Model Predictive Control

I. Model representations for the dynamic system:

1. Impulse response model :

   By using the convolution computation, a linear dynamic system can be represented by using its impulse response sequence and is known as the *impulse response model*:

$$y_k = \sum_{i=1}^{\infty} h_i u_{k-i} \tag{1}$$

   For practical computation, the so call *finite impulse response* (**FIR**) representation is used, i. e.:

$$y_k = \sum_{i=1}^{N_m} h_i u_{k-i} \tag{2}$$

2. Step response model :

   The impulse response train $\{h_1, h_2, \cdots, h_\infty\}$ is related to the step reponse sequence $\{a_1, a_2, \cdots, a_{ss}\}$ in the following manner:

$$\begin{aligned} a_1 &= h_1 \\ a_2 &= h_1 + h_2 \\ &\cdots \\ &\cdots \\ a_n &= h_1 + h_2 + \cdots + h_n \\ &\cdots \end{aligned}$$

   Thus, the impulse response model can be written in terms of step response sequence and is called *step response model*

$$y_k = a_1 \Delta u_{k-1} + a_2 \Delta u_{k-2} + \cdots + a_\infty \Delta u_{k-\infty} \tag{3}$$

   But, for practical computation, only finite terms in the above equation are adopted. During trancation, N_m should be able to take into account most steady state conditions. The resulting trancated *finite step response model* is known as the (**FSR**) model, i.e.:

$$y_k = a_1 \Delta u_{k-1} + a_2 \Delta u_{k-2} + \cdots + a_{N_m} \Delta u_{k-N_m} + a_{N_m} u_{k-N_m-1} \tag{4}$$

3. Remark : Quite often, it is very easy to drop the last term of equation(4) and result an erroneous representation for the **FSR** model.

II. Models for Predicting Future outputs:

The output of the system in the future time horizon is computed using prediction models. Usually, there are two kinds of prediction models: the uncorrected and corrected ones. The uncorrected model is also referred as open-loop prediction model in literature. In order to clarify different predictive estimations, the following notations will be used in the text that follows.

- y_k = the output at the present moment.
- $\bar{y}_k$= the model output at present moment.
- y_{k+j} = the output at j^{th} step ahead into the future.
- $\hat{y}_{k+j}$ = estimation of y_{k+j}
- $\hat{y}^o_{k+j}$ = uncorrected form for $\hat{y}_{k+j}$
- $\hat{y}^c_{k+j}$ = corrected form for $\hat{y}_{k+j}$
- $\hat{y}_{(k+j|k)}$ = estimation of y_{k+j} by assuming $\Delta u_{k+j} = 0$ for $j > 0$
- $\hat{y}^o_{(k+j|k)}$ = uncorrected form for $\hat{y}_{(k+j|k)}$
- $\hat{y}^c_{(k+j|k)}$= corrected form for $\hat{y}_{(k+j|k)}$

1. **Uncorrected Prediction Models:**

   The uncorrected prediction models use inputs in the past to compute the future outputs:

   (a) Uncorrected Prediction using **FIR** :

   $$\hat{y}^o_{k+j} = \sum_{i=1}^{N_m} h_i u_{k+j-i} \tag{5}$$

   (b) Uncorrected Prediction using **FSR** :

   $$\begin{aligned} \hat{y}^o_{k+j} &= a_1 \Delta u_{k+j-1} + a_2 \Delta u_{k+j-2} + \cdots + a_{N_m} \Delta u_{k+j-N_m} + \\ &\quad + a_{N_m} u_{k+j-N_m-1} \end{aligned} \tag{6}$$

2. **Corrected Prediction Models:**

   In general, the corrected prediction for the output is given as follows:

   $$\hat{y}^c_{k+j} = \hat{y}^o_{k+j} + \psi[\ y_k - \hat{y}^o_k\ ] \tag{7}$$

   where, $\psi[\ y_k - \hat{y}^o_k\ ]$ is a function of prediction error that is detected at this present moment. In many predictive control systems, the function ψ is taken as:

   $$\psi[\ y_k - \hat{y}^o_k\ ] = y_k - \hat{y}^o_k$$

so that

$$\hat{y}^c_{k+j} = y_k + [\ \hat{y}^o_{k+j} - \hat{y}^o_k\ ] \tag{8}$$

III. Long-Range Prediction Models (LRPM)

In literature, there are various representations for long-range output prediction. However, they are equivalent in the sense of Eq.(8). They all consist of two parts: the part related to the future input moves and the part related to past input moves. In the following, we shall provide two definitions for future input moves:

$$\delta u_{k+j} = u_{k+j} - u_{k-1} \tag{9}$$

$$\Delta u_{k+j} = u_{k+j} - u_{k+j-1} \tag{10}$$

1. LRPM based on FIR representation :

   By successively applying Eq.(8) and the FIR, we have:

$$\begin{aligned}
\hat{y}^c_{k+1} &= y_k + h_1\delta u_k + h_2\Delta u_{k-1} + h_3\Delta u_{k-2} + \cdots + h_{N_m}\Delta u_{k-N_m+1} \\
\hat{y}^c_{k+2} &= y_k + h_1\delta u_{k+1} + h_2\delta u_k \\
&\quad + h_2\Delta u_{k-1} + h_3(\Delta u_{k-1} + \Delta u_{k-2}) + h_4(\Delta u_{k-2} + \Delta u_{k-3}) + \cdots \\
&\quad + h_{N_m}(\Delta u_{k-N_m+2} + \Delta u_{k-N_m+1}) \\
\hat{y}^c_{k+3} &= y_k + h_1\delta u_{k+2} + h_2\delta u_{k+1} + h_3\delta u_k \\
&\quad + h_2\Delta u_{k-1} + h_3(\Delta u_{k-1} + \Delta u_{k-2}) \\
&\quad + h_4(\Delta u_{k-1} + \Delta u_{k-2} + \Delta u_{k-3}) + h_5(\Delta u_{k-2} + \Delta u_{k-3} + \Delta u_{k-4}) + \cdots \\
&\quad + h_{N_m-2}(\Delta u_{k-N_m+5} + \Delta u_{k-N_m+4} + \Delta u_{k-N_m+3}) \\
&\quad + h_{N_{m-1}}(\Delta u_{k-N_m+4} + \Delta u_{k-N_m+3} + \Delta u_{k-N_m+2}) \\
&\quad + h_{N_m}(\Delta u_{k-N_m+3} + \Delta u_{k-N_m+2} + \Delta u_{k-N_m+1}) \\
&\quad \cdots \\
&\quad etc.
\end{aligned} \tag{11}$$

Let,

$$\hat{Y}(k+1) = \left[\ \hat{y}_{k+1},\quad \hat{y}_{k+2},\quad \cdots,\quad \hat{y}_{k+p}\ \right]^T$$

so that,

$$\hat{Y}^c(k+1) = \left[\ \hat{y}^c_{k+1},\quad \hat{y}^c_{k+2},\quad \cdots,\quad \hat{y}^c_{k+p},\ \right]^T$$

and,

$$\hat{Y}^o(k+1) = \left[\ \hat{y}^o_{k+1},\quad \hat{y}^o_{k+2},\quad \cdots,\quad \hat{y}^o_{k+p},\ \right]^T$$

$$\delta U(k) = \begin{bmatrix} \delta u_k, & \delta u_{k+1}, \cdots, & \delta u_{k+p-1} \end{bmatrix}^T$$

$$\Delta U(k) = \begin{bmatrix} \Delta u_k, & \Delta u_{k+1}, \cdots, & \Delta u_{k+p-1} \end{bmatrix}^T$$

$$\Delta U^*(k-1) = \begin{bmatrix} \Delta u_{k-1}, & \Delta u_{k-2}, \cdots, & \Delta u_{k-N_m+1} \end{bmatrix}^T$$

$$[\mathbf{1}]_{p\times 1} = [1,\ 1,\ 1, \cdots, 1]^T_{p\times 1}$$

Then,

$$\hat{Y}^c(k+1) = [\mathbf{1}]_{p\times 1}\ y_k + \mathbf{H_1}\ \delta U(k) + \mathbf{H_2}\ \Delta U^*(k-1) \tag{12}$$

Where,

$$\mathbf{H_1} = \begin{bmatrix} h_1, & 0 & 0 & \cdots & 0 \\ h_2, & h_1, & 0 & \cdots & 0 \\ h_3, & h_2, & h_1, & \cdots & 0 \\ & \cdots & & \cdots & \\ & \cdots & & \cdots & \\ h_p & h_{p-1} & h_{p-2} & \cdots & h_1 \end{bmatrix} \tag{13}$$

$$\begin{aligned} \mathbf{H_2} &= \begin{bmatrix} h_2 & h_3 & \cdots & h_{N_m-1} & h_{N_m} \\ h_2+h_3 & h_3+h_4 & \cdots & h_{N_m-1}+h_{N_m} & h_{N_m} \\ h_2+h_3+h_4 & h_3+h_4+h_5 & \cdots & h_{N_m-1}+h_{N_m} & h_{N_m} \\ & & \cdots & \cdots & \\ \sum_{i=2}^{p+1} h_i & \sum_{i=3}^{p+2} h_i & \cdots & h_{N_m-1}+h_{N_m} & h_{N_m} \end{bmatrix} \\ &= \begin{bmatrix} a_2-a_1 & \cdots & \cdots & \cdots & a_{N_m-1}-a_{N_m-2} & a_{N_m}-a_{N_m-1} \\ a_3-a_1 & \cdots & \cdots & a_{N_m-1}-a_{N_m-3} & a_{N_m}-a_{N_m-2} & a_{N_m}-a_{N_m-1} \\ a_4-a_1 & \cdots & a_{N_m-1}-a_{N_m-4} & a_{N_m}-a_{N_m-3} & a_{N_m}-a_{N_m-2} & a_{N_m}-a_{N_m-1} \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p+1}-a_1 & \cdots & \cdots & \cdots & a_{N_m}-a_{N_m-2} & a_{N_m}-a_{N_m-1} \end{bmatrix} \end{aligned}$$

The uncorrected LRPM for the output is given as:

$$\hat{Y}^o(k+1) = \mathbf{H_1}\ \delta U(k) + A^*\Delta U^*(k-1) + [\mathbf{1}]_{p\times 1} a_{N_m} u_{k-N_m} \tag{14}$$

Where,

$$A^* = \begin{bmatrix} a_2 & a_3 & \cdots & \cdots & a_{N_m-1} & a_{N_m} \\ a_3 & a_4 & \cdots & a_{N_m-1} & a_{N_m} & a_{N_m} \\ a_4 & a_5 & \cdots & a_{N_m} & a_{N_m} & a_{N_m} \\ & & \cdots & a_{N_m} & a_{N_m} & a_{N_m} \\ a_p+1 & a_{p+2} & \cdots & a_{N_m} & a_{N_m} & a_{N_m} \end{bmatrix} \tag{15}$$

2. <u>LRPM based on FSR representation</u> :

Since,

$$\delta U(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ & & & \cdots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \Delta U(k) \tag{16}$$

Thus, it is straightforward to substitute the above euqation into Eq.(12) to obtain:

$$\hat{Y}^c(k+1) = [\mathbf{1}]_{p\times 1}\ y_k + \mathbf{A}\ \Delta U(k) + \mathbf{H_2}\ \Delta U^*(k-1) \tag{17}$$

Where,

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ a_p & a_{p-1} & a_{p-2} & \cdots & a_1 \end{bmatrix} \tag{18}$$

or, to obtain the following uncorrected LRP from Eq.(14):

$$\hat{Y}^o(k+1) = \mathbf{A}\ \Delta U(k) + A^* \Delta U^*(k-1) + [\mathbf{1}]_{p\times 1}\ a_{N_m}\ u_{k-N_m} \tag{19}$$

3. **<u>LRPM based on state space representation</u>**:

Let $\hat{Y}(k+1|k)$ is definded as:

$$\hat{Y}^o(k+1|k) = \left[\ \hat{y}^o_{(k+1|k)},\ \ \hat{y}^o_{(k+2|k)},\ \ \cdots,\ \ \hat{y}^o_{(k+p|k)}\ \right]^T \tag{20}$$

Where, $y^o_{(k+j|k)}$ designates the estimation of y_{k+j} using all the information available up to the instant k and assuming no input moves in the future time horizon.

By giving the definitions of the following matrices:

$$\begin{aligned} \tilde{\Phi} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \\ \mathbf{F} &= \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{p-1} & a_p \end{bmatrix}^T \\ \mathbf{1}_p &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \end{aligned}$$

and by the following equation:

$$\hat{y}^o_{(k+j|k)} = \hat{y}^o_{(k+j|k-1)} + a_j \Delta u_k; \quad j \ = \ 1, 2, \cdots, p$$

We can write:

$$\begin{aligned}
\hat{Y}^o(k+1|k-1) &= \tilde{\Phi}\hat{Y}^o(k|k-1) + \mathbf{1}_p\ \hat{y}^o_{(k+p|k-1)} &(21)\\
\hat{Y}^o(k|k-1) &= \Phi\ \hat{Y}^o(k-1|k-2) + \mathbf{F}\Delta u_{k-1} + \mathbf{1}_p\ \hat{y}^o_{(k+p-1|k-2)} &(22)\\
\hat{Y}^o(k+1) &= \Phi\ \hat{Y}^o(k|k-1) + \mathbf{A}\ \Delta U(k) &(23)
\end{aligned}$$

where,

$$\begin{aligned}
\hat{y}^o_{(k+p|k-1)} &= a_{p+1}\Delta u_{k-1} + a_{p+2}\Delta u_{k-2} + \cdots + a_{N_m}\Delta u_{k+p-N_m}\\
&\quad + a_{N_m}\sum_{j=N_m+1-p}^{\infty}\Delta u_{k-j}\\
\hat{y}^o_{(k+p-1|k-1)} &= a_p\Delta u_{k-1} + a_{p+1}\Delta u_{k-2} + \cdots + a_{N_m}\Delta u_{k+p-1-N_m}\\
&\quad + a_{N_m}\sum_{j=N_m+2-p}^{\infty}\Delta u_{k-j}
\end{aligned}$$

Notice that, for a proper value of p,

$$\begin{aligned}
\hat{y}^o_{(k+p|k-1)} - \hat{y}^o_{(k+p-1|k-1)} &= (a_{p+1}-a_p)\Delta u_{k-1} + (a_{p+2}-a_{p+1})\Delta u_{k-2}\\
&\quad + \cdots \approx 0 &(24)
\end{aligned}$$

Thus, for, a proper large value of p, Eq.(21) can be written as:

$$\hat{Y}^o(k+1|k-1) = \Phi\ \hat{Y}^o(k|k-1) \tag{25}$$

Where, $\tilde{\Phi}$ is changed to Φ as follows:

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ & & & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{26}$$

So that Eq.(21) $\sim$ Eq.(23) becomes:

$$\begin{aligned}
\hat{Y}^o(k+1|k-1) &= \Phi\ \hat{Y}^o(k|k-1) &(27)\\
\hat{Y}^o(k|k-1) &= \Phi\ \hat{Y}^o(k-1|k-2) + \mathbf{F}\ \Delta u_{k-1} &(28)\\
\hat{Y}^o(k+1) &= \Phi\ \hat{Y}^o(k|k-1) + \mathbf{A}\ \Delta U(k) &(29)
\end{aligned}$$

In this formulation, $\hat{Y}^o(k|k-1)$ can then be treated as state variable and $\hat{Y}^o(k+1)$ becomes the uncorrected LRPM.

The corrected LRPM can then be formulated by making use of the state representation in Eq.(27) $\sim$ Eq.(29). In the conventional DMC, the LRPM is simply formulated by adding a correcting term to $\hat{Y}^o(k+1)$, i.e.:

$$\begin{aligned}\hat{Y}^c(k+1) &= \Phi\, \hat{Y}^o(k|k-1) + \mathbf{A}\Delta U(k) + [\mathbf{1}]_{p\times 1}\; [y_k - \hat{y}^o_{(k|k-1)}] \quad (30)\\ \hat{Y}^o(k|k-1) &= \Phi\, \hat{Y}^o(k-1|k-2) + \mathbf{F}\; \Delta u_{k-1} \quad (31)\\ \hat{y}^o_{(k|k-1)} &= \mathbf{C}\; \hat{Y}^o(k|k-1)\end{aligned}$$

Recently, Morari and Lee proposed the following LRPM in formulating MPC, i.e.:

- Updated Based on Measurement:

  $$\hat{Y}^*(k|k) = \hat{Y}(k|k-1) + \mathbf{K}\; [y_k - \hat{y}_{(k|k-1)}] \quad (32)$$

- Model Prediction:

  $$\hat{Y}(k+1|k) = \Phi\; \hat{Y}^*(k|k) + \mathbf{F}\; \Delta u_k \quad (33)$$

- Long Range Prediction:

  $$\hat{Y}^c(k+1) = \Phi\; \hat{Y}^*(k|k) + \mathbf{A}\; \Delta U(k) \quad (34)$$

The first two steps in the above LRPM can be combined into one, i.e.:

$$\hat{Y}(k+1|k) \;=\; \Phi\; \hat{Y}(k|k-1) + \mathbf{F}\; \Delta u_k + \mathbf{K}_f\; [y_k - \hat{y}_{(k|k-1)}] \quad (35)$$

Thus, it resembles a filtering problem of the following linear system and

$$\begin{aligned} Z(k+1) &= \Phi\; Z(k) + \mathbf{F}\; \Delta u_k + \mathbf{T}\; \Omega \\ y(k) &= \mathbf{C}\; Z(k) + v \end{aligned} \quad (36)$$

Where, Ω and v are white noise disturbances with Q_1 and Q_2 as their covariance matrices. The coefficient matrix $\mathbf{K_f}$ in Eq.(35) serves as a filter constant. And, the coefficient matrix $\mathbf{T}$ is determined by the way how disturbance model in the LRPM is considered. Morari and Lee suggested that the filter constant $\mathbf{K}_f$ is taken as:

$$\mathbf{K}_f = \mathbf{K} = [\mathbf{1}]_{p\times 1}\; f \quad (37)$$

where, f becomes a tuning parameter.

IV. Modeling the output disturbance in the existing LRPM's

If model is perfect, according to IMC, the disturbance at the output is the difference between the process output, y, and the model output, $\bar{y}$.

$$y_k = \bar{y}_k + d_k = \hat{y}^o_k + d_k$$

So that

$$
\begin{aligned}
Y(k+1) &= \bar{Y}^o(k+1) + \begin{bmatrix} d_{k+1} \\ d_{k+2} \\ \cdot \\ \cdot \\ \cdot \\ d_{k+p} \end{bmatrix} \\
&= \hat{Y}^o(k+1) + D(k+1) \qquad (38)
\end{aligned}
$$

Where,

$$D(k+1) = [\ d_{k+1}, d_{k+2}, \cdots, d_{k+p}\ ]^T$$

In this section, we shall show that either LRPM of the conventional DMC or of Morari and Lee's MPC, $\hat{Y}^c(k+1)$ can be expressed in the form of Eq.(39) with different definition for $\hat{D}(k+1|k)$ which is considered as a vector of predicted output disturbance extended into the prediction horizon, i.e.:

$$\hat{Y}(k+1) = \hat{Y}^o(k+1) + \hat{D}(k+1|k) \qquad (39)$$

Where,

$$\hat{D}(k+1|k) = [\ \hat{d}_{(k+1|k)}, \hat{d}_{(k+2|k)}, \cdots, \hat{d}_{(k+p|k)}\ ]^T$$

and $\hat{d}_{(k+j|k)}$ measns the estimation of $\hat{d}$ based on all available information up to the moment k.

1. $\hat{D}(k+1|k)$ for LRPM of conventional DMC:

Let us define :

$$\hat{d}^o_{(k|k)} = y_k - \hat{y}^o_{(k|k-1)}$$

and notice that:

$$\hat{Y}^o(k+1) = \Phi\ \hat{Y}^o(k|k-1) + \mathbf{A}\ \Delta U(k)$$

Thus, according to Eq.(30), we have:

$$\begin{aligned}\hat{y}^c_{k+j} &= \hat{y}^o_{k+j} + [y_k - \hat{y}^o_{(k|k-1)}] \\ &= \hat{y}^o_{k+j} + \hat{d}^o_{(k|k)}; \quad j > 0\end{aligned}$$

Thus, we conclude that:

$$\hat{Y}^c(k+1) = \hat{Y}^o(k+1) + [\mathbf{1}]_{p\times 1}\ \hat{d}^o_{(k|k)} \tag{40}$$

So that

$$\hat{D}(k+1|k) = [\mathbf{1}]_{p\times 1}\ \hat{d}^o_{(k|k)} \tag{41}$$

and

$$\hat{d}_{(k+i|k)} = \hat{d}^o_{(k|k)}, \qquad i = 1, ...p$$

Therefore, the future values of disturbance at the moment k in this LRPM are estimated based on zero-order extrapolation from the value at this current instant, k.

2. $\hat{D}(k+1|k)$ for LRPM of Morari and Lee:

By the assumption of a perfect model for the process and no disturbance entering at the output before instant $k-1$, we can start with assigning:

$$\hat{Y}(k|k-1) = \hat{Y}^o(k|k-1) = Y(k)$$

and

$$y_k - \hat{y}_{(k|k-1)} = y_k - \hat{y}^o_{(k|k-1)} = \hat{d}^o_{(k|k)}$$

By Eq.(32), we have:

$$\begin{aligned}\hat{y}^*_{(k|k)} &= \hat{y}_{(k|k-1)} + f\ \hat{d}^o_{(k|k)} = \hat{y}^o_{(k|k-1)} + f\ \hat{d}^o_{(k|k)} \\ \hat{y}^*_{(k+1|k)} &= \hat{y}_{(k+1|k-1)} + f\ \hat{d}^o_{(k|k)} = \hat{y}^o_{(k+1|k-1)} + f\ \hat{d}^o_{(k|k)} \\ \hat{y}^*_{(k+2|k)} &= \hat{y}_{(k+2|k-1)} + f\ \hat{d}^o_{(k|k)} = \hat{y}^o_{(k+2|k-1)} + f\ \hat{d}^o_{(k|k)} \\ \hat{y}^*_{(k+3|k)} &= \hat{y}_{(k+3|k-1)} + f\ \hat{d}^o_{(k|k)} = \hat{y}^o_{(k+3|k-1)} + f\ \hat{d}^o_{(k|k)} \\ . & \quad \cdots\end{aligned}$$

In other words, we have:

$$\hat{Y}^*(k|k) = \hat{Y}(k|k-1) + f\ [\ \mathbf{1}\ ]_{p\times 1}\ \hat{d}^o_{(k|k)} = \hat{Y}^o(k|k-1) + f\ [\ \mathbf{1}\ ]_{p\times 1}\ \hat{d}^o_{(k|k)}$$

and, according to Eq.(33), we have:

$$
\begin{aligned}
\hat{y}_{(k+1|k)} &= \hat{y}^*_{(k+1|k)} + a_1 \Delta u_k = \hat{y}^o_{(k+1|k)} + f\ \hat{d}^o_{(k|k)} \\
&= \hat{y}^o_{k+1} + f\ \hat{d}^o_{(k|k)} \\
\hat{y}_{(k+2|k)} &= \hat{y}^*_{(k+2|k)} + a_2 \Delta u_k = \hat{y}^o_{(k+2|k)} + f\ \hat{d}^o_{(k|k)} \\
\hat{y}_{(k+3|k)} &= \hat{y}^*_{(k+3|k)} + a_3 \Delta u_k = \hat{y}^o_{(k+3|k)} + f\ \hat{d}^o_{(k|k)} \\
. &\quad \dots \\
\hat{y}_{(k+j|k)} &= \hat{y}^*_{(k+j|k)} + a_j \Delta u_k = \hat{y}^o_{(k+j|k)} + f\ \hat{d}^o_{(k|k)}
\end{aligned}
\tag{42}
$$

or,

$$
\hat{Y}(k+1|k) = \Phi \hat{Y}^*(k|k) + \mathbf{F}\ \Delta u_k = \hat{Y}^o(k+1|k) + [\mathbf{1}]_{p\times 1}\ f\ \hat{d}^o_{(k|k)}
$$

Therefore,

$$
\begin{aligned}
\hat{Y}^c(k+1) &= \hat{Y}^o(k+1|k-1) + [\ \mathbf{1}\ ]_{p\times 1}\ f\ \hat{d}^o_{(k|k)} + \mathbf{A}\ \Delta U(k) \\
&= \hat{Y}^o(k+1) + [\mathbf{1}]_{p\times 1}\ f\ \hat{d}^o_{(k|k)} \\
&= \hat{Y}^o(k+1) + [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k|k)}
\end{aligned}
\tag{43}
$$

Where,

$$
\hat{d}_{(k|k)} = f\ \hat{d}^o_{(k|k)}
$$

Similarly, we can move the time origin, which stands for the current moment, from k to $k' = k+1$ to give:

$$
\begin{aligned}
\hat{y}^*(k'|k') &= \hat{y}_{(k+1|k)} + f\ [y_{k+1} - \hat{y}_{(k+1|k)}] \\
&= \hat{y}^o_{k+1} + f\ \hat{d}^o_{(k|k)} + f\ [y_{k+1} - \hat{y}^o_{k+1} - f\ \hat{d}^o_{(k|k)}] \\
&= \hat{y}^o_{(k+1|k)} + f\ \hat{d}^o_{(k+1|k+1)} + f(\ 1-f\ )\ \hat{d}^o_{(k|k)} \\
\hat{y}^*(k'+1|k') &= \hat{y}^o_{(k+2|k)} + f\ \hat{d}^o_{(k+1|k+1)} + f\ (\ 1-f\ )\ \hat{d}^o_{(k|k)} \\
\hat{y}^*(k'+2|k') &= \hat{y}^o_{(k+3|k)} + f\ \hat{d}^o_{(k+1|k+1)} + f\ (\ 1-f\ )\ \hat{d}^o_{(k|k)} \\
\dots &\quad \dots \\
\hat{y}^*(k'+j|k') &= \hat{y}^o_{(k+j+1|k)} + f\ \hat{d}^o_{(k+1|k+1)} + f\ (\ 1-f\ )\ \hat{d}^o_{(k|k)} \qquad (44) \\
\dots &\quad \dots
\end{aligned}
\tag{45}
$$

i.e.,

$$
\hat{Y}^*(k'|k') = \hat{Y}^o(k'|k'-1) + [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k'|k')}
$$

where,

$$\begin{aligned}\hat{d}_{(k'|k')} &= f\ \hat{d}^o_{(k'|k')} + f(\ 1-f\ )\ \hat{d}^o_{(k'-1|k'-1)} \\ &= f\ \hat{d}^o_{(k'|k')} + (1-f)\ \hat{d}_{(k'-1|k'-1)}\end{aligned}$$

Furthermore, we have:

$$\hat{Y}(k'+1|k') = \hat{Y}^o(k'+1|k') + [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k'|k')}$$

and

$$\begin{aligned}\hat{Y}^c(k'+1) &= \hat{Y}^o(k'+1|k'-1) + [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k'|k')} + \mathbf{A}\Delta U(k') \\ &= \hat{Y}^o(k'+1) + [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k'|k')}\end{aligned} \tag{46}$$

We can proceed the same procedure to obtain $\hat{d}(k'|k')$ at $k' = k+j$ for any j, i.e.:

$$\begin{aligned}\hat{d}_{(k'|k')} &= f\ \hat{d}^o_{(k'|k')} + f(1-f)\ \hat{d}^o_{(k'-1|k'-1)} \\ &\quad + f(1-f)^2\ \hat{d}^o_{(k'-2|k'-2)} \\ &\quad + f(1-f)^3\ \hat{d}^o_{(k'-3|k'-3)} \\ &\quad + \cdots + f(1-f)^j\ \hat{d}^o_{(k'-j|k'-j)} \\ &= f\ \hat{d}^o_{(k'|k')} + (1-f)\ \hat{d}_{(k'-1|k'-1)}\end{aligned} \tag{47}$$

where, $k' = k+j, \quad j \ = \ 1,\ 2,\ \cdots.$

Therefore, we can conclude:

$$\hat{d}_{(k'|k')} = \frac{1-\tilde{f}}{1-\tilde{f}q^{-1}} \hat{d}^o_{(k'|k')}; \quad \tilde{f} = 1-f$$

That means the estimated output disturbance, $\hat{d}_{(k'|k')}$ is the output of a first order filter deriven by $d^o_{(k'|k')}$.

As for prediction, i.e. $\hat{d}_{(k+j|k)}$, it is found from Eq.(46) that:

$$\hat{D}(k'+1|k') = [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k'|k')} \tag{48}$$

so that

$$\hat{d}_{(k'+1|k')} = \hat{d}_{(k'+2|k')} = \cdots = \hat{d}_{(k'|k')}$$

and

$$\hat{Y}^c(k'+1) = \hat{Y}^o(k'+1) + [\mathbf{1}]_{p\times 1}\ \hat{d}(k'|k') \tag{49}$$

Notice that Eq.(41) and Eq.(48) become identical, if f is taken as one. On the other hand, if $f = 0$, $\hat{Y}^c(k'+1)$ in Eq.(48) becomes $\hat{Y}^o(k'+1)$.

V. Analytical State-space LRPM

By Eq.(41) and Eq.(48), it is found that the future values of the disturbance estimated at each current moment k is considered by using constant extrapolation. As a result, the prediction scheme is similar to a Smith predictor of which the output disturbance is considered unchanged in the future.

To estimate $\hat{d}_{(k+j|k)}$ extend into the future time horizon, prediction of the output disturbance must be incorporated, i.e.:

$$\begin{aligned} \hat{d}_{(k+1|k)} &= p_1[\ \hat{d}^o_{(k|k)}\ ] \\ \hat{d}_{(k+2|k)} &= p_2[\ \hat{d}^o_{(k|k)}\ ] \\ \hat{d}_{(k+3|k)} &= p_3[\ \hat{d}^o_{(k|k)}\ ] \\ \cdots & \quad \cdots \\ \hat{d}_{(k+s|k)} &= p_s\ [\ \hat{d}^o_{(k|k)}\ ] \end{aligned}$$

The disturbance predictor, $p_i, i = 1, 2, 3, \cdots$ has to be formulated according to a dynamic model that can represent the changing disturbance.

Assume that the output disturbance is given as follows:

$$d_{k+1} = \psi\ d_k + (1-\psi)\ \ell_k$$

where, ℓ is a unknown input.

Let

$$\mathbf{S_1} = \begin{bmatrix} \psi \\ \psi^2 \\ \psi^3 \\ \cdot \\ \cdot \\ \psi^p \end{bmatrix}; \quad \mathbf{S_2} = \begin{bmatrix} 1 \\ \psi \\ \psi^2 \\ \psi^3 \\ \cdot \\ \cdot \\ \psi^{p-1} \end{bmatrix} (1-\psi); \quad \mathbf{I}_\Delta = \begin{bmatrix} 1 & 0 & \cdot & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 \\ 1 & 1 & \cdot & \cdots & 1 \end{bmatrix}$$

Assume that a step change of ℓ is introduced at $k-1$, and is held constant thereafter. It is, then, easy to obtain:

$$\hat{D}(k+1|k) = [\mathbf{1}]_{p\times 1}\ \hat{d}_{(k|k)} + \mathbf{I}_\Delta\ \{\ \mathbf{S}_1 \Delta \hat{d}_{(k|k)} + \mathbf{S}_2 \Delta \hat{\ell}_k \} \tag{50}$$

Where,

$$\Delta\ell(k) = \hat{d}_{(k|k)} - \psi \hat{d}_{(k-1|k-1)} = [1 - \psi q^{-1}]\ \hat{d}_{(k|k)}$$

Thus the prediction of $\hat{Y}^c(k+1)$ can be obtained according to the following:

- Updated Based on Measurement:

$$\hat{Y}^*(k|k) = \hat{Y}(k|k-1) + [\mathbf{1}]_{p\times 1}\ [y_k - \hat{y}_{(k|k-1)}] \tag{51}$$

- Model Prediction:

$$\hat{Y}(k+1|k) = \Phi \hat{Y}^*(k|k) + \mathbf{F}\Delta u_k \tag{52}$$

- Long Range Prediction:

$$\hat{Y}^c(k+1) = \Phi\ \hat{Y}^*(k|k) + \mathbf{A}\ \Delta U(k) + \mathbf{I}_\Delta\ \{\ \mathbf{S}_1 + \mathbf{S}_2\ (1 - \psi q^{-1})\}\ [y_k - \hat{y}_{(k|k-1)}] \tag{53}$$

According to the result in the previous section, we have:

$$\hat{d}_{(k|k)} = \hat{d}^o_{(k|k)} = y_k - \hat{y}^o_{(k|k-1)}$$

$$y_k - \hat{y}_{(k|k-1)} = \hat{d}^o_{(k|k)} - \hat{d}^o_{(k-1|k-1)} = \Delta \hat{d}^o_{(k|k)}$$

and

$$\hat{Y}(k+1|k) = \hat{Y}^o(k+1|k) + [\mathbf{1}]_{p\times 1}\ \hat{d}^o_{(k|k)}$$

Consequently, Eq.(54) becomes:

$$\hat{Y}^c(k+1) \quad = \quad \hat{Y}^o(k+1) + \hat{D}(k+1|k)$$

Where $\hat{D}(k+1|k)$ is given by Eq.(51). Under the assumption of:

$$\ell_{k-j} = 0\ ; \quad j \geq 2 \quad \textit{and} \quad \ell_{k+j} = \ell_{k-1}\ ; \quad j \geq 0$$

the estimation of $\hat{D}(k+1|k)$ of Eq.(51) can be shown to be equivalent to the analytical predictor of Won and Seborg [1986] as follows:

By the assumption, we have:

$$\Delta\hat{\ell}_{k+j} = 0 \quad for\ j \geq 0$$

$$\begin{aligned}
\hat{d}_{(k+1|k)} &= \hat{d}_{(k|k)} + \psi\Delta\hat{d}_{(k|k)} + (1-\psi)\ \Delta\hat{\ell}_k \\
&= \psi\ \hat{d}_{(k|k)} + [\hat{d}_{(k|k)} - \psi\ \hat{d}_{(k-1|k-1)}]
\end{aligned}$$

$$\begin{aligned}
\hat{d}_{(k+2|k)} &= \hat{d}_{(k|k)} + (\psi+\psi^2)\ \Delta\hat{d}_{(k|k)} + [(1-\psi)+\psi(1-\psi)]\ \Delta\hat{\ell}_k \\
&= \hat{d}_{(k|k)} + (\psi+\psi^2)\ [\hat{d}_{(k|k)} - \hat{d}_{(k-1|k-1)}] \\
&= \psi^2\ \hat{d}_{(k|k)} + \hat{d}_{(k|k)} - \psi\ \hat{d}_{(k-1|k-1)} + \psi\hat{d}_{(k|k)} - \psi^2\ \hat{d}_{(k-1|k-1)} \\
&= \psi^2\ \hat{d}_{(k|k)} + (1+\psi)\ [\hat{d}_{(k|k)} - \psi\hat{d}_{(k-1|k-1)}]
\end{aligned}$$

$$\begin{aligned}
\hat{d}_{(k+3|k)} &= \hat{d}_{(k|k)} + (\psi+\psi^2+\psi^3)\ \Delta\hat{d}_{(k|k)} + [(1-\psi)+\psi(1-\psi)+\psi^2(1-\psi)]\ \Delta\hat{\ell}_k \\
&= \hat{d}_{(k|k)} + (\psi+\psi^2+\psi^3)\ [\hat{d}_{(k|k)} - \hat{d}_{(k-1|k-1)}] \\
&= \psi^3\ \hat{d}_{(k|k)} + \hat{d}_{(k|k)} - \psi\ \hat{d}_{(k-1|k-1)} + \psi\ [\hat{d}_{(k|k)} - \psi\ \hat{d}_{(k-1|k-1)}] \\
&\quad +\psi^2\ [\hat{d}_{(k|k)} - \psi\ \hat{d}_{(k-1|k-1)}] \\
&= \psi^3\ \hat{d}_{(k|k)} + [1+\psi+\psi^2]\ [\hat{d}_{(k|k)} - \psi\ \hat{d}_{(k-1|k-1)}]
\end{aligned}$$

Thus, by the same way, we have:

$$\hat{d}_{(k+n|k)} = \psi^n\ \hat{d}_{(k|k)} + \frac{1-\psi^n}{1-\psi}(1-\psi q^{-1})\ \hat{d}_{(k|k)}$$

which is known as the analytical predictor of Wong and Seborg [1986].

It has to be noticed that $\hat{Y}^o(k+1)$ can also be obtained from the FIR, FSR formula, i.e.:

$FIR:$

$$\hat{Y}^o(k+1) = \mathbf{H}_1 \; \delta U(k) + \mathbf{A}^* \; U^*(k-1) + [\mathbf{1}]_{p\times 1} \; a_{N_m} \; u_{k-N_m} \tag{54}$$

$FSR:$

$$\hat{Y}^o(k+1) = \mathbf{A} \; \Delta U(k) + \mathbf{A}^* \; U^*(k-1) + [\mathbf{1}]_{p\times 1} \; a_{N_m} \; u_{k-N_m} \tag{55}$$

VI. Model Predictive Control based on the LRPMs

We have shown that the LRPM can generally be expressed as:

$$\hat{Y}^c(k+1) = \hat{Y}^o(k+1) + \hat{D}(k+1|k)$$

The major difference between each form of LRPM is the modeling of $\hat{D}(k+1|k)$ at each different moment k. For example, the LRPM of conventional DMC uses:

$$\hat{D}(k+1|k) = [\mathbf{1}]_{p\times 1}\ \hat{d}^o_{(k|k)}$$

while the LRPM of Morari and Lee uses:

$$\hat{D}(k+1|k) = [\mathbf{1}]_{p\times 1}\ \frac{1-\tilde{f}}{1-\tilde{f}q^{-1}}\ \hat{d}^o_{(k|k)}$$

And, finally, the Analytical State-space LRPM uses:

$$\hat{D}(k+1|k) = \begin{bmatrix} \psi + (1-\psi)\ \frac{1-\psi q^{-1}}{1-\psi} \\ \psi^2 + (1-\psi^2)\ \frac{1-\psi q^{-1}}{1-\psi} \\ \cdot \\ \cdot \\ \cdot \\ \psi^p + (1-\psi^p)\ \frac{1-\psi q^{-1}}{1-\psi} \end{bmatrix} \hat{d}^o_{(k|k)};$$

Based on those LRPM, derivation for the model predictive control algorithm is then straightforward. As has been mentioned, all expressions for $\hat{Y}^o(k+1)$ are equivalent, we will adopt the FSR representation for $\hat{Y}^o(k+1)$ in the following, since it is more popular to those who fimilar with MPC.

$$\begin{aligned} \hat{Y}^c(k+1) &= \hat{Y}^o(k+1) + \hat{D}\ (k+1|k) \\ &= \mathbf{A}\ \Delta U(k) + \mathbf{A}^*\ U^*(k-1) + \hat{D}(k+1|k) + [\mathbf{1}]_{p\times 1}\ a_{N_m}\ u_{k-N_m} \end{aligned} \quad (56)$$

The control objective is to minimize the performance index of the following:

$$J = \left[R(k+1|k) - \hat{Y}^c(k+1)\right]^T \mathbf{Q}_1 \left[R(k+1|k) - \hat{Y}^c(k+1)\right] + \Delta U^T(k)\ \mathbf{Q}_2\ \Delta U(k) \quad (57)$$

and subject to the constraints:

$$\begin{aligned}
&U_{min} \leq U(k) \leq U_{max} \\
&\Delta U_{min} \leq \Delta U(k) \leq \Delta U_{max} \\
&Y_{min} \leq \hat{Y}(k+1) \leq Y_{max}
\end{aligned} \tag{58}$$

The $R(k+1|k)$ designates the reference trajectory generated according to all the information up to k.

The control law for the above model predictive control system becomes:

$$\Delta u_k = [\ 1\ 0\ 0\ \cdots\ 0]\ \ \Delta U(k) \tag{59}$$

For unconstrained case, an explicit form for the control law, Δu_k can be easily obtained:

$$\Delta u_k = [\ 1\ 0\ 0\ \cdots\ 0\ ]\ \left[\ \mathbf{A}^T \mathbf{Q}_1 \mathbf{A} + \mathbf{Q_2} \right]^{-1} \mathbf{A}^T\ \mathbf{Q}_1\ E^* \tag{60}$$

Where,

$$E^* = R(k+1|k) - \mathbf{A}^*\ U^*(k-1) - \hat{D}(k+1|k)$$