Chapter 2

Linear Algebra

Objective

Demonstrate solution methods for systems of linear equations. Show that a system of equations can be represented in matrix-vector form.

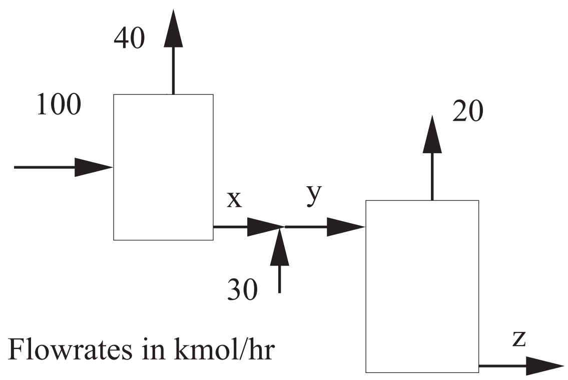

Figure 2.1: Two distillation columns in series.

2.1 Example System

Two distillation columns in series with a additional feed stream mixing in with the bottoms stream of the first column. The flow rate of three streams are unknown. As indicated in the Figure 2.1, the flow rate of streams x, y, and z are unknown. No reaction is taking place. The steadystate flow rates must be calculated.

Basic Mass Balance:

$$accumulation = in - out + created - destroyed$$

Mass Balance on first column (In this case, assume steady state: $accumulation = 0$):

$$0 = 100 - 40 - x$$

Mass balance on mixing point:

$$0 = x + 30 - y$$

Mass balance on second column:

$$0 = y - 20 - z$$

Three linear equations:

$$\begin{aligned} 0 &= 100 - 40 - x \\ 0 &= x + 30 - y \\ 0 &= y - 20 - z \end{aligned}$$

Note that you could write too many equations. You could write an overall balance:

$$0 = 100 - 40 - 20 - z$$

Ending up with an overspecified system of equations, 4 equations, 3 unknowns. Stick with the three equations from above for now.

Note that these are linear equations. The unknown variables have constant linear coefficients, nonlinear terms do not appear (no x^2, no $\sqrt{x}$, no e^x).

You can rearrange the set of three equations (without the overall balance equation) to get all the variable terms on the left side and the constants on the right. After some The set of equations can be written as:

$$\begin{aligned} 1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20 \end{aligned} \tag{2.1}$$

As we will see later, this can be more compactly written as:

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

You may already realize that the solution to this problem is $x = 60$, $y = 90$, and $z = 70$. For more complex systems, this is not quite so easy. To solve the three linear equations simultaneously in a general manner, you can perform row reduction using three possible row operations:

RULES

1. Add (or subtract) one row to (or from) another
2. Multiply or divide a row by a scalar value (any real scalar$\neq 0$)
3. Swap position of rows

Typically you would perform these operations until you have a triangular representation (all 0's above or below the diagonal). The triangular form allows for quick solution.

The set of linear equations in Equation 2.1 can be compactly written using only the coefficients as:

$$\begin{array}{rrr|r} 1 & 0 & 0 & 60 \\ -1 & 1 & 0 & 30 \\ 0 & -1 & 1 & -20 \end{array}$$

We need to perform steps 1-3 to get the system of equations in triangular form with ones on the diagonal and zeros below the diagonal, like

$$\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}$$

We can look at the original system of equations and realize that we must get zeros in position 2,1 (row 2, column 1) and position 3,2 (row 3, column 2). You can multiply row 2 by -1 using Rule 2:

$$\begin{array}{rrr|r} 1 & 0 & 0 & 60 \\ 1 & -1 & 0 & -30 \\ 0 & -1 & 1 & -20 \end{array}$$

Next, swap position of rows 2 and 3 using Rule 3 to get:

$$\begin{array}{rrr|r} 1 & 0 & 0 & 60 \\ 0 & -1 & 1 & -20 \\ 1 & -1 & 0 & -30 \end{array}$$

Then, subtract row 1 from row 3 using Rule 1 to get:

$$\begin{array}{rrr|r} 1 & 0 & 0 & 60 \\ 0 & -1 & 1 & -20 \\ 0 & -1 & 0 & -90 \end{array}$$

Then, multiply rows 2 and 3 by -1 using Rule 2:

$$\begin{array}{rrr|r} 1 & 0 & 0 & 60 \\ 0 & 1 & -1 & 20 \\ 0 & 1 & 0 & 90 \end{array}$$

Subtract row 2 from row 3 using Rule 1 again to get:

$$\begin{array}{rrr|r} 1 & 0 & 0 & 60 \\ 0 & 1 & -1 & 20 \\ 0 & 0 & 1 & 70 \end{array}$$

Now, all coefficients below the diagonal are 0. The solution can be found quickly. From equation 3 (row 3), $z = 70$. Using equation 2 (row 2) $y - z = 20$, but you know that $z = 70$ so $y = 90$. Equation 1 (row 1) gives $x = 60$, so the overall solution is $x = 60$, $y = 90$, and $z = 70$.

CHECK SOLUTIONS: You can plug your solution back into the original three equations and verify that the equations are satisfied. **THIS WILL HELP YOU ON EXAMS.**

Note that the general Gaussian elimination or row reduction method specifies that you start with column 1 and perform operations until all coefficients below the diagonal are 0, then move to column 2 and perform operations until all coefficients below the diagonal are zero, etc.

2.2 Linear Equations - Special Cases

In general, there are three possibilities for a "square" set of linear equations.

2.2.1 Case A - One solution

Consider a simpler system: $x + y = 1$ and $x - y = 1$. Graphically, you can plot the two lines and look for the intersection of two lines which occurs at $x = 1$, $y = 0$. The system of equations is:

$$\begin{array}{rr|l} 1 & 1 & 1 \\ 1 & -1 & 1 \end{array}$$

Subtracting row 1 from row 2 gives:

$$\begin{array}{rr|l} 1 & 1 & 1 \\ 0 & -2 & 0 \end{array}$$

This implies $-2y = 0$ or $y = 0$ and $x + y = 1$ or $x = 1$ as you already realized.

In 3 dimensions (3 unknowns) each row represents a plane. Two equations can intersect to give a line, and a line can intersect with a third plane to give a point, the single solution (in a single solution case).

2.2.2 Case B - No solution

Consider the system $x + y = 1$ and $x + y = 2$. Graphically, this represents two lines that never intersect.

$$\begin{array}{rr|l} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}$$

Note that column 1 and column 2 are identical. Subtracting row 1 from row 2 gives:

$$\begin{array}{rr|l} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}$$

You know that $0x + 0y = 1$ cannot be true. For a "square" system, if Gaussian elimination results in a 0 on the diagonal, this may be the case.

2.2.3 Case C - Many solutions

Consider the system $x + y = 1$ and $2x + 2y = 2$. Graphically, this represents two lines that are coincident.

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}$$

Subtracting twice the value of row 1 from row 2 gives:

$$\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}$$

These equations are consistent. $0x + 0y = 0$ and $x + y = 1$ are consistent. There is no single solution, as many solutions make the equation $x + y + 1$ consistent.

2.3 Nonsquare Systems

The original example was for a "square" system with 3 unknowns and 3 equations. You may often end up with more (or fewer) equations than unknowns.

Consider the original set of equations:

$$\begin{aligned} 1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20 \end{aligned}$$

One additional equation can be specified using a mass balance on the entire system, $0 = 100 + 30 - 40 - 20 - z$.

$$\begin{aligned} 1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20 \\ 0x + 0y + 1z &= 70 \end{aligned} \tag{2.2}$$

These four linear equations are not "linearly independent." You can test this by using row operations to make two rows identical. Simultaneously adding row 1 and row 3 to row 2 will make row 2 the same as row 4.

$$\begin{aligned} 1x + 0y + 0z &= 60 \\ 0x + 0y + 1z &= 70 \\ 0x - 1y + 1z &= -20 \\ 0x + 0y + 1z &= 70 \end{aligned} \tag{2.3}$$

This set of equations can still be satisfied using the original solution $x = 60$, $y = 90$, and $z = 70$. In other cases, having more equations than unknowns may complicate the solution process a bit.

2.3.1 Reconciliation and Nonsquare Systems

For curve fitting, parameters that appear linearly can be formulated as a nonsquare solution to a linear algebraic system of equations. Given that you have some (scalar valued) measured value, y, that depends on a process parameter, x. Assume the model takes the form:

$$y = mx + b \tag{2.4}$$

Technically, you only need two data points to find m and b, the model parameters. Assuming that you have more than two data points, we often desire to determine the "best-fit" for the line. These parameters minimize the sum of the square of the model error. For an experiment with four data points:

$$\begin{aligned} y(1) &= m\,x(1) + b \\ y(2) &= m\,x(2) + b \\ y(2) &= m\,x(3) + b \\ y(4) &= m\,x(4) + b \end{aligned} \tag{2.5}$$

Here, you know values of y and x but m and b are your unknown values. This can be written as a set of equations:

$$\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix} = \begin{bmatrix} x(1) & 1 \\ x(2) & 1 \\ x(3) & 1 \\ x(4) & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

You can get the "best-fit" solution to this overspecified set of equations using the psuedo-inverse of the matrix:

$$x = (A^T A)^{-1} A^T b$$

2.4 Vectors

A group of unknown (or known) values can be "stacked" to form a vector. In the example problem, the unknowns x, y, and z can be described by the vector $\underline{x}$:

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The solution to the problem has a known value and can be written as a vector $\underline{x}_{soln}$:

$$\underline{x}_{soln} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

Note that the underbar is used to distinguish between $\underline{x}$ (the vector) and x the unknown. A vector is NOT limited to 2 or 3 unknowns (dimension of the vector).

2.5 The Matrix

A matrix is similar to a vector, having 2 dimensions. One may think of it as a group of vectors augmented together. A Matrix has a size, $m \times n$ representing m rows and n columns. The values for m and n are sometimes written as subscripts for the matrix. For example, the 2x3 matrix $\underline{\underline{A}}_{2\times 3}$ with two rows and three columns may have values:

$$\underline{\underline{A}}_{2\times 3} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

Note that each of the six elements has two indices. The first index is the row, the second is the column. For the applications in this class, a matrix will have constant coefficient values. Some example matrices:

$$\underline{\underline{A}}_{2\times 3} = \begin{bmatrix} 0 & -2 & 1 \\ 5 & 1 & 0.2 \end{bmatrix} \; \underline{\underline{B}}_{3\times 3} = \begin{bmatrix} 6 & 0 & 0 \\ -2 & 0 & -1 \\ 3 & -1 & 5 \end{bmatrix}$$

Square Matrix - A matrix with indices equal ($m = n$).

Note: A vector can be seen as a special matrix having only 1 column.

Transpose - The transpose operator swaps the indices of a matrix (or vector). For example, for $\underline{\underline{A}}_{2\times 3}$ as before:

$$\left(\underline{\underline{A}}_{2\times 3}\right)^T = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix}$$

Example. For the matrix $\underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\underline{\underline{A}}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Finally, one can take the transpose of a vector. For $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\underline{x}^T = [x\, y\, z] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T$$

Row Vector - The transpose of a vector is also known as a row vector.

Dot Product - The dot product of two vectors is the sum of the product of the elements taken individually. Examples:

$$\underline{x} \cdot \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + y^2 + z^2$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1x + 2y + 3z$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$$

Matrix Multiplication - Two matrices can be multiplied together. For example $\underline{\underline{A}}_{m \times n}$ can be multiplied by $\underline{\underline{B}}_{n \times j}$. Matrix $\underline{\underline{A}}$ has m rows and n columns, while $\underline{\underline{B}}$ has n rows and j columns.

$$\underline{\underline{A}}_{m \times n} = \begin{bmatrix} \dots & r_1 & \dots \\ \dots & r_2 & \dots \\ & \vdots & \\ \dots & r_m & \dots \end{bmatrix}$$

Here, each row up to r_m is a row vector with n elements.

$$\underline{\underline{B}}_{n \times j} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ c_1 & c_2 & \dots & c_j \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Here, each column up to column c_j is a vector (column vector) with n elements. To compute $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$ or simply $\underline{\underline{A}} \times \underline{\underline{B}}$ or just $\underline{\underline{A}}\ \underline{\underline{B}}$

$$\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j} = \begin{bmatrix} r_1^T \cdot c_1 & r_1^T \cdot c_2 & \dots & r_1^T \cdot c_j \\ r_2^T \cdot c_1 & r_2^T \cdot c_2 & \dots & r_2^T \cdot c_j \\ \vdots & \vdots & & \vdots \\ r_m^T \cdot c_1 & r_m^T \cdot c_2 & \dots & r_m^T \cdot c_j \end{bmatrix}$$

Method - To compute $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$, the result will have j columns. The first column of the result is computed by taking the dot product of $\underline{\underline{B}}_{1 \times j}$ (first column of $\underline{\underline{B}}$) with the transpose of all the rows of $\underline{\underline{A}}$. The second column of the result is computed by taking the dot product of $\underline{\underline{B}}_{2 \times j}$ (second column of $\underline{\underline{B}}$) with the transpose of all the rows of $\underline{\underline{A}}$. Repeat up to the j^{th} column of $\underline{\underline{B}}$ which produces the j^{th} column of the result.

Note: In general, $\underline{\underline{A}}\ \underline{\underline{B}} \neq \underline{\underline{B}}\ \underline{\underline{A}}$.

Conformable - In order to multiply $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$ the "inner" dimensions must be equal. In $\underline{\underline{A}}_{m \times n} \underline{\underline{B}}_{n \times j}$, if the first matrix has n columns and the second matrix must n rows.

Matrix Multiplication Examples:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 + 10 \\ 4 + 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+2y \\ x+y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 4-6 & 3 \\ 2+2 & -1 \\ 10+0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \\ 10 & 0 \end{bmatrix}$$

2.6 Column Example

Consider again the equations from the original distillation column example:

$$\begin{aligned} 1x + 0y + 0z &= 60 \\ -1x + 1y + 0z &= 30 \\ 0x - 1y + 1z &= -20 \end{aligned}$$

Notice that the variables (with constant coefficients) are on the left side and constant values are on the right hand side. This set of linear equations can be represented in the compact notation $\underline{\underline{A}}\,\underline{x} = \underline{b}$ where

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 60 \\ 30 \\ -20 \end{bmatrix}$$

Identity Matrix - The identity matrix has values of one on the diagonal and zeros elsewhere. It is defined as $\underline{\underline{I}}$ and for a square matrix $\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}$ and $\underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}$.

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.6.1 How to solve sets of linear equations

We need a solution to the matrix equation $\underline{\underline{A}}\,\underline{x} = \underline{b}$. You cannot "divide" by a matrix:

$$\underline{x} \neq \underline{b}/\underline{\underline{A}}$$

There is no "division" operator for a matrix. Instead, an inverse is defined for some square matrices such that

$$\underline{\underline{A}}\ \left(\underline{\underline{A}}\right)^{-1} = \underline{\underline{I}}$$

Also,

$$\left(\underline{\underline{A}}\right)^{-1}\ \underline{\underline{A}} = \underline{\underline{I}}$$

Now, to solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ for $\underline{x}$

First, multiply on the left by $\left(\underline{\underline{A}}\right)^{-1}$

$$\left(\underline{\underline{A}}\right)^{-1}\underline{\underline{A}}\,\underline{x} = \left(\underline{\underline{A}}\right)^{-1}\underline{b}$$

Realizing that $\left(\underline{\underline{A}}\right)^{-1}\ \underline{\underline{A}} = \underline{\underline{I}}$ replace $\left(\underline{\underline{A}}\right)^{-1}\ \underline{\underline{A}}$ with $\underline{\underline{I}}$.

$$\underline{\underline{I}}\,\underline{x} = \left(\underline{\underline{A}}\right)^{-1}\underline{b}$$

Now, realizing $\underline{\underline{I}}\,\underline{x}$ is $\underline{x}$, the solution is

$$\underline{x} = \left(\underline{\underline{A}}\right)^{-1}\underline{b}$$

Note that multiplying on the right will not lead to a solution.

$$\underline{\underline{A}}\,\underline{x}\ \left(\underline{\underline{A}}\right)^{-1} = \underline{b}\ \left(\underline{\underline{A}}\right)^{-1}$$

2.6.2 How determine a matrix inverse

To solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$, you need to know $\left(\underline{\underline{A}}\right)^{-1}$. We are going to use row reduction to calculate $\left(\underline{\underline{A}}\right)^{-1}$. Start with $\underline{\underline{A}} \mid \underline{\underline{I}}$. use row reduction techniques until $\underline{\underline{A}}$ is $\underline{\underline{I}}$. $\left(\underline{\underline{A}}\right)^{-1}$ if it exists will be on the right where $\underline{\underline{I}}$ was originally.

Inverse Example

Solve the following for $\underline{x}$ using $\left(\underline{\underline{A}}\right)^{-1}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\underline{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

For this procedure, one must first calculate $\left(\underline{\underline{A}}\right)^{-1}$. Set up $\underline{\underline{A}} \mid \underline{\underline{I}}$ as:

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}$$

Use row reduction to get

$$\begin{array}{cc|cc} 1 & 0 & ? & ? \\ 0 & 1 & ? & ? \end{array}$$

Then verify that $\underline{\underline{A}}\ \left(\underline{\underline{A}}\right)^{-1} = \underline{\underline{I}}$. Use $\left(\underline{\underline{A}}\right)^{-1}$ to calculate $\underline{x}$ using $\underline{x} = \left(\underline{\underline{A}}\right)^{-1}\underline{b}$. Verify solution again to be safe.

START
Start by using row reduction on

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}$$

Multiply row 2 by 1/3 to get :

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & \frac{4}{3} & 0 & \frac{1}{3} \end{array}$$

Then subtract row 1 from row 2 to get:

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -\frac{2}{3} & -1 & \frac{1}{3} \end{array}$$

Now, multiply row 2 by -3/2 to get:

$$\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}$$

To get the left side looking like the identity matrix, subtract 2 times row 2 from row 1. Note that this is a compound use of row reduction rules.

$$\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}$$

You now have $\left(\underline{\underline{A}}\right)^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Now verify that $\underline{\underline{A}} \left(\underline{\underline{A}}\right)^{-1} = \underline{\underline{I}}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1(-2) + 2(\frac{3}{2}) & 1(1) + 2(-\frac{1}{2}) \\ 3(-2) + 4(\frac{3}{2}) & 3(1) + 4(-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You may also verify that $\left(\underline{\underline{A}}\right)^{-1} \underline{\underline{A}} = \underline{\underline{I}}$

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2+3 & -4+4 \\ \frac{3}{2} - \frac{3}{2} & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, compute the solution, $\underline{x} = \left(\underline{\underline{A}}\right)^{-1} \underline{b}$.

$$\underline{x} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -10+6 \\ \frac{15}{2} - 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4\frac{1}{2} \end{bmatrix}$$

Again, verify the solution is the solution to the original equations:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underline{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 4\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -4+9 \\ -12+18 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Just as expected...

2.6.3 Steady State Control Example

Two pumps are used to fill two tanks. The pumps usually operate at 50%, keeping the tanks at levels of 75 inches and 80 inches respectively.It is known that a 1% increase in pump 1 increases the height of tank 1 by 5 inches and the height of tank 2 by 3 inches. For a 1% change in pump 2, the height of tank 2 increases by 4 inches. It is desired to change the operating levels of the tanks to 110 inches and 89 inches.

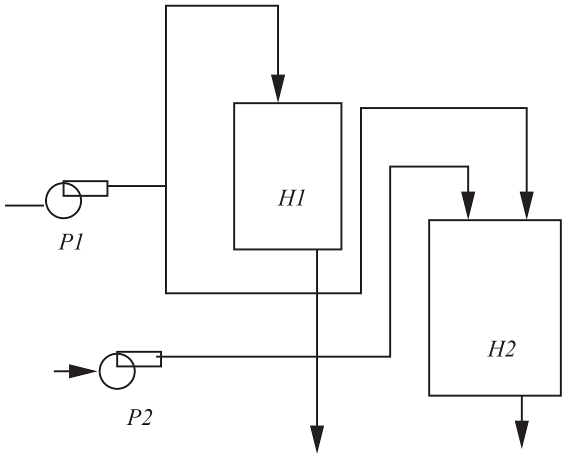

Figure 2.2: Pump / Tank example

What do you know:

$$\begin{aligned} 5\,\Delta P_1(\%) &= \Delta H_1(inches) \\ 3\,\Delta P_1(\%) + 4\,\Delta P_1(\%) &= \Delta H_2(inches) \end{aligned}$$

You know the target (reference, setpoint) for H_1 and H_2 as 110 and 89. This translates into $\Delta H_1 = 110 - 75 = 35$ and $\Delta H_2 = 89 - 80 = 9$. You need to increase tank 1 by 35 inches and increase tank 2 by 9 inches. You do not know the final values of the pump speeds. You do know the original steadystate values, 50% and 50%, realizing that:

$$P_{final} = P_{ss} + \Delta P$$

You can now set up linear equations to solve for ΔP_1 and ΔP_2, then calculate the final values for the pump speeds.

$$\begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} = \begin{bmatrix} \Delta H_1 \\ \Delta H_2 \end{bmatrix}$$

2.7 Visualization

Each row in $\underline{\underline{A}}\,\underline{x} = \underline{b}$ is a single linear equation. For a 2D problem ($\underline{x}$ with 2 elements / unknowns) the equation defines a line in the (x, y) plane. Two equations define two lines, and the unique solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$ is the point $\underline{x}$ where the lines intersect. In some cases, there may be many solutions to $\underline{\underline{A}}\,\underline{x} = \underline{b}$ and in some cases there may be no solutions to $\underline{\underline{A}}\,\underline{x} = \underline{b}$.

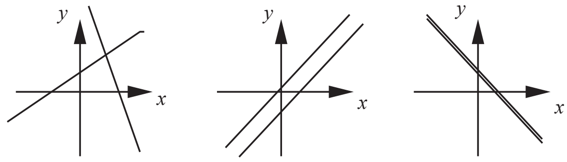

Figure 2.3: Three 2D examples with two equations. Each equation (row) represents a line. The first case has one solution, the second case has no solution, and the third case has many solutions.

For a 3D problem, each row defines the equation for a plane in 3 space. The intersection of 2 non-parallel planes is a line in 3 space, and the intersection of a line and a plane in 3 space is a point. Again, in some cases there may be a single solution, many solutions, or no solutions.

For higher dimensions, each equation defines a ***hyperplane*** in a n dimensional space, $\mathbb{R}^n$.

2.7.1 Linear Transform

A vector in $\mathbb{R}^n$ means x has n elements. Matrix multiplication of a matrix of size $m \times n$ times a vector of size $n \times 1$ "maps" the vector from $\mathbb{R}^n$ to $\mathbb{R}^m$.

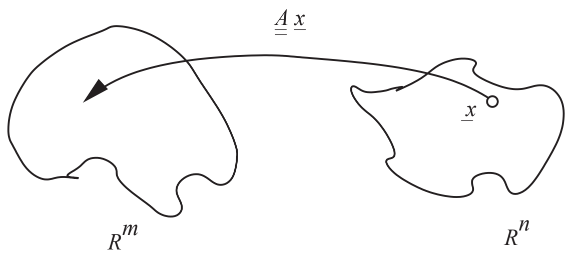

Figure 2.4: Matrix multiplication as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

2.7.2 Range

The range of a matrix is the space of all possible points that may be mapped to in a matrix multiplication of that matrix times an unknown vector.

Range Example 1

For example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

can only map to points on the line $x + y$ in 3D as follows.

$$\underline{\underline{A}}\,\underline{x} = 2x + 2y + 0z$$

The columns of the matrix define possible directions for the matrix to transform a vector. In this example, columns 1 and 2 are the same, and column 3 is the zero vector. $\underline{\underline{A}}\,\underline{x}$ where $\underline{x}$ takes any real value will always be on the line defined by the direction $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Range Example 2

In another example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

can only map to a variety of points in 3D as follows.

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} z$$

Again, the columns of the matrix define possible directions for the matrix to transform a vector. In this example, only points in the directions of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can be reached when multiplying $\underline{\underline{A}}\,\underline{x}$. These two directions form a plane in 3 dimensional space.

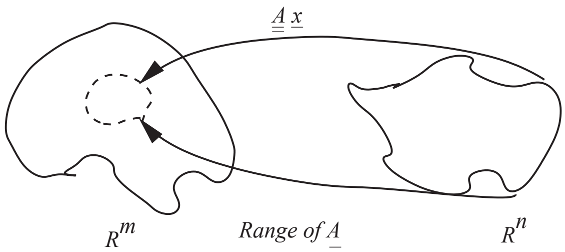

Figure 2.5: Range of $\underline{\underline{A}}$ as space in $\mathbb{R}^m$ of all possible mappings from $\mathbb{R}^n$ using matrix multiplication.

Range Example 3

In another example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

can only map to a variety of points in 3D as follows.

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} z$$

Here, column 3 is linearly dependent upon columns 1 and 2. This means that you can find some combination of columns 1 and 2 that give column 3. Column 3 lies in the plane defined by columns 1 and column 2.

Underlying point: For $\underline{\underline{A}}\,\underline{x} = \underline{b}$ to have a solution, the $\underline{b}$ must be in the range of $\underline{\underline{A}}$.

For the last examples, if $\underline{b} = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix}$ (if $\underline{b}$ has element in the z position) there will not be a solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$. In such a case, the possible range of $\underline{\underline{A}}$ does not include $\underline{b}$.

Range Example 4

In another example, the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

can map to all of the points in 3D as follows.

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} z$$

Here, column 3 is NOT linearly dependent upon columns 1 and 2. This means that you can find some combination of columns 1, 2, and 3 that give any point in 3 dimensions.

Rank - The rank of a matrix is the number of linearly independent columns. For a square matrix of size $n \times n$, there is a unique solution if there are n independent columns. The matrix would have rank n.