\underline{\sigma}$ as large as possible.

Singular values for performance: In general, it is reasonable to require that the gain $\|e(\omega)\|_2 / \|r(\omega)\|_2$ remains small for any direction of $r(\omega)$, including the worst-case direction which gives a gin of $\bar{\sigma}(S(j\omega))$. Let $1/|w_p(j\omega)|$ represent the maximum allowable magnitude of $\|e(\omega)\|_2 / \|r(\omega)\|_2$ at each frequency, This results in the following performance requirement:

$$\bar{\sigma}(S(j\omega)) < \frac{1}{|w_p(j\omega)|},\ \forall\omega \iff \sigma\left\{w_p S(j\omega)\right\} < 1,\ \forall\omega \iff \|w_p S(j\omega\|_\infty < 1$$

Typical weight function is given asL

$$w_p(s) = \frac{s/M + \omega_b}{s + \omega_b A}$$

which means $1/|w_p|$ equals $A \le 1$ at low frequencies, and equals to $M \ge 1$ at high frequencies, and the asymptote crosses 1 at ω_b (the bandwidth frequency).