Introduction to MVC design

Performance Issues

[Definition] Well-posed system

A feedback system is said to be well-posed if all closed loop transfer functions are well defined and proper.

[Lemma:] The control system as shown in the figure is well-posed iff: $I + K(\infty)G(\infty)$ is invertible. (notice that $\widehat{K} = -K$)

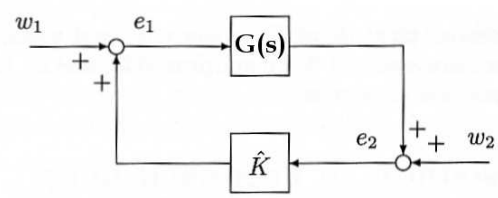


Figure : Internal Stability Analysis Diagram

Notice: $I - G(\infty)K(\infty)$ is invertible $\Leftrightarrow$ $\begin{bmatrix} I & K(\infty) \\ -G(\infty) & I \end{bmatrix}$ is invertible

IF $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$; -K=$\begin{bmatrix} \widehat{\mathrm{A}} & \widehat{\mathrm{B}} \\ \widehat{\mathrm{C}} & \widehat{\mathrm{D}} \end{bmatrix}$, then $G(\infty) = D$ and $K(\infty) = -\widehat{D}$

The well-posed condition becomes:

$\begin{bmatrix} I & -\widehat{D} \\ -D & I \end{bmatrix}$ is invertible.

[Definition] Internal stability

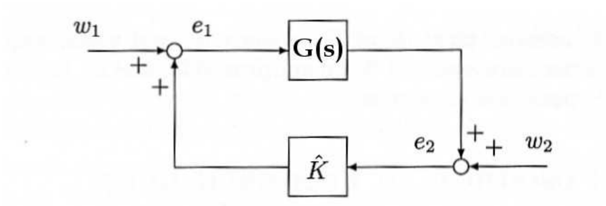


Figure : Internal Stability Analysis Diagram

$$Plant:\ G=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad Controller:\ \hat{K}=\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$

From the block diagram, it is easy to obtain:

$$\begin{bmatrix} I & -\hat{K} \\ -G & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}=\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \text{(A)}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}+\begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} C & 0 \\ 0 & \hat{C} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}+\begin{bmatrix} D & 0 \\ 0 & \hat{D} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}=\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}+\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \text{(B)}$$

The last two equations can be re-written as:

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}=\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}+\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}+\begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Thus,

$$\tilde{A}=\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix}+\begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}$$

The closed-loop system with given stablizable and detectable realization of G and K is called internal stable iff $\widehat{A}$ is a Hurwitz matrix.

Compare Eq(A) with Eq(B), we conclude that the well-posedness condition implies that $(I-D\hat{D})=(I-G\hat{K})(\infty)$ is invertible.

[Lemma] The system as shown is internally stable iff:

$$\begin{bmatrix} I & K \\ -G & I \end{bmatrix}^{-1}=\begin{bmatrix} \left(I+KG\right)^{-1} & -K\left(I+GK\right)^{-1} \\ G(I+KG)^{-1} & (I+GK)^{-1} \end{bmatrix}$$

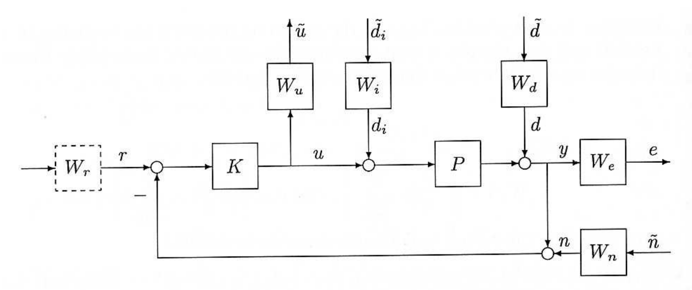
$$=\begin{bmatrix} I-K\left(I+GK\right)^{-1}G & -K\left(I+GK\right)^{-1} \\ \left(I+GK\right)^{-1}G & \left(I+GK\right)^{-1} \end{bmatrix}\in RH_\infty$$

[Note: H_∞ space is a sub-spaceof L_∞ with functions that are analytic and bounded in the open-right-half plane. The H_∞ – norm is defined as:

$$\|F\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}[F(s)] = \sup_{\omega\in R} \bar{\sigma}[F(j\omega)]$$

The real rational subspace of H_∞ is denoted as RH_∞ which consists of all proper and real rational stable transfer matrices.]

H_2 and H_∞ Performance



1. H_2-optimal control

Let the disturbance $\tilde{d}(t)$ can be approximated as an impulse with random

input direction: $\tilde{d}(t) = \eta\,\delta(t)$ and $E\{\eta\eta^*\} = I$

where E denotes the expectation. One may choose to minimize the expected energy of error e due to the disturbance $\tilde{d}$:

$$E\{\|e\|_2^2\} = E\left\{\int_0^\infty |e|^2\, dt\right\} = \|W_e S_o W_d\|_2^2$$

where, $S_o = (I + GK)^{-1}$

Alternatively, if $\tilde{d}(t)$ can be modeled as white noise, so that $S_{\tilde{d}\tilde{d}} = I$, then,

$e = W_e S_o W_d \tilde{d}$ and $S_{ee} = E\{ee^*\} = (W_e S_o W_d) E(d\tilde{d}^*)(W_e S_o W_d)^* = (W_e S_o W_d) S_{d\tilde{d}} (W_e S_o W_d)^*$

and, we may chose to minimize $\|e\|_p^2$

$$\|e\|_p^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\{S_{ee}(j\omega)\}d\omega = \| W_e S_o W_d \|_2^2$$

In general, a controller minimizing only $\|e\|_p^2$ can lead to a very large control signal u that could cause saturation of the actuators. Hence, for a realistic controller design, we may use the following criterion:

$$E\left\{\|e\|_2^2 + \rho^2 \|u\|_2^2\right\} = \left\| \begin{matrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{matrix} \right\|_2^2$$

In the following, assume all the weighting matrices are identity matrices.

(1) **SISO case:** The controller K is determined such that the integral square error is minimized *for a particular input v*. That is:

$$\min_K \|e\|_2^2 = \min_K \frac{1}{2\pi}\int_{-\infty}^{\infty} e^*(j\omega)\, e(j\omega) d\omega = \min_K \frac{1}{2\pi}\int_{-\infty}^{\infty} | s(j\omega) w(j\omega) |^2 \, d\omega$$

where, $v(j\omega) = w(j\omega) v'(j\omega)$; $|v'(j\omega)|^2 = 1$

(2) **MIMO case:** In analogy to the SISO case, the controller K is determined to minimize the H_2-norm for the error vector:

$$\min_K \|E\|_2^2 = \min_K \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\{E^*(j\omega)\, E(j\omega)\} d\omega$$

or,

$$\min_K \|W_2 E W_1\|_2^2 = \min_K \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\{(W_2 E W_1)^* (W_2 E W_1)\, E\}_{(j\omega)} d\omega$$

The H_2-optimal control can be interpreted as the minimization of the 2-norm of sensitivity operator with input weight W_1 and output weight W_2.

2. **H_∞-optimal control**

(1) SISO case:

Assume that:

$$V = \left\{ v : \|v'\|_2^2 = \left\| \frac{v}{w} \right\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \frac{v(j\omega)}{w(j\omega)} \right|^2 d\omega \le 1 \right\}$$

Each input in V gives rise to an error e. The H_∞-optimal control is designed to minimize the worst error which can result from any $v \in V$, that is:

$$\min_K \sup_{v \in V} \|e\|_2 = \min_K \sup_{v \in V} \|s w v'\|_2$$

The worst error can be bounded for a set of bounded input V' as follows:

$$\sup_{v' \in V'} \|swv'\|_2^2 = \sup_{v' \in V'} \frac{1}{2\pi} \int_{-\infty}^{\infty} |swv'|^2 \, d\omega$$

$$\leq \sup_{\omega \in R} |sw|^2 \sup_{v' \in V'} \frac{1}{2\pi} \int_{-\infty}^{\infty} |v'|^2 \, d\omega$$

$$\leq \sup_{\omega \in R} |sw|^2 = \|sw\|_\infty$$

where, $V' = \{v' : \|v'\|_2^2 \leq 1\}$

Thus,

$$\min_K \sup_{v \in V} \|swv'\|_2 \leq \min_K \sup_{\omega \in R} \|sw\|_2 = \min_K \|sw\|_\infty$$

(2) MIMO case:

Let

$$V = \left\{ v' : \|W_1^{-1} v\|_2^2 \leq 1 \right\} \quad \text{and,}$$

$$V' = \{v' : \|v'\|_2^2 \leq 1\}$$

The controller K is to be designed to minimize the worst normalized error e', that is:

$$\min_K \max_{v \in V} \|e'\|_2 = \min_K \max_{v' \in V'} \|W_2 E W v'\|_2$$

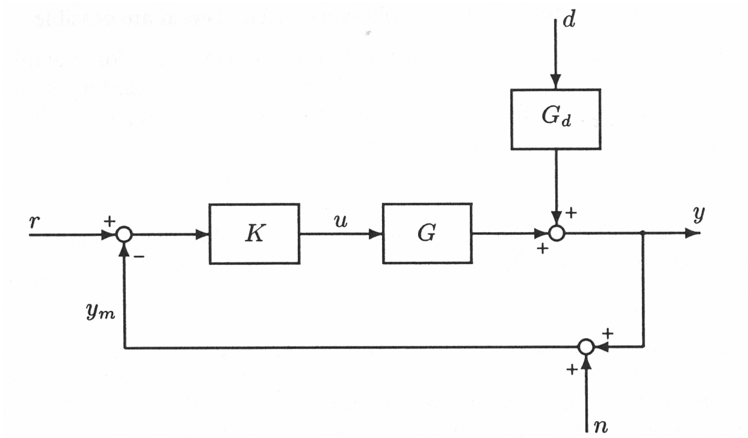
$$\leq \min_K \sup_{\omega} \bar{\sigma}\left(W_2 E W_1(j\omega)\right) = \min_K \|W_2 E W_1\|_\infty$$

Notice that if W_1 and W_2 are scalars, it means that $\|W_2 E W_1\|_\infty$ is bounded to lie below some constant value. The H_∞ performance requirement is usually written as:

$$\|W_2 E W_1\|_\infty < 1$$

Synthesis methods

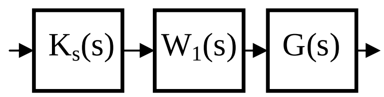
Consider the simple feedback system as shown in the following figure:



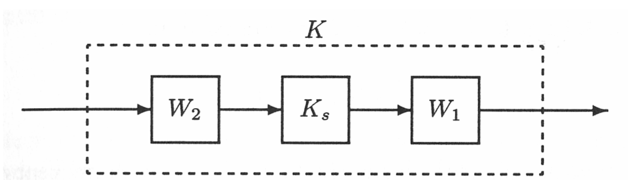
The most common approach is to use pre-compensator, $W_1(s)$, which counteracts the interactions in the plant and results in a *new shaped* plant:

$$G_s(s) = G(s)W_1(s)$$

Which is more diagonal and easier to control. After finding $W_1(s)$, a diagonal controller $K_s(s)$ is design to control this *new shaped* plant.



A more general framework for MVC design is to include another post compensator, $W_2(s)$ in the overall controller K(s) as the following:



In other words,

$$K(s) = W_2(s)K_s(s)W_1(s)$$

1. The design approach that uses Nyquist Array technique (such as: DNA, INA) of Rosenbrock (1974) and Characteristic loci of MacFarlane and Kouvaritakis (1977) are of this category. In this approach, Both of $W_1(s)$ and $W_2(s)$ are design so as to make the process diagonal dominant.
2. The decoupling control is another of this approach. In this approach, $W_2(s)$=I, and $W_1(s) = G^{-1}(s),\ \ K_s(s) = f(s)I$

   The decoupling control is appealing, but there are several difficulties:

   (1). It may be very sensitive to modeling error and model uncertainties.

(2). The requirement of decoupling and the use of inverse-based controller may not be desirable for disturbance rejection.

(3). The issue of RHP zero.

3. SVD-controller

   SVD-controller is a special case of a pre- and post-compensator design. Here, $W_1(s) = V_o$ and $W_2(s) = U_o^T$, where, U_o and V_o are obtained from a SVD of $G_o = G(j\omega_o) = U_o \Sigma V_o^T$. By selecting $K_s(s) = f(s)\Sigma_o^{-1}$, a decoupling design is achieved.

4. Mixed-sensitivity H_∞ design (S / KS)

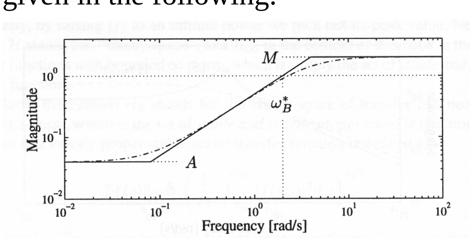
   The objective of this design is to minimize the H_∞-norm of

$$N = \begin{bmatrix} W_P S \\ W_u KS \end{bmatrix}$$

Where, S is a sensitivity function of the system, and KS is the transfer function matrix from the set-point R to u. A reasonable initial choice for W_u-weight is $W_u = I$, and a common choice of the W_p-weight is a diagonal matrix with

$$w_{p,i} = \frac{s / M_i + \omega_{B,i}^*}{s + \omega_{B,i}^* A_i}.$$

Selecting $A_i \ll 1$ ensures approximate integral action with $S(0) \approx 0$. Often, M_i is selected about 2for all outputs, and $\omega_{B,i}^*$ is approximately the bandwidth requirement and may be different for each output. The shape of $|W_p(j\omega)|$ is given in the following:



In the SISO system, usually, we require: $\| w_p(j\omega) S(j\omega) \|_\infty < 1$

Inverse and Direct Nyquist Arrays

Rosenbrock extended the Nyquist stability and design concepts to MIMO systems containing significant interaction. The methods are known as the inverse and direct Nyquist array (INA and DNA) methods. As an extension from the SISO Nyquist stability and design concepts, these methods use frequency response approach.

Frequency response techniques are theoretically less attractive than optimal controllers resulting from state-space analysis, but their simplicity, high stability, and low noise sensitivity make them quite attractive from a pratical point of view. The INA and DNA generally require the use of a digital computer with graphics capability and interactive computer-aided design facilities.

1. Notations

Consider the following system:

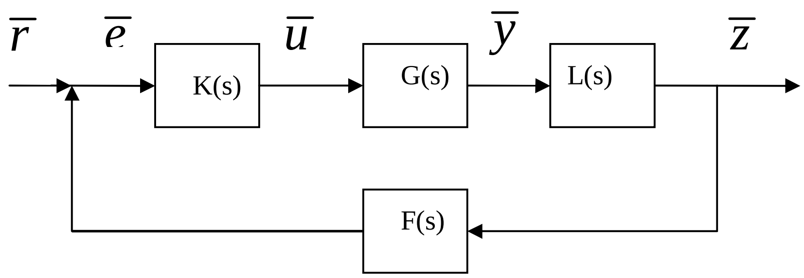


Figure 1

$$\bar{z} = LGK\bar{e} = LGK(\bar{r} - F\bar{z})$$

Where, if [I+LGK] is not identically zeo,

$$\bar{z} = [I_k + LGKF]^{-1} LGK\bar{r}$$

Alternatively, we have:

$$\bar{e} = \bar{r} - FLGK\bar{e} \quad \Rightarrow \quad \bar{e} = [I + FLGK]^{-1}\bar{r}$$

or,

$$\bar{z} = LGK[I_k + LGKF]^{-1}\bar{r}$$

Define: Q=LGK

$$\bar{z} = H\bar{r} \;\; ; \;\; H = [I_k + QF]^{-1} Q = Q[I_k + FQ]^{-1} = H(Q, F)$$

Note:

(i) $\begin{vmatrix} I_k & F \\ -Q & I_k \end{vmatrix} = |I_k + QF| = |I_k + FQ|$

Thus, $|I+FQ|^{-1}$ exists if $|I+QF|^{-1}$ exists.

(ii) H(Q,0)=Q

(iii) Y=H[Q,F]X

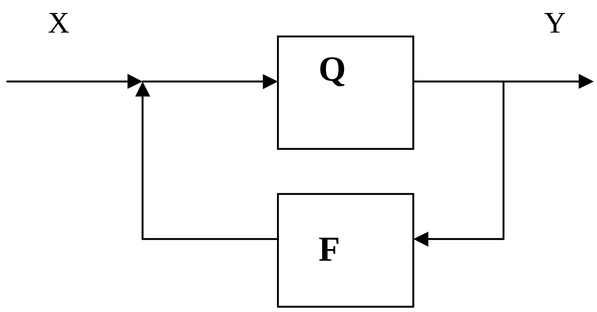


Figure 2

Let: $K(s)=K_1(s)\ K_2(s)$, and $L(s)=L_2(s)\ L_1(s)$, where, K_2 and L_2 are diagonal matrices. Thus, comparing the following two equivalent block diagrams,

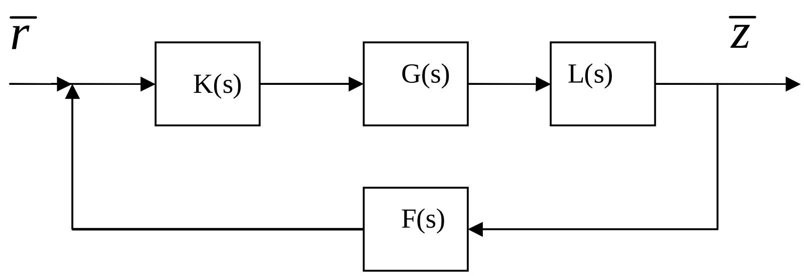


Figure 3

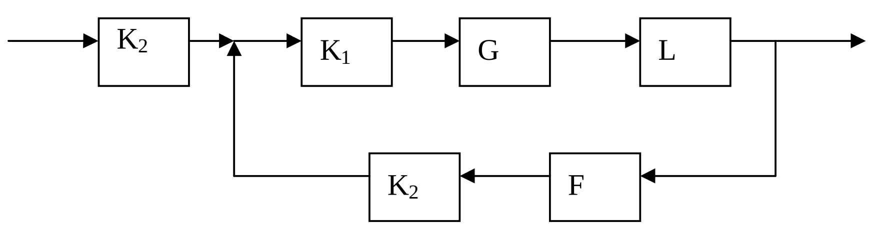


Figure 4

we have:

$$H[Q,F] = H[Q,K_2F]K_2$$

Similarly, by referring to the following diagram,

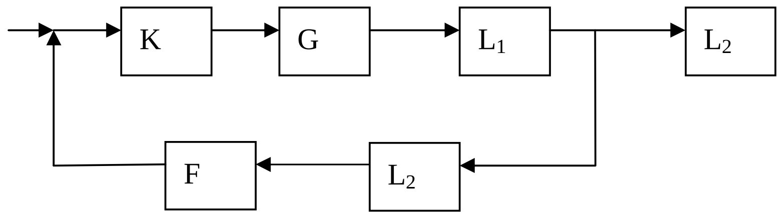


Figure 5

we have: $H[Q,F] = L_2 H[Q, L_2 F]$

In Figure 4, the product K_2F can be renamed as F since there are diagonal. Denote the diagonal entries as $f_{i.}$.

The two manipulations in Figure 4 and Figure 5 are important when there is concern about actuator or transducer failure.

(1) When transducer error is impoerant, use Figure 4. Where L is set equal to the identity matrix.. Then K_2 is assimilated with F. Setting any elements of K_2F equal to zero shows effects of the corresponding transducer failures.
(2) When actuator failure is important, use figure 5. Where, K is set equal to the identity matrix. The L2 is assimilated with F. Setting any elements of L2K equal to zero shows the effects of the corresponding actuator failure.

2. General Feedback System Stability

$$H = [I_k + QF]^{-1} Q = Q[I_k + FQ]^{-1}$$

$$|H| = \left|(I_k + QF)^{-1} Q\right| = \left|(I_k + QF)^{-1}\right| |Q| = \frac{|Q|}{|I_k + QF|}$$

$|I_k + QF|$ = close loop characteristic polynomial = clc$p$

In other words, according to stability criterion, we have:

$$N_{\{H\}} = N_{\{Q\}} - (Z - P_o)$$

In other words,

$$Z - P_o = N_{\{Q\}} - N_{\{H\}}$$

Thus, if the system is stble, we shall have:

$N_{\{Q\}} - N_{\{H\}} = -P_o$ or $N_{\{H\}}$ - $N_{\{Q\}} = P_o$ **(Criterion A)**

Similarly, since

$$H = [I_k + QF]^{-1} Q = Q[I_k + FQ]^{-1}$$

It is equivalent to have:

$$H^{-1} = Q^{-1}[I_k + QF] = Q^{-1} + F \quad \Rightarrow \quad \widehat{H} = |\widehat{Q}| + F$$

and $\dfrac{|\widehat{H}|}{|\widehat{Q}|} = |I_k + QF|$

Thus,

$$N_{\{\widehat{H}\}} - N_{\{\widehat{Q}\}} = Z - P_o$$

So, if the system is stable, we have:

$N_{\{\widehat{Q}\}} - N_{\{\widehat{H}\}} = P_o$ **(Criterion B)**

3. Nyquist array and MIMO stability theorems

It is difficult to apply criterion A and criterion B to determine the stability of multivariable systems, since the origin encirclements by the mappings due to the determinants of matrices Q and H (or Q^{-1} and H^{-1}) are required. Rosenbrock's Nyquist array techniques utilize modifications of the criteria, which are valid if the matrices are diagonally dominant.

3.1 Diagonal Dominance

A rational m by m matrix Z(s) is diagonally dominant on the Nyquist contour, D, if (for all s on D and for all I, i=1,2,…,m) it is diagonally row dominant or diagonally column dominant.

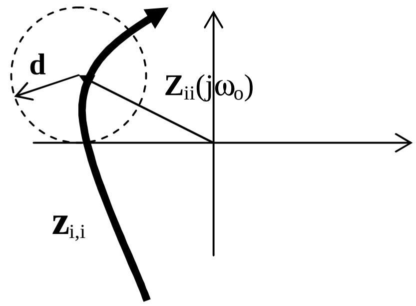
Z(s) is diagonally row dominant if:

$$|z_i(s)| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |z_{ij}(s)| = d_i(s)$$

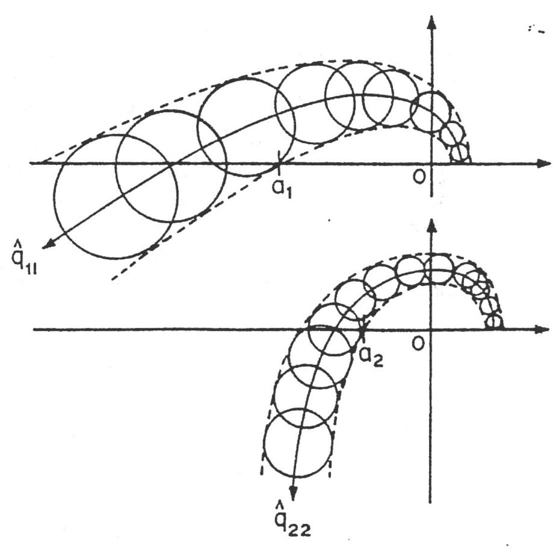
Z(s) is diagonally column dominant if:

$$|z_j(s)| > \sum_{\substack{i=1 \\ i \neq j}}^{m} |z_{ij}(s)| = d_j(s)$$

As shown in the following figure, the diagonal dominance means the origin of the Z-plane will be located outside the disk which is centered on zi,i(jw) with radius equaling to d_i or d_i'.



Let $Z(s)=\{z_{i,j}(s),\ I,j=1,2,\ldots,m\}$. As s travels along the Nyquist contour, The corresponding circles (centered at $z_{i,j}(j\omega)$ with radius $d_i(\omega)$ or $d_i'(\omega)$) sweep out what is called a "Gershgorin band".



If each of the band associated with all I of the diagonal elements excludes the origin, then Z(s) is diagonal dominant.

[Theorem] Let Z(s) be diagonal dominance on C, which is any closed elementary contour having on it no pole of $z_{ii}(s)$, i=1,2,…,m. Let $z_{ii}(s)$ maps C into Γ_i, i=1,2,…,m, and |Z(s)| maps C into Γ_z,. Let origin encirclement by Γ_i be Ni times, and let origin encirclement by Γ_z be N_z.

Then,
$$N_z = \sum_{i=1}^{m} N_i$$

[Theorem---INA] Let the Gershgorin bands based on the diagonal elements of Q^{-1} exclude the origin and the point ($-f_i$,0). Let these bands encircle the origin $N_{\hat{q}_i}$ times clockwise and encircle the points ($-f_i$,0) $N_{\hat{h}_i}$ times clockwise. Then the system is stable iff:

$$\sum_{i=1}^{m} N_{\{\hat{q}_i\}} - \sum_{i=1}^{m} N_{\{\hat{h}_i\}} = P_o$$

[Proof]

Since,

$$H = [I + FQ]^{-1}Q \quad \Rightarrow \quad H^{-1} = Q^{-1}[I + QF]$$
$$\Rightarrow \left|I + QF\right| = \frac{\left|\hat{H}\right|}{\left|\hat{Q}\right|}$$

we have:

$$N[\hat{H};0] - N[\hat{Q};0] = N[\hat{Q} + F;0] - N[\hat{Q};0] = -P_o$$

The fact that Gershgorin bands based on the diagonal elements of $\hat{Q}$ exclude the origin implies that $\hat{Q}$ is diagonal dominant. Similarly, The fact that Gershgorin bands based on the diagonal elements of $\hat{Q}$ exclude the ($-f_i$,0) implies that $\hat{H}$ is diagonal dominant. Thus,

$$N[\hat{H};0] = \sum_{i=1}^{m} N[\hat{h}_{i,i};0] = \sum_{i=1}^{m} N[\hat{q}_{i,i} + f_i;0] = \sum_{i=1}^{m} N\ [\hat{q}_{i,i};(-f_i,0)]$$
$$= \sum_{i=1}^{m} N_{\hat{h}_{i,i}}$$

Similarly,

$$N[\hat{Q};0] = \sum_{i=1}^{m} N_{\hat{q}_{i,i}}$$

Thus, the Nyquist stability criterion becomes:

$$\sum_{i=1}^{m} N_{\{\hat{q}_i\}} - \sum_{i=1}^{m} N_{\{\hat{h}_i\}} = P_o$$

[Theorem---DNA] Let F=diag{f_i}, where f_i are real and non-zero, and F^{-1}+Q be diagonal dominant on D. Let $q_{i,i}$ map D into Γ_i which encircles ($-f_i^{-1}$,0) N_i times, i=1,2,…m. Then the closed-loop system is asymptotically stable iff: $\Sigma N_i = -P_o$.

[Proof]

$$N[\ |I+QF|;\ 0\ ] = Z - P_o \quad \Rightarrow \quad N[\ |F^{-1}+Q|;\ 0] = Z - P_o$$
$$\Rightarrow \ N[Q;\ F^{-1}] = Z - P_o$$

Thus, if system is stable, Z=0, so,

$$N[Q+F^{-1};0] = -P_o$$

Since $F^{-1}+Q$ is diagonal dominant,

$$N[\ |F^{-1}+Q|;\ 0] = \sum_{i=1}^{m} N_i[f_i^{-1}+q_{i,i};0] = -P_o$$
$$\Rightarrow \sum_{i=1}^{m} N[q_{i,i};\ -f^{-1}] = -P_o$$
$$\Rightarrow \sum_{i=1}^{m} N_i = -P_o$$

[Ostrowski's Theorem] Let the m×m rational matrix Z(s) be row [resp. column] dominant for $s=s_o$ on C. Then $Z(s_o)$ has an inverse $\hat{Z}$ (s_o) and for i=1,2,…,m,

$$\left|\hat{z}_{i,i}^{-1}(s_o) - z_{i,i}(s_o)\right| < \phi_i(s_o)d_i(s_o) < d_i(s_o)$$
$$\left[resp. < \phi_i{}'(s_o)d_i{}'(s_o) < d_i{}'(s_o)\right]$$

where,

$$\phi_i(s_o) = \max_{j;\,j\neq i} \frac{d_j(s_o)}{\left|z_{j,j}(s_o)\right|}; \qquad \left(resp, \quad \phi_i^{'}(s_o) = \max_{j;\,j\neq i} \frac{d_j^{'}(s_o)}{\left|z_{j,j}(s_o)\right|} \right)$$

[Theorem A] Let $\hat{Q}$ and $\hat{H}$ be dominant on C. For each s on D the diagonal element $h_{i,i}$ of H[Q,F] satisfy:

$$\left|h_{i,i}^{-1}(s) - (f_i + \hat{q}_{i,i})\right| < \phi_i(s)d_i(s) < d_i(s)$$
$$or \qquad \left|h_{i,i}^{-1}(s) - (f_i + \hat{q}_{i,i})\right| < \phi_i^{'}(s)d_i^{'}(s) < d_i^{'}(s)$$

according as $\hat{H} = F + \hat{Q}$ is row or column dominant.

[Proof] This theorem is a direct result fro the Ostrowski's theorem by substituting $\hat{z}_{i,i}^{-1}$ with $h_{i,i}^{-1}$, and $z_{i,j}$ with $f_i + \hat{q}_{i,i}$ (i.e. $\hat{h}_{i,j}$).

Notice that,

$$\hat{h}_{i,i}[Q, diag\{f_1, f_2, \ldots. f_{i-1}, 0, f_{i+1}, \ldots, f_m\}] + f_j = \hat{h}_{i,i}[Q, diag\{f_1, f_2, \ldots. f_{i-1}, f_i, f_{i+1}, \ldots, f_m\}]$$

Designate $h_{i,i}[Q, diag\{f_1, f_2, \ldots. f_{i-1}, 0, f_{i+1}, \ldots, f_m\}] = h_i$ and

$$H[Q,\ diag\{f_1, f_2, \ldots. f_{i-1}, 0, f_{i+1}, \ldots, f_m\}] = H_i$$

Because of $\hat{H}\ [\ Q,\ F\}] = F + \hat{Q}$ and because of the Ostrowski theorem,

$$\left|h_{i,i}^{-1}(s) - (f_i + \hat{q}_{i,i})\right| < \phi_i(s) d_i(s) < d_i(s)$$

$$or \qquad \left|h_{i,i}^{-1}(s) - (f_i + \hat{q}_{i,i})\right| < \phi_i^{'}(s) d_i^{'}(s) < d_i^{'}(s)$$

Similarly, because of $\hat{H}[Q,\ diag\{f_1, f_2, \ldots. f_{i-1}, 0, f_{i+1}, \ldots, f_m\}] = \hat{Q}$ and because of the Ostrowski theorem, we have:

$$\Rightarrow \left|h_i^{-1} - \hat{q}_i\right| < \phi_i(s) d_i(s) < d_i(s)$$

This means the inverse transfer function viewing from the ith input to the ith output lies within the ith Ostrowski band. Thus, if H and Hi are diagonal dominant, we can use the Ostrowski bands to analysize the stability of a closed-loop system, and design the ith loop based on h_i^{-1}(s).

The Ostrowski bands have two implications:

(1). They locate the inverse of transfer function $h_i^{-1}(s)$. If we wish to design a single-loop compensator for the ith lop, we must design it for $h_i^{-1}(s)$. As loop gains (i.e. $f_1,\ldots f_{i-1}, f_{i+1}, \ldots f_m$) vary, and dominance is maintained, $h_i^{-1}(s)$ lies within the appropriate Ostrowski band, evaluated for that gains.

(2). They are used to determine the stability margins of the loops. We may determine appropriate gain and phase margins, or appropriate values of M, if we know $h_i^{-1}(s)$, which is within the i th Ostrowski band.

Reasons for using $\hat{Q}$:

(1). The relation $\hat{H} = F + \hat{Q}$ gives an easy transition from open-loop to close-loop properties.

(2). There appears to be a tendency for $\hat{Q}$ to be more dominant than Q.

(3). For some given s=jω that the distance from (-fi,0) to $\hat{q}_{i,i}(i\omega)$ in all loops except the jth one becomes infinitely large, the width of the Ostrowski band for the jth loop shrinks to zero at s=jω.

Achieving dominance

There are various methods of achieving or increasing dominance.

1. Elementary operations.
2. Pseudo-diagonalization
3. Approximate inversion.

The INA and DNA methods reduce loop interactions by determining a precompensator, **K**(s) and possibly a postcompensator, **L(s)**, so that $\mathbf{Q^{-1}(s)=[L(s)G(s)K(s)]^{-1}}$ or $\mathbf{Q(s)=[L(s)G(s)K(s)]}$ is diagonal dominant. When dominance has been achieved, single-loop controllers may be implemented as required to meet design specification. In many instances, only precompensator is needed. The precompensator, **K(s)**, is required to have elements whose poles are in the open left half plane.

Elementary operations.

The earliest and most widely used method to achieve dominance is to use elementary row and column operations to build **K** and **L** matrices. The precompensator K(s) can be written as $\mathbf{K_a(s)K_b(s)K_c(s)}$. Similarly, **L(s)** is written as $\mathbf{L_c(s)L_b(s)L_a(s)}$. Where, matrices with subscript "c" designate non-singular parts of **K(s)** and **L(s)**; those with subscripts designate the operational matrices that add a multiple of one column to another that they postmultiply.

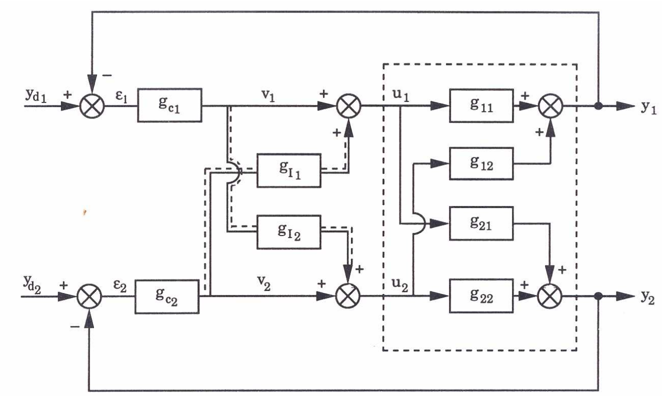
Decoupling Control

The main objective in decoupling control is to compensate for the effect of interactions brought about by cross-coupling of the process variables. In the ideal case, the decoupler causes the control loops to act as if totally independent of one another, thereby reducing the controller tuning task to that of tuning several non-interacting controllers. There are different types of decoupling control:

1. Dynamic decoupling: Design decoupler $G_I(s)$ to eliminate interactions from all loops. In other words, the open-loop transfer function matrix, $G(s)G_I(s)$, achieves being diagonal for all frequencies.
2. Steady-state decoupling: Design a decoupler so that $G(0)G_I(0)$ is diagonal.
3. Partial decoupling: Design $G_I(s)$ to eliminate interactions in a subset of the control loops.

Simplified Decoupling:

Consider the following 2×2 system.



$$y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s)$$
$$y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s)$$

$$u_1(s) = v_1(s) + g_{I1}(s)v_2(s);$$
$$u_2(s) = v_2(s) + g_{I2}(s)v_1(s);$$
$$v_1(s) = g_{c,1}(s)e_1(s), \quad v_2(s) = g_{c,2}(s)e_2(s)$$

Then,

$$y_1(s) = \left[g_{11}(s) + g_{12}(s)g_{I_2}(s)\right]v_1(s) + \left[g_{11}(s)g_{I_1}(s) + g_{12}(s)\right]v_2(s)$$
$$y_2(s) = \left[g_{21}(s) + g_{22}(s)g_{I_2}(s)\right]v_1(s) + \left[g_{22}(s) + g_{21}(s)g_{I_1}(s)\right]v_2(s)$$

To have ideal decoupling, it is required that

$$g_{11}(s)g_{I_1}(s)+g_{12}(s)=0,\ \ g_{22}(s)g_{I_2}(s)+g_{21}(s)=0$$

In other words,

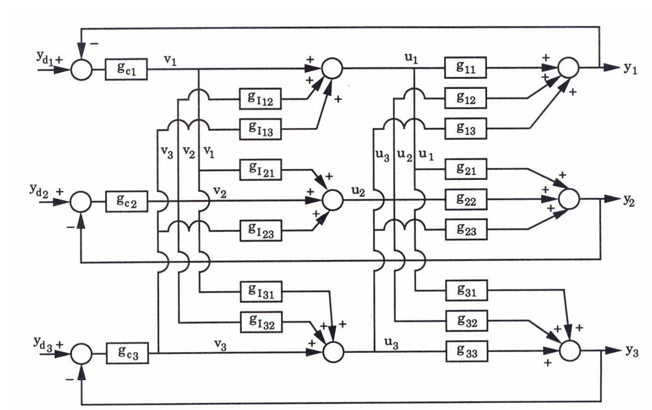
$$g_{I_1}(s)=-\frac{g_{12}(s)}{g_{11}(s)},\ g_{I_2}(s)=-\frac{g_{21}(s)}{g_{22}(s)}$$

By this,

$$y_1(s)=\left[g_{11}(s)-\frac{g_{12}(s)g_{21}(s)}{g_{22}(s)}\right]v_1(s),$$

$$y_2(s)=\left[g_{22}(s)-\frac{g_{21}(s)g_{12}(s)}{g_{11}(s)}\right]v_2(s)$$

When dealing with systems larger than 2×2, the simplified decoupling approach becomes very tedious. For example, in a 3×3 system, there are $N(N-1)$ decouplers to be designed and implemented as in the following block diagram.



Generalized decoupling

A more general procedure for decoupler design is as follows:

$$y(s)=G(s)u(s),\ u(s)=G_I(s)v(s),\ \text{so that}\ y(s)=G(s)G_I(s)v(s)$$

In order to eliminate the interactions,

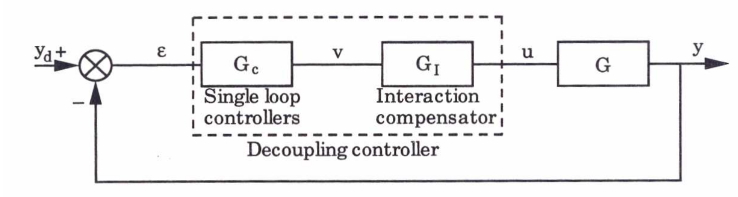
$$G(s)G_I(s)=G_R(s)=diag\{g_{r,i}(s)\}$$

so that

$$G_I(s)=G^{-1}(s)G_R(s)$$

Notice that, the $G_I(s)$ is a simplified decoupling system is:

$$G_I(s)=\begin{bmatrix} 1 & g_{I_1}(s) \\ g_{I_2}(s) & 1 \end{bmatrix}$$



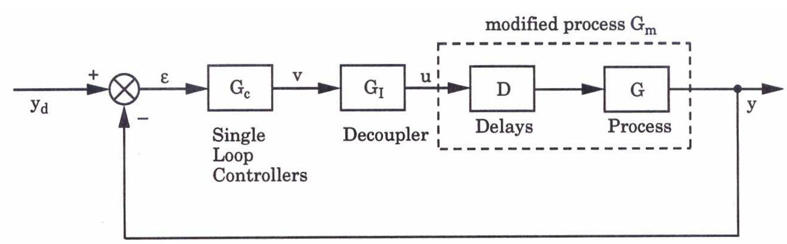
Limitations to the application of decoupling

(1) Causality: In order to ensure causality in the compensator, it is necessary that the time delay structure in G(s) be such that the smallest delay in each row occurs on the diagonal. If not, it needs to compensate the process with additional dealy time as shown in the following figure, that is

$G_m(s)=G(s)D(s)$

(2) Stability: It needs to ensure G(s) has no RHP zero. If not, $G_R(s)$ must be adjust to contai the RHP zero.

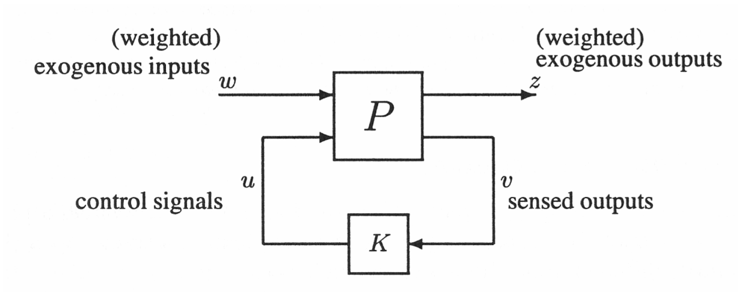
(3) Robustne: In general the diagonal controllers shold be detune to ensure the system's stability robustness by relaxing the controller to be more conservative.



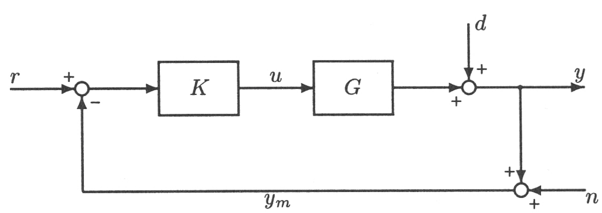
H_∞-Design

1. Formulation of generalized plant for control systems.

A grneral control formulation of Doyle (1983; 1984) makes use of the general control configuration of the following:



Where, P is the generalized plant and K is the controller. To find the generalized plant for a one-degree freedom control system, consider the control the following block diagram:

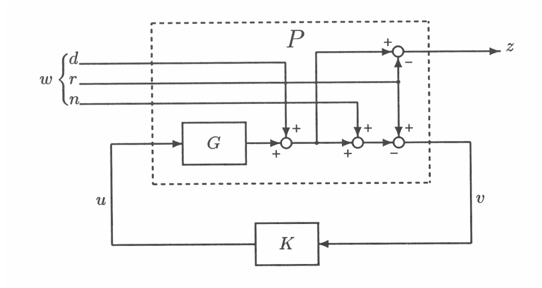


The first step is to identify the signals for the generalized plant:

$$W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix}; \quad z = e = y - r, \quad v = r - y_m = r - y - n$$

$$z = y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu = \begin{bmatrix} I & -I & 0 & G \end{bmatrix} \begin{bmatrix} W \\ u \end{bmatrix}$$

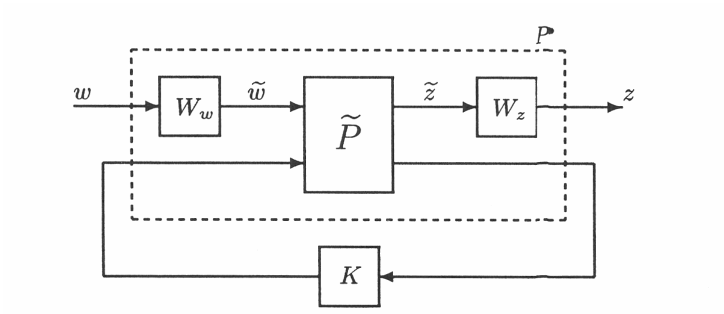
$$v = r - ym = r - Gu - d - n = -Iw_1 + Iw_2 - Iw_3 - Gu = \begin{bmatrix} -I & I & -I & G \end{bmatrix} \begin{bmatrix} W \\ u \end{bmatrix}$$



Which are equivalent to

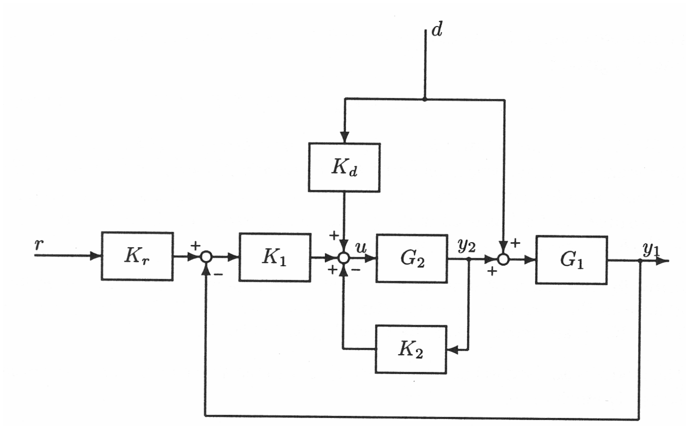
$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \begin{bmatrix} W \\ u \end{bmatrix}$$

To get a meaningful controller synthesis problem, for example, in terms of H_2 and H_∞ norms, weights W_z and W_W are included, and the general configuration becomes the one as shown below:



[Notice that the vector v consists of all the inputs to the controllers.]

Example: Write the generalized plant for the following system:



Notice that: $w = [d \quad r]^T$; $z = y_1 - r$; $v = [r \quad y_1 \quad y_2 \quad d]^T$

$$u = Kv; \quad K = \begin{bmatrix} K_1 K_r & -K_1 & -K_2 & K_d \end{bmatrix}$$

Thus by inspection, the generalized plant is: $P = \begin{bmatrix} G_1 & -I & G_1 G_2 \\ 0 & I & 0 \\ G_1 & 0 & G_1 G_2 \\ 0 & 0 & G_2 \\ I & 0 & 0 \end{bmatrix}$

2. Stacked S/T/KS problem for H_∞ -design:

Consider an H_∞ -problem where we want to bound $\bar{\sigma}(S)$ for performance, $\bar{\sigma}(T)$ for robustness and avoid sensitivity to noise, and $\bar{\sigma}(KS)$ to penalize large input. The requirements may be combined into a stacked H_∞ -problem of the following:

$$\min_K \|N(K)\|_\infty , \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix}$$

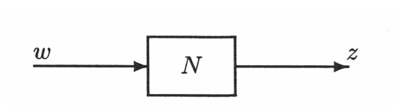
Let $z = N\ w$ and from which, we have:

$z_1 = W_u u \quad \rightarrow$ for penalizing the use of input

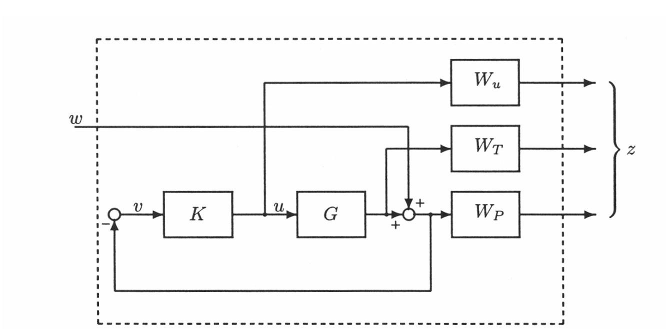
$z_2 = W_T G u \quad \rightarrow$ for stability robustness

$z3 = W_P w + W_P G u \quad \rightarrow$ for performance

$v = -w - Gu$



Thus the corresponding block diagram becomes:



And, the generalized plant becomes:

$$P = \begin{bmatrix} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ -I & -G \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$Z = P_{11} w + P_{12} u$$
$$v = P_{21} w + P_{22} u$$

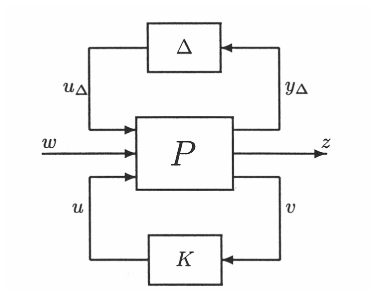
Closing the loop by letting $u = Kv$, the transfer function from w to z becomes:

$$z = Nw, \quad N = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} \triangleq F_l[P, K\}$$

Where, $F_l[P, K]$ is called a lower linear fractional transformation (LFT) of P with K as the parameter.

A generalized control configuration including model uncertainty

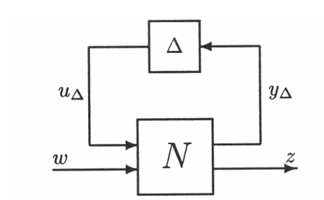
The generalized control configuration can be extended to include model uncertainty as shown in the following figure:

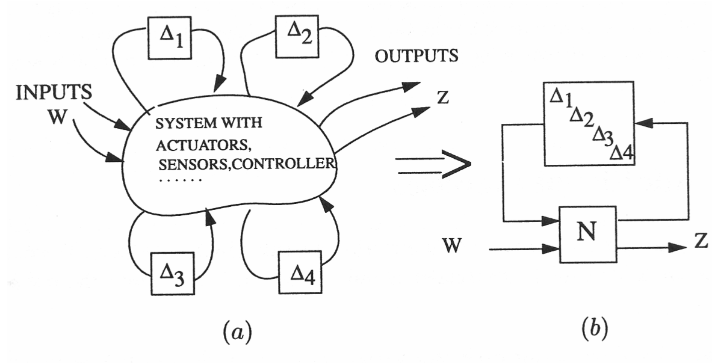


The generalized plant P can be partitioned to be compatible to the controller K. In other words,

$$\begin{bmatrix} \begin{bmatrix} y_\Delta \\ z \end{bmatrix} \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix} \\ u \end{bmatrix}; \; u = Kv \quad \Rightarrow \begin{bmatrix} Z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} W \\ u \end{bmatrix}$$

$$\Rightarrow Z = NW; \quad N = F_l[P, K] = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$





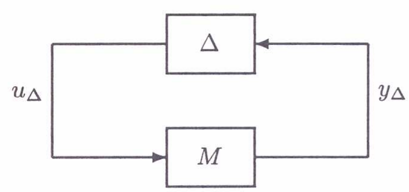
(a) (b)

Similarly,

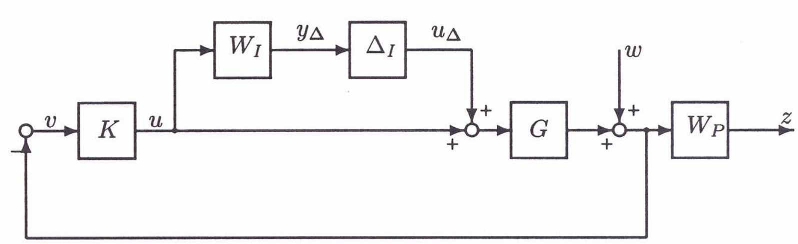
$$\begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix}, \quad u_\Delta = \Delta y_\Delta ,$$

$$Z = F_\Delta [N, \Delta] w = \left\{ N_{22} + N_{21}\Delta \left(I - N_{11}\Delta \right)^{-1} N_{12} \right\} w$$

To analyze the robust stability of $F_\Delta[N,\Delta]$, one should focus on the inverse of $I - N_{11}\Delta$, i.e. $[I - N_{11}\Delta]^{-1}$. For this, the system of the following is considered:



Obtaining P, N and M



LMI synthesis for processes with model uncertainties

LMI problems

A linear matrix inequality is a matrix inequality of the form:

$$F(\zeta) \triangleq F_o + \sum_{i=1}^{m} \zeta_i F_i > 0$$

where $\zeta \in R^m$ is the variable, and $F_i = F_i^T \in R^{n\times m}, i = 1, 2, \cdots, m$ are given.

The inequality symbol in the above equation means that $F(\zeta)$ is positive definite.

The set $\{\zeta \mid F(\zeta) > 0\}$ is convex. For many problems, the variables are matrices,

e.g.,

$$A^T P + PA < 0$$

where $A \in R^{n\times n}$ is given and $P = P^T$ is variable. The problem is: " the LMI $A^T P + PA < 0$ in P"

LMI feasibility problem: Given an LMI $F(\zeta) > 0$, the corresponding LMI Problem

(LMIP) is to find ζ^{feas} such that $F(\zeta^{feas}) > 0$.

Eigen Value Problem(EVP): The EVP is to minimimize the maximum eigen value of a matrix, subject to an LMI, or:

$$\underset{\text{w.r.t. } \zeta \text{ and } \lambda}{Minimize} \quad \{\lambda\}$$

subject to: $\lambda I - A(\zeta) > 0, \quad B(\zeta) > 0$

Here, A and B are symmetric matrices that depend affinely on the variable ζ. This is a convex optimization problem.

As an example of EVP:

Minimize γ

Subject to $A^T P + PA + C^T C + \gamma PBB^T P < 0$

The above EVP problem is equivalent to the following problem:

Minimize γ

Subject to $\begin{bmatrix} -A^T P - PA - C^T C & PB \\ B^T P & \gamma I \end{bmatrix} > 0$

LMI and Passivity

The system (A, B, C, D) of the following

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is passive, i.e.,

$$\int_0 [u(t)]^T \, y(t)dt \geq 0$$

if and only if there exists a matrix P>0 such that

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0, \quad , P > 0, \quad D^T + D > 0$$

The passivity is equivalent to the transfer function matrix H being positive real, which means that

$$H(s) + H(s)^* \geq 0 \quad \text{for all } \mathrm{Re}\{\, s\} > 0, \quad H(s) = C(sI - A)^{-1}B + D$$

Minimizing Condition number by scaling:

Let $A \in R^{p\times q}$ with $p \geq q$. Then

$$\kappa(A) = \frac{\overline{\sigma}(A)}{\underline{\sigma}(A)} = \frac{\sqrt{\lambda_{\max}(A^T A)}}{\sqrt{\lambda_{\min}(A^T A)}}$$

Consider the following problem:

$Min \;\; \kappa(LAR)$

$L \in R^{p\times q}$, diagonal and nonsingular

$R \in R^{q\times q}$, diagonal and nonsingular

There exist non-singular, diagonal L and R and $\mu > 0$ such that

$$\mu I \leq (LAR)^T (LAR) \leq \mu\gamma^2 I$$

By absorbing $1/\sqrt{\mu}$ into L, it becomes

$$I \leq (LAR)^T (LAR) \leq \gamma^2 I$$

which is the same as:

$$(RR^T) - 1 \leq A^T (L^T L) A \leq \gamma^2 (RR^T)^{-1}$$

And this is equivalent to the existence of diagonal P,Q, with P>0, Q>0, and

$$Q \leq A^T PA \leq \gamma^2 Q$$

Thus the problem becomes

$Min \;\; \gamma^2$

$P \in R^{p\times p}$, diagonal and nonsingular, P>0

$Q \in R^{q\times q}$, diagonal and nonsingular

$Q \leq A^T PA \leq \gamma^2 Q$

Analysis and design of uncertain control systems using LMIs

The set Π is described by the following state equations:

$$\dot{x} = A(t)x + B_u(t)u + B_w(t)w, \quad x(0) = x_0$$
$$z = C_z(t)x + D_{zu}(t)u + D_{zw}(t)w,$$

where the matrices are unknown except for the fact that they satisfy

$$\begin{bmatrix} A(t) & B_u(t) & B_w(t) \\ C_z(t) & C_{zu}(t) & C_{zw}(t) \end{bmatrix} \in \Omega \subseteq R^{(n+n_z)\times(n+n_u+n_w)}$$

is a convex set of a certain type. When one or more integer n_u,n_w,n_z equal zero means the corresponding variable is not used. For example, when $n_u = n_z = n_w = 0$, the set Π is described by $\left\{ \dot{x} = A(t)x \middle| \; A(t) \in \Omega \right\}$. The Ω has many choices for a number of common control system models: LTI systems, polytopic systems (PS), norm-bound systems, structured norm bound systems, systems with parametric perturbations, systems with structured and bounded LTI perturbations, etc.

For illustration purpose, polytopic system models arise when the uncertain plant is modeled as a LTI system with state space matrices is given as follows.

A polytopic Ω is described as a convex hull of its vertices:

$$Co \left\{ \begin{bmatrix} A1(t) & B_{u,1}(t) & B_{w,1}(t) \\ C_{z,1}(t) & C_{zu,1}(t) & C_{zw,1}(t) \end{bmatrix}, \cdots, \begin{bmatrix} A_l(t) & B_{u,l}(t) & B_{w,l}(t) \\ C_{z,l}(t) & C_{zu,l}(t) & C_{zw,l}(t) \end{bmatrix} \right\}$$

with the definition of a convex full of the following:

$$Co\{G_1, \cdots, G_l\} \triangleq \left\{ G : G = \sum_{i=1}^{l} \lambda_i G_i ; \lambda_i \geq 0, \sum_{i=1}^{l} \lambda_i = 1 \right\}$$

Example:

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -a_1(t) & -a_2(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t); \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

with $a_1(t) \in [-1 \;\; 1]$; $a_2(t) \in [-2 \;\; 2]$ for all $t \geq 0$.

The corresponding polytopic convex full is:

$$A \in \left\{ \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \right\}$$

Stability of Polytopic Systems

Consider a PS system: $\dot{x} = A(t)x, \quad A(t) \in Co\{A_1, A_2, \cdots, A_l\}$

A sufficient condition for this system to converge to zero is the existence of a quadratic positive function $V(x) = x^T Px$ such that $\frac{dV(x(t))}{dt} < 0$. Since

$$\frac{dV(x(t))}{dt} = x^T(t)\left[A^T(t)P + PA(t)\right]x(t)$$

A sufficient condition is the existence of a P satisfying the following conditions:

$$P > 0, \quad A^T(t)P + PA(t) < 0, \quad A(t) \in Co\{A_1, A_2, \cdots, A_l\}$$

If such a p exists, the PS is quadratically stable.

The above condition is equivalent to

$$P > 0, \quad A_i^T(t)P + PA_i(t) < 0, \quad i = 1, 2, \cdots, l$$

which is an LMI in P. Thus determining quadratic stability is an LMIP.

Quadratic stability

A sufficient condition for the quadratic stability is the existence of a quadratic function $V(x) = x^T Px,\ P > 0$ that decreases along every nonzero stable trajectory of the LDI (linear differential inclusion) system:

$\dot{x} = A(t)x;\ A(t) \in \Omega$.

Since. $\dot{V} = x^T[A^T(t)P + PA(t)]x$

The necessary and sufficient condition for QS is:

$$P > 0, \quad A^T(t)P + PA(t) < 0 \ \text{ for all } A(t) \in \Omega$$

(1). For LTI system:

$$P > 0, \quad A^T P + PA < 0$$

(2). For Polytopic LDI system

$$P > 0, \quad A_i^T P + PA_i < 0, \quad i = 1, 2, \cdots, L$$

Stabilizing state-feedback synthesis for polytopic systems

Consider the system with state feedback:

$$\dot{x} = A(t)x, \quad A(t) \in Co\{A_1, A_2, \cdots, A_l\}, \quad u = Kx(t)$$

The system is quadratically stable, if P and K exist so that:

$$P > 0, \quad \left(A_i + B_i K\right)^T P + P\left(A_i + B_i K\right) < 0, \quad i = 1, 2, \cdots, l$$

This matrix inequality is not jointly convex in P and K. However, with bijective transformation $Y \triangleq P^{-1}, \ W \triangleq kP^{-1}$, the equation can be rewritten as:

$$Y > 0, \quad \left(A_i + B_i W Y^{-1}\right)^T Y^{-1} + Y^{-1}\left(A_i + B_i W\right) Y^{-1} < 0, \quad i = 1, \cdots, l$$

Multiplying the inequality on the left and right by Y yields an LMI in Y and W

$$Y > 0, \quad Y_i A_i^T + W^T B_i^T + A_i Y + B_i W < 0, \quad i = 1, \cdots, l$$

If this LMI in Y and W has a solution, then the Lyapunov function V proves the quadratic stability of the closed-loop system with state-feedback. In other words, one can synthesize a linear state-feedback for the PS by solving an LMIP.

Robust Stability and Performance

The various sources of model uncertainty may be grouped into the following:

1. Parametric uncertainty: Praameter uncertainty is quantitied by assuming that each uncertain parameter α is bounded within som region $[\alpha_{\min}, \alpha_{\max}]$, that is

   $\alpha = \alpha(1 + r_\alpha \Delta), r_\alpha = (\alpha - \alpha_{\min})/(\alpha_{\max} - \alpha_{\min})$, Δ is are a scalar satisfying $|\Delta| \le 1$
2. Neglected and unmodelled dynamics uncertainty: This type of uncertainty is more difficult to quantify, but it is suited to use frequency domain representation.
3. Lumped uncertainty: Here the uncertainty description represents one or seversl sources combined into a single lumped perturbation of a chosen structure (e.g. input uncertainty, output uncertainty, or input-output uncertainty, etc.) The frequency domain representation is well suited for this type of uncertainty,

Notice that lumped perturbation form is used to represent the all types of modelling errors, and, unstructured perturbations are often used to get a simple uncertainty model. It is used to define unstructured uncertainty as the use of a full complex perturbation matrix Δ in the following forms:

$$\Pi_A : G_A = G + E_A$$
$$\Pi_I : G_I = G + E_I$$
$$\Pi_O : G_O = G + E_O$$

Each representation can be represented by multiplicative form. In other words,

$$G = \bar{G}(I + w_A \Delta_A), \quad \|\Delta_A\|_\infty \le 1; \quad G = (I + w_O \Delta_O)\bar{G}, \quad \|\Delta_O\|_\infty \le 1$$

$$G = \bar{G}(I + w_I \Delta_I), \quad \|\Delta_I\|_\infty \le 1$$

Each individual perturbation is assumed to be stable and is normalized,

$\bar{\sigma}(\Delta_i(j\omega)) \le 1 \ \forall \omega$

The maximum singular value of a block diagonal matrix is equal to the largest maximum singular values of the individual blocks. As a result, for $\Delta = diag\{\Delta_i\}$, it follows that $\bar{\sigma}_i(\Delta_i(j\omega)) \le 1 \ \forall \omega$ and $\forall i \Leftrightarrow \|\Delta\|_\infty \le 1$

Definitions of robust stability and robust performance

1. Robust stability (RS): With a given controller K, the system remains stable for all plants in the set of of uncertainty.
2. Robust performance (PS): If RS is satisfied, the transfer function from exogenous inputs w to outputs z remains reasonable performance for all plants in the uncertainty set.

In terms of the $N\Delta$-structure, the requirements for stability and performance can be summarized as follows:

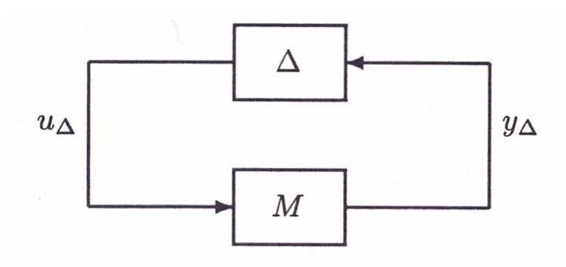
$NS \quad \Leftrightarrow \quad N$ is internal stable.

$NP \quad \Leftrightarrow \quad \|N_{22}\|_\infty < 1;$ and NS

$\text{RS} \quad \Leftrightarrow \quad F = F_u(N,\Delta)$ is stable $\forall\Delta,\ \|\Delta\|_\infty \le 1;$ and NS

$RP \quad \Leftrightarrow \quad \|\text{F}\|_\infty < 1,\ \forall\Delta,\ \|\Delta\|_\infty \le 1;$ and NS

Robust stability of the $M\Delta$-structure



Theorem 1 Determinant stability condition Assume that the nominal system M(s) and the perturbations Δ, such that if Δ' is an allowed perturbation then so is $c\Delta'$ where c is any real scalar that $|c| \le 1$. Then the $M\Delta$-system is stable for all allowed perturbations if and only if the Nyquist plot of $\det\{I - M\Delta\}$ does not encircle the origin for each Δ, and

$$N\left\{0,\det\left(I-\Delta(j\omega)\right)\right\}=0,\ \forall\omega,\forall\Delta$$
$$\Leftrightarrow \det\left(I-\Delta(j\omega)\right)\neq 0,\ \forall\omega,\forall\Delta \qquad \text{(A)}$$
$$\Leftrightarrow\ \lambda_i(\mathrm{M}\Delta)\neq 1,\ \forall \mathrm{i},\forall\omega,\forall\Delta \qquad \text{(B)}$$

First, assume that for some $\Delta^{'}$, $\det\left\{I-M\Delta^{'}\right\}\neq 0\ \forall\omega$ and the image of $\det\left\{I-M\Delta\right\}$ ecircles the origin as s traverses the Nyquist contour. Because the Nyquist contour and its image are closed, there exists another $\Delta^{''}$ such that $\Delta^{''}=\varepsilon\Delta^{'}$, $\varepsilon\in[0,1]$, and with an $\omega^{'}$ such that $\det\left\{I-M\Delta^{''}(j\omega^{'})\right\}=0$. This contradict with the assumption that $\det\left(I-\Delta(j\omega)\right)\neq 0,\ \forall\omega,\forall\Delta$. This proved part (A).

$$\det\left\{I-M\Delta\right\}=\prod_i\lambda_i(I-M\Delta)=\prod_i(1-\lambda_i(M\Delta))\neq 0$$

$$\Rightarrow \det\left\{I-M\Delta\right\}\neq 0$$

which implies $\lambda_i(\mathrm{M}\Delta)\neq 1,\ \forall \mathrm{i},\forall\omega,\forall\Delta$. This proves (B).

Theorem 2 Spectral radius condition for complex perturbation. Assume:

(1) The nominal system and M(s) of the perturbations $\Delta(\mathrm{s})$ are stable,

(2) The class of perturbations, Δ, shat that if $\Delta^{'}$ is an allowed perturbation the so is $c\Delta^{'}$, where c is any complex scalar such that $|c|\leq 1$

Then, the $M\Delta$-system is stable for all allowed perturbations if and only if $\rho(M\Delta(j\omega))<1,\ \forall\omega,\forall\Delta$,

oe equivalently $\max\limits_{\Delta}\rho(M\Delta(j\omega))<1,\forall\omega$

Proof:

Assume that $\det\left(I-M\Delta(j\omega)\right)\neq 0,\ \forall\omega,\forall\Delta$ and there exists a perturbation such that $\rho(M\Delta)\geq 1$at some frequency. Then $\lambda_i\left(M\Delta^{'}\right)\geq 1$ for some i, and there always exists another perturbation in that set, $\Delta^{''}=\varepsilon\Delta^{'}$ where ε is a complex scalar with $|\varepsilon|\leq 1$ such that $\lambda_i\left(M\Delta^{'}\right)=1$ so that $\det\left(I-M\Delta(j\omega)\right)=0$ at some frequency, and, this contradict with the assumption that $\det\left(I-M\Delta(j\omega)\right)\neq 0,\ \forall\omega,\forall\Delta$. Thus, the theorem is proved.

RS Lemma for complex unstructured uncertainty. Let Δ be the set of all complex matrices such that $\bar{\sigma}(\Delta) \le 1$, the following is true:

$$\max_{\Delta} \rho(M\Delta) \le \bar{\sigma}(M)$$

Proof:

$$\max_{\Delta} \rho(M\Delta) \le \max_{\Delta} \bar{\sigma}(M\Delta) \le \bar{\sigma}(M)\bar{\sigma}(\Delta) \le \bar{\sigma}(M)$$

[The first part of inequality is due to: $\underline{\sigma} \le |\lambda| \le \bar{\sigma}$ is true for each λ]

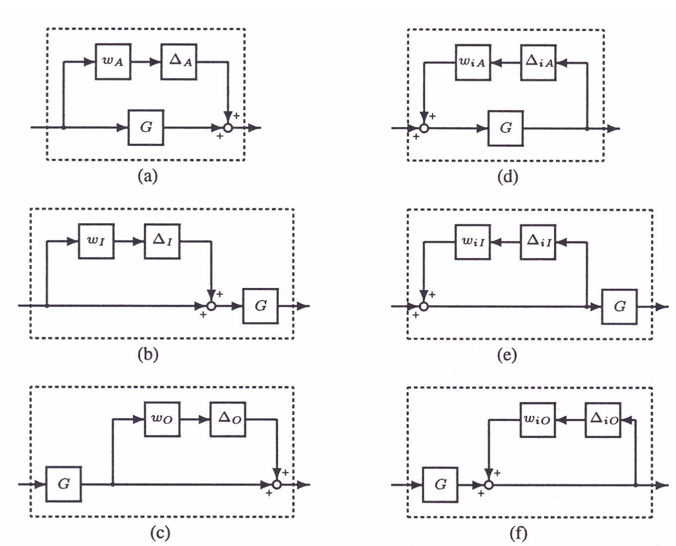
Theorem 3 Assume that the nominal system M(s) is stable and that the perturbations Δ are stable. The $M\Delta$ -system is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \le 1$ if and only if

$$\bar{\sigma}(M(j\omega)) < 1 \ \ \forall\omega \quad \Leftrightarrow \quad \|M\|_\infty < 1$$

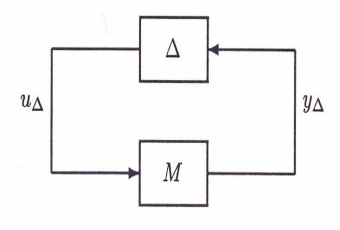
Application of the unstructured RS-condition

For each of the six single unstructured perturbations in the following figure,

$E = W_1 \Delta W_2$



And, in terms of $M\Delta$-structure of the following:



$$M = W_1 M_0 W_2$$

Where, M_0 is given by:

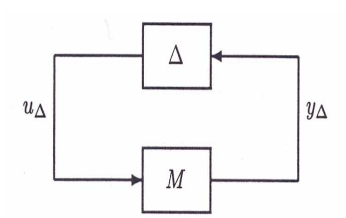
$$\begin{aligned}
G &= \bar{G} + E_A: & M_0 &= K(I+\bar{G}K)^{-1} = KS \\
G &= \bar{G}(I+E_I): & M_0 &= K(I+\bar{G}K)^{-1}\bar{G} = T_I \\
G &= (I+E_o)\bar{G}: & M_0 &= \bar{G}K(I+\bar{G}K)^{-1} = T \\
G &= \bar{G}(I-E_{iA})^{-1}: & M_0 &= (I+\bar{G}K)^{-1}\bar{G} = S\bar{G} \\
G &= \bar{G}(I-E_{iI})^{-1}: & M_0 &= (I+K\bar{G})^{-1} = S_I \\
G &= (I-E_{iO})^{-1}\bar{G}: & M_0 &= (I+\bar{G}K)^{-1} = S
\end{aligned}$$

The RS theorem yields

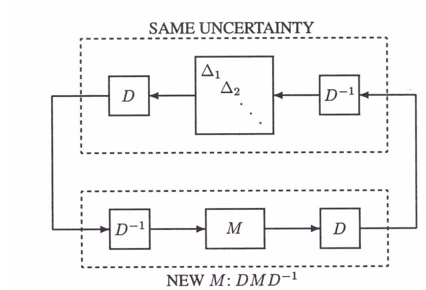
$$RS \quad \Leftrightarrow \quad \|W_1 M_0 W_2(j\omega)\|_\infty < 1, \ \forall\omega$$

For example, $G = \bar{G}(I+E_I)$, $\|\Delta_I\|_\infty \le 1 \Leftrightarrow \|W_I T_I\|_\infty < 1$

The Structured Singular value



$\Leftrightarrow$



Unstructured uncertainty Structured uncertainty

Consider the presence of structured uncertainty, where $\Delta = diag\{\Delta_i\}$ is a block diagonal. To test for the RS of the system, the $M-\Delta$-structure is used. That is:

$$RS \text{ if } \bar{\sigma}\left(M(j\omega)\right) < 1 \quad \forall\omega$$

The figures shown above are two $M-\Delta$-structures for the same system, where $D = diag\left\{d_i I_i\right\}$. The question is whether we can take advantage of the fact that $\Delta = diag\left\{\Delta_i\right\}$ is structured to obtain a more tight RS-condition. In the right figure, the inputs and the outputs to M and Δ are re-scaled. With the chosen form, $\Delta_i^{new} = d_i \Delta_i d_i^{-1}$ and $M^{new} = DMD^{-1}$, the RS condition becomes:

$$RS \text{ if } \bar{\sigma}\left(M^{new}(j\omega)\right) = \bar{\sigma}\left(DM(j\omega)D^{-1}\right) < 1 \quad \forall\omega$$

The most improved RS-condition is obtained by minimizing at each frequency the scaled singular value:

$$RS \text{ if } \underset{\mathrm{D}(\omega)}{\mathrm{Min}}\, \bar{\sigma}\left(M^{new}(j\omega)\right) = \bar{\sigma}\left(D(\omega)M(j\omega)D(\omega)^{-1}\right) < 1 \quad \forall\omega$$

The structured singular value is a function which provides a generalization of the singular value and the spectral radius. A simple statement is: “The smallest structured Δ (measured in terms of $\bar{\sigma}(\Delta)$ which makes $\det\left\{I - M\Delta\right\} = 0$. Then the inverse of this $\bar{\sigma}(\Delta)$ is called as the structured singular value”.

Mathematically,

$$\left[\mu(M)\right]^{-1} = \min_{\Delta}\{\bar{\sigma}(\Delta) \,|\, \det\left(I - M\Delta\right) = 0 \text{ for structured } \Delta\}, \text{ or}$$

$$\frac{1}{\mu(M)} = \min_{\Delta}\{\bar{\sigma}(\Delta) \,|\, \det\left(I - M\Delta\right) = 0 \text{ for structured } \Delta\}$$

Example: (This example is to show that μ depends on the structure of Δ.)

$$M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0.894 & 0.447 \\ -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 3.162 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}^*$$

The perturbation Δ

$$\Delta = v_1 u_1^T = \frac{1}{3.162} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \begin{bmatrix} 0.894 & -0.447 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 \\ -0.1 & -0.1 \end{bmatrix}$$

In fact, for the matrix M, the smallest diagonal Δ which makes $\det\left\{I - M\Delta\right\} = 0$ is:

$$\Delta = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \bar{\sigma}(\Delta) = 0.333,$$ and, thus, $\mu(M) = 3$, when Δ is diagonal.

When all the blocks in Δ are complex,

$$\mu(M) = Max_{\{\Delta, \bar{\sigma}(\Delta) \le 1\}} \rho(M\Delta)$$

The following are the properties of μ for complex perturbations:

Properties of μ

1. $\mu(\alpha M) = |\alpha| \mu(M)$ for any real α.
2. Let $\Delta = diag\{\Delta_1, \Delta_2\}$ and M is partitioned accordingly.

   Then, $\mu_\Delta(M) \ge Max\left\{\mu_{\Delta_1}(M_{11}),\ \mu_{\Delta_2}(M_{11})\right\}$
3. Δ full matrix: $\mu(M) \le \bar{\sigma}(M)$
4. $\rho(M) \le \mu(M) \le \bar{\sigma}(M)$
5. For any unitary matrix U with the same structure as Δ,

   $\mu(MU) = \mu(M) = \mu(UM)$
6. Any matrix D which commutes with Δ, ($\Delta D = D\Delta$),

   $\mu(DM) = \mu(MD); \quad \mu(DMD^{-1}) = \mu(M)$

Robust stability with structured uncertainty

According to theorem 1, we already have:

$$RS \iff \det\left(I - \Delta(j\omega)\right) \ne 0, \quad \forall\omega, \forall\Delta,\ \sigma(\Delta(j\omega)) \le 1, \forall\omega$$

To find the factor k_m by which the system is robust stable, the Δ is scaled by k_m, and look for the smallest k_m which yields borderline instability, that is:

$$\det\{I - k_m M\Delta\} = 0$$

From the definition of μ, this value is $k_m = 1/\mu(M)$.

Theorem 4 RS for block-diagonal perturbations Assume that nominal M and Δ are stable. Then, the $M\Delta$-system is stable for all allowed Δ with $\bar{\sigma}(\Delta) \le 1, \ \forall\omega$, if and only if: $\mu(M(j\omega)) < 1, \ \forall\omega$