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Preface

This is a collection of class notes, handouts, homework assignments, and exam problems developed over the past years teaching courses in Process Systems Engineering at the University of South Carolina. Most of the material relates to ECHE 550, Chemical Process Dynamics and Control. This course covers the basics of dynamic modeling, solution and analysis of ordinary differential equations using Laplace methods, feedback control, and some advanced control topics. Information is also included from other courses, specifically ECHE 589 Intermediate Process Control. The intermediate course includes more advanced topics, such as numerical optimization and discrete time dynamic modeling.

This offering is not provided as a text book for a course. Many important topics are not covered in sufficient detail, while some derivations are provided in excruciating depth. This is expected to provide extra depth and additional examples for topics that may be lacking in other text books. Additionally, practice problems are provided and tutorial materials on Matlab/Simulink are included.

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Chapter 1
Mathematics Review

Objectives

This is a review of various mathematical topics that you probably have seen in previous mathematics courses. Complete the problems where indicated with “EXERCISE”. Some topics are just mentioned, without specific review questions.

Function of One Variable

You should understand the basic concept of a mathematical function / algebraic function. In this course, we examine process dynamics. Things change with time, so some value like the pressure in a tank, \( P \) could be some function of time, \( P(t) \) or \( P(t) = f(t) \), or in a specific case \( P(t) = 5\sin(3t + 2) \). In other cases, you could have a parameter that changes with temperature, like a chemical reaction rate. This could be expressed as \( r(T) = k_0 e^{\frac{E}{RT}} \) where \( E, k, \) and \( R \) are assumed constant.

The mathematical function provides a mapping. A scalar function maps one value to another value. \( f(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \). This can be considered a input-output relationship. Some people also use the analogy of a “black-box” you put some number in (the independent variable) and another comes out (the dependent variable).

Some examples as functions of time:
\[
\begin{align*}
  f(t) &= \sin(5t) \\
  f(t) &= 2t \\
  f(t) &= e^{3t}
\end{align*}
\]

Additional examples including constants
\[
\begin{align*}
  f(t) &= ce^{\tau}, \ \tau > 0, \ c > 0 \\
  \text{Reaction rate } r \text{ as a function of temperature } T: \\
  r(T) &= k_0 e^{\frac{E}{RT}}
\end{align*}
\]

Note that \( k_0, R, \) and \( E \) are constants.

You should know how to graph functions without the use of a calculator or computer, specifically any function of time (time as independent variable). For some functions, you may want to pick a variety of values of \( t \) and evaluate the function values, then graph \( f(t) \) vs. \( t \). For a sum of functions \( f = f_1 + f_2 \) you can plot \( f_1 \) and \( f_2 \) and add them point by point. When you multiply two functions, \( f = f_1 f_2 \) you can graph \( f_1 \) and \( f_2 \)
then multiply them at each point. In many cases, you need only graph the “interesting” points of the response where something significantly changes. Interesting points could be at $t = 0, t = 1, or t = \infty$. For trigonometric functions, multiples of $\pi/2$ may be “interesting”.

1. **EXERCISE, graph the following functions BY HAND:**

   (a) $f(t) = e^{3t}$  
   (b) $f(t) = e^{-3t}$  
   (c) $f(t) = e^{-0.3t}$  
   (d) $f(t) = \sin(t) + 2t$  
   (e) $f(t) = 2t + t^2$  
   (f) $f(t) = e^t - t^3 + \sin(t) - 1$  
   (g) $f(t) = t(\sin(t))$

**Function of Two or More Variables**

Sometimes, a value will be a function of multiple different values. Again, the mathematical function provides a mapping. A function can also map one value to another value. $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, $n > 0$.

Example, in a topographic (elevation) map, elevation is a function of map position: $ELEVATION = z = f(x, y)$ or $z = -(x^2 + y^2)$.

Example, reaction rate expression as a function of concentrations and temperature: $r = f(C_A, C_B, T) = 3.0 \, e^{\frac{A}{RT}} \, C_A^2 \, C_B$

Example, distance from the point $(4, 2)$  
$d = f(x, y) = \sqrt{(x - 4)^2 + (y - 2)^2}$

2. **Exercise:**

   (a) What is the function describing points on a circle of radius $r$ as a function of $x$ and $y$?  
   (b) What is the function describing points on a sphere of radius $r$ as a function of $x, y,$ and $z$?  
   (c) What is the function that determines the distance from the point $(3, -1, 2)$?  
   (d) Assuming an ideal gas, what is the function for pressure of a gas as a function of volume, temperature, and moles of gas?  
   (e) Given that you have a function of only two variables with points in $x \times y$ and a specified function of $x$ and $y$, you should realize that $f(x, y)$ gives values that can be plotted in 3 dimensions. This surface (manifold) specifies the function. Try to sketch $z = f(x, y) = x^2 + y^2$ in 3D.
Solving Equations of One Variable

If you have a function of one variable, you may be able to find a solution to the equation \( f(x) = 0 \). This means you find a value of \( x \) that satisfies the equation. The values of \( x \) that satisfy the equation are also called the roots of the equation.

Sometimes you can easily solve the equation analytically. This means that you get a closed-form expression for the solution that satisfies the equation. For the function \( f(x) = x^3 \), \( x = 0 \) is the solution to the equation. For the quadratic equation, \( ax^2 + bx + c \), the roots are \( x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \). YOU SHOULD KNOW THIS EQUATION. Note, imaginary roots do not mean that something is incorrect. In many process systems engineering problems, roots of a polynomial should have imaginary components.

In many cases, you may have a simple polynomial function that requires the roots to be found. This means, given \( f(x) \), what are values of \( x \) to make \( f(x) = 0 \)?

- In the general case, \((x - r_1)(x - r_2)\ldots(x - r_n) = 0\), roots = \( r_1, r_2, \ldots r_n \) that satisfy \( f(x) = 0 \)
- In the specific second order case, quadratic equation for: \( ax^2 + bx + c = 0 \) with 2 roots at \( x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \)
- There are analytical expressions for roots of polynomials up to fifth order, but they are in general very, very complex.

There are a variety of numerical methods to find roots. You may have a very complex nonlinear expression, \( f(x) \), that is not easily factored into roots or solved directly. To solve \( f(x) = 0 \) you can graph the expression, then examine the graph to locate zero crossings at the values of \( x \) that satisfy the function. Using the bisection method, you can evaluate the function at two points, \( x_L \) and \( x_R \), \( x_R \geq x_L \). Assuming that \( f(x_L) \leq 0 \) and \( f(x_U) \geq 0 \), you know a root must lie in the region \( x_L \leq x \leq x_R \). Bisect the region to find \( x_M = x_L + \frac{x_R-x_L}{2} \) and evaluate the function at \( x_M \). Update bounds, keeping the region that must contain a solution.

3. **EXERCISE**: Find analytically solutions to the following equations:

   (a) \( f(x) = (x - 3)(x^2 - x + 12) \)
   (b) \( f(x) = (x^2 + 6x + 8)(x - 4)x \)
   (c) \( f(x) = 2x^2 + 3x + 5 \)
   (d) \( f(x) = x^2 + x + 10 \)
   (e) \( f(x) = x^2 + 3x \)

4. **EXERCISE**: Find numerical solutions that satisfy the following equations.

   (a) \( f(x) = e^x - x^3 + \sin(x) - 1 \), multiple different solutions, \( x \) in radians
   (b) \( f(\omega) = \pi + \tan^{-1}(20\omega) - 2\omega \), \( \omega \) in radians
Solving Equations of Multiple Variables

In some cases, you have multiple unknown values. Using a degree of freedom analysis, you must have as many independent equations as unknown values in order to find a solution. For example, given the following equations:

\[ 5 = x + e^y \]
\[ 2 = yx \]

You can say that \( x = 5 - e^y \) using equation 1, then \( y = \frac{2}{x} \) or \( y = \frac{2}{5-e^y} \). Now, the second equation is \( f(y) = \frac{2}{5-e^y} - y \) which can be satisfied if \( f(y) = 0 \), so try to find \( y \) such that \( f(y) = 0 \), if it exists. Once you find a value for \( y \) that satisfies the function, you can determine values for \( x \) from equation 1. Alternatively, for a 2D nonlinear case, you can plot \( f_1(x, y) = 0 \) and \( f_2(x, y) = 0 \) and determine the points where the two lines intercept. In this example, there are two separate solutions.

5. **EXERCISE**: Find solutions to the following equations:

\[ 3 = x^2y \]
\[ 4 = x + \frac{1}{y} \]

Check your solution to make sure your values for \( x \) and \( y \) satisfy both equations.

6. **EXERCISE**: Find solutions to the following equations:

\[ 4 = x^2 + y^2 \]
\[ 0 = x^2 - y \]

**Slope of a Line**

You must be able to find the slope (derivative) of a function, \( \frac{df}{dt}(t) \) or \( \frac{df}{dx}(x) \) given the function and know the derivative of simple functions. Note that the derivative of a function of time is also a function of time! You should also remember how to use the chain rule!
7. **EXERCISE**, calculate the derivative of the following functions:

(a) \( f(t) = 3t^3 + 2t + 7 \)
(b) \( f(t) = \sqrt{t} \)
(c) \( f(t) = e^t \)
(d) \( f(t) = \sin(at) + 3t^2 \)
(e) \( f(x) = (e^{ax})^2 \)
(f) \( f(t) = \sin(3t^4) \)

The derivative evaluated at a point, \( \frac{df}{dt}(t) \bigg|_{t=ts} \) is the slope of the function \( f(t) \) at time \( t = ts \). This also defines the slope of the line tangent to \( f(t) \) at time \( t = ts \).

8. **EXERCISE**

(a) Graph \( f(t) = t^2 + t \) and find the value of function and the value of the slope for \( f(t) = t^2 + t \) at \( t = 0, t = -1, t = 1 \)

(b) Find the slope of the following line

![Graph showing valve position and steady state pressure](image)

**Basic Algebra Properties**

You should know how to solve basic equations using algebraic properties, such as the distributive property, \( ax + bx = (a + b)x \).

In some cases, you will have to solve an equation that includes a variety of constants. To solve for \( x \) in the equation \( ax = by + cx \) with constants \( a, b, c \). First, get terms with \( x \) on one side: \( ax - cx = by \), then use distributive property: \( (a - c)x = by \), finally divide to solve for \( x \) in terms of \( y \) and some constants: \( x = \frac{by}{(a-c)} \)
9. **EXERCISE**: Solve the following equation for \( f(x) = 0 \):

\[
f(x) = (2x + 5x) + 6x - x(2 + 3x)
\]

10. **EXERCISE**: Solve the following equation for \( x \).

\[
a x = xy + d + 3
\]

11. **EXERCISE**: Evaluate the follow fractions with different denominators just to make sure you know what a common denominator is.

   (a) \( \frac{2}{3} + \frac{5}{7} \)
   
   (b) \( \frac{1}{6} + \frac{2}{5} \)

**Partial Fraction Expansion**

Partial Fraction Expansion of fractions with polynomials in the numerator and denominator allows for simplification of complex polynomials. Use your preferred method simplify complex fractions involving polynomials.

12. **EXERCISE** Find \( A \) and \( B \) in the following expression

\[
\frac{5x + 2}{(2x + 1)(3x + 2)} = \frac{A}{2x + 1} + \frac{B}{3x + 2}
\]

**Determinant of a Matrix**

13. **EXERCISE**: find the determinant of the following matrices without using a calculator or computer:

   (a) 
   \[
   \begin{bmatrix}
   1 & 2 \\
   3 & 4 
   \end{bmatrix}
   \]

   (b) 
   \[
   \begin{bmatrix}
   1 & 2 \\
   2 & 4 
   \end{bmatrix}
   \]

   (c) 
   \[
   \begin{bmatrix}
   1 & 1 & 0 \\
   1 & 1 & 2 \\
   1 & 2 & 4 
   \end{bmatrix}
   \]

   (d) 
   \[
   \begin{bmatrix}
   0 & 1 & 2 \\
   4 & -1 & -1 \\
   -1 & -2 & 1 
   \end{bmatrix}
   \]
Multiple Linear Equations

You can find the solution of multiple linear equations by row reduction / Gaussian elimination. Linear equations are simple coefficients and variables (no $x^2$ terms, no $e^x$ terms, just $ax = b$ with $a$ and $b$ constant coefficients.) You have learned a variety of ways to solve systems of linear equations, but a standard method is often called row reduction or Gaussian elimination.

14. **EXERCISE:** Solve the following set of linear equations by hand:

\[
\begin{align*}
1x + 1y + 1z &= 0 \\
1x + 2y + 3z &= 1 \\
3x + 3y + 1z &= 2
\end{align*}
\]

Scalar values

Numbers can be a constant scalar (3, -0.1, $e$, $\pi$) or a variable scalar $(x, y, z)$. These are just basic real numbers.

Vector Values

There are many examples of vectors in 3 dimensions, \[
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.
\]

Note that you are not limited to 3 dimensions, you could specify all four concentrations in a reactor at a given time:

\[
\begin{bmatrix}
C_A \\
C_B \\
C_C \\
C_D
\end{bmatrix}
\]

Or you could specify all flow rates in a process at some time:

\[
[F_1 F_2 F_3 F_4 F_5 F_6 F_7 F_8]^T
\]

Partial Derivative / Gradient of a Multivariable Function

Partial derivative as slope of the tangent surface in direction of one variable.

15. **EXERCISES**

(a) What is $\frac{\delta f}{\delta x}$ of $f(x, y) = x^2 + y^2$?

(b) What is $\frac{\delta f}{\delta x}$ of $f(x, y) = x^2 + y^2$ evaluated at $x = 1$, $y = 1$?

(c) What is the gradient of $f(x, y) = x^2 + y^2$? 

\[
\begin{bmatrix}
\frac{\delta f}{\delta x} \\ 
\frac{\delta f}{\delta y}
\end{bmatrix}
\]
(d) What is the equation of the plane tangent to \( f(x, y) = x^2 + y^2 \) at \( x = 2, y = 1 \)?

Integration of a Function

16. **EXERCISE** Integrate the following basic functions:

(a) \( f(x) = x^2 \)
(b) \( f(x) = e^{ax} \)
(c) \( f(x) = \sin(ax) \)
(d) \( f(x) = \cos(bx) \)
(e) \( f(x) = \ln(x) \) (Integration by parts)
(f) \( f(x) = x^2e^x \) (Integration by parts)

Differential Equations

Basic differential equations such as:

\[
\frac{dy}{dx} = ay
\]

This can be solved by separation of variables,

\[
\frac{dy}{y} = adx
\]

Integrating to get \( \ln(y) = ax + c \). Assuming \( c = 0 \), \( y = e^{ax} \) where \( y \) is a function of \( x \) and the constant \( a \). The same differential equation can be put in the form

\[
\frac{df}{dt}(t) = af(t)
\]

with a solution \( f(t) = ce^{at} \). Given that you assume \( f(t) = ce^{at} \) so you can determine that \( \frac{df}{dt} = cae^{at} = af(t) \). Therefore \( f(t) = e^{at} \) is the differential equation solution. You can easily determine the value for \( c \) using the initial condition for \( f(t = 0) \), \( f(t = 0) = c e^{a0} = c = 1 = c \). If you also know the initial value for \( df/dt \) at time \( t = 0 \) you can find the value for \( a \).

\[
\frac{df}{dt}(t = 0) = af(t = 0)
\]

[17.] **EXERCISE**: Alternatively, you can collect data and determine values for \( a \) and \( c \). Assuming you have measurements \( y(t) \) at sample times \( t \) and \( y(t) \) is in the form \( y(t) = ce^{at} \). Find \( a \) and \( c \).
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**EXERCISE:** Assuming $t$ is in units of seconds and $y(t)$ is in units of cm, determine the units of $a$ and $c$.

18. **EXERCISE:** Go to http://www.ncsu.edu/felder-public/ILSdir/ilsweb.html and take the learning style test. **Record your four results.**
   Go to http://www.ncsu.edu/felder-public/ILSdir/styles.htm and read about your learning style.

19. **EXERCISE:** Fill out the Google Docs form at http://tinyurl.com/eche550form
Chapter 2
Linear Algebra

Objective

Demonstrate solution methods for systems of linear equations. Show that a system of equations can be represented in matrix-vector form.

\[ x \]
\[ y \]
\[ z \]

Figure 2.1: Two distillation columns in series.

2.1 Example System

Two distillation columns in series with a additional feed stream mixing in with the bottoms stream of the first column. The flow rate of three streams are unknown. As indicated in the Figure 2.1, the flow rate of streams \( x \), \( y \), and \( z \) are unknown. No reaction is taking place. The steadystate flow rates must be calculated.

Basic Mass Balance:

\[
\text{accumulation} = \text{in} - \text{out} + \text{created} - \text{destroyed}
\]

Mass Balance on first column (In this case, assume steady state: \( \text{accumulation} = 0 \)):

\[ 0 = 100 - 40 - x \]
Mass balance on mixing point:

\[ 0 = x + 30 - y \]

Mass balance on second column:

\[ 0 = y - 20 - z \]

Three linear equations:

\[
\begin{align*}
0 &= 100 - 40 - x \\
0 &= x + 30 - y \\
0 &= y - 20 - z
\end{align*}
\]

**Note that you could write too many equations.** You could write an overall balance:

\[ 0 = 100 - 40 - 20 - z \]

Ending up with an overspecified system of equations, 4 equations, 3 unknowns. Stick with the three equations from above for now.

**Note that these are linear equations.** The unknown variables have constant linear coefficients, nonlinear terms do not appear (no \( x^2 \), no \( \sqrt{x} \), no \( e^x \)).

You can rearrange the set of three equations (without the overall balance equation) to get all the variable terms on the left side and the constants on the right. After some The set of equations can be written as:

\[
\begin{align*}
1x + 0y + 0z &= 60 \\
-1x + 1y + 0z &= 30 \\
0x - 1y + 1z &= -20
\end{align*}
\]

As we will see later, this can be more compactly written as:

\[ A\hat{x} = \hat{b} \]

You may already realize that the solution to this problem is \( x = 60 \), \( y = 90 \), and \( z = 70 \). For more complex systems, this is not quite so easy. To solve the three linear equations simultaneously in a general manner, you can perform row reduction using three possible row operations:

**RULES**

1. Add (or subtract) one row to (or from) another
2. Multiply or divide a row by a scalar value (any real scalar\( \neq 0 \))
3. Swap position of rows
Typically you would perform these operations until you have a triangular representation (all 0’s above or below the diagonal). The triangular form allows for quick solution.

The set of linear equations in Equation 2.1 can be compactly written using only the coefficients as:

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 60 \\
-1 & \quad 1 & \quad 0 & \quad 30 \\
0 & \quad -1 & \quad 1 & \quad -20
\end{align*}
\]

We need to perform steps 1-3 to get the system of equations in triangular form with ones on the diagonal and zeros below the diagonal, like

\[
\begin{align*}
1 & \quad a & \quad b & \quad d \\
0 & \quad 1 & \quad c & \quad e \\
0 & \quad 0 & \quad 1 & \quad f
\end{align*}
\]

We can look at the original system of equations and realize that we must get zeros in position 2,1 (row 2, column 1) and position 3,2 (row 3, column 2). You can multiply row 2 by \(-1\) using Rule 2:

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 60 \\
1 & \quad -1 & \quad 0 & \quad -30 \\
0 & \quad -1 & \quad 1 & \quad -20
\end{align*}
\]

Next, swap position of rows 2 and 3 using Rule 3 to get:

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 60 \\
0 & \quad -1 & \quad 1 & \quad -20 \\
1 & \quad -1 & \quad 0 & \quad -30
\end{align*}
\]

Then, subtract row 1 from row 3 using Rule 1 to get:

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 60 \\
0 & \quad -1 & \quad 1 & \quad -20 \\
0 & \quad -1 & \quad 0 & \quad -90
\end{align*}
\]

Then, multiply rows 2 and 3 by \(-1\) using Rule 2:

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 60 \\
0 & \quad 1 & \quad -1 & \quad 20 \\
0 & \quad 1 & \quad 0 & \quad 90
\end{align*}
\]

Subtract row 2 from row 3 using Rule 1 again to get:

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 60 \\
0 & \quad 1 & \quad -1 & \quad 20 \\
0 & \quad 0 & \quad 1 & \quad 70
\end{align*}
\]

Now, all coefficients below the diagonal are 0. The solution can be found quickly. From equation 3 (row 3), \(z = 70\). Using equation 2 (row 2) \(y - z = 20\), but you know
that \( z = 70 \) so \( y = 90 \). Equation 1 (row 1) gives \( x = 60 \), so the overall solution is \( x = 60, y = 90, \) and \( z = 70 \).

CHECK SOLUTIONS: You can plug your solution back into the original three equations and verify that the equations are satisfied. **THIS WILL HELP YOU ON EXAMS.**

Note that the general Gaussian elimination or row reduction method specifies that you start with column 1 and perform operations until all coefficients below the diagonal are 0, then move to column 2 and perform operations until all coefficients below the diagonal are zero, etc.

### 2.2 Linear Equations - Special Cases

In general, there are three possibilities for a “square” set of linear equations.

#### 2.2.1 Case A - One solution

Consider a simpler system: \( x + y = 1 \) and \( x - y = 1 \). Graphically, you can plot the two lines and look for the intersection of two lines which occurs at \( x = 1, y = 0 \). The system of equations is:

\[
\begin{array}{c|c}
1 & 1 \\
1 & -1 \\
\end{array}
\]

Subtracting row 1 from row 2 gives:

\[
\begin{array}{c|c}
1 & 1 \\
0 & -2 \\
\end{array}
\]

This implies \(-2y = 0\) or \( y = 0 \) and \( x + y = 1 \) or \( x = 1 \) as you already realized.

In 3 dimensions (3 unknowns) each row represents a plane. Two equations can intersect to give a line, and a line can intersect with a third plane to give a point, the single solution (in a single solution case).

#### 2.2.2 Case B - No solution

Consider the system \( x + y = 1 \) and \( x + y = 2 \). Graphically, this represents two lines that never intersect.

\[
\begin{array}{c|c}
1 & 1 \\
1 & 1 \\
\end{array}
\]

Note that column 1 and column 2 are identical. Subtracting row 1 from row 2 gives:

\[
\begin{array}{c|c}
1 & 1 \\
0 & 0 \\
\end{array}
\]

You know that \( 0x + 0y = 1 \) cannot be true. For a “square” system, if Gaussian elimination results in a 0 on the diagonal, this may be the case.
2.2.3 Case C - Many solutions

Consider the system \( x + y = 1 \) and \( 2x + 2y = 2 \). Graphically, this represents two lines that are coincident.

\[
\begin{array}{cc|c}
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}
\]

Subtracting twice the value of row 1 from row 2 gives:

\[
\begin{array}{cc|c}
1 & 1 & 1 \\
0 & 0 & 0 \\
\end{array}
\]

These equations are consistent. \( 0x + 0y = 0 \) and \( x + y = 1 \) are consistent. There is no single solution, as many solutions make the equation \( x + y + 1 \) consistent.

2.3 Nonsquare Systems

The original example was for a “square” system with 3 unknowns and 3 equations. You may often end up with more (or fewer) equations than unknowns.

Consider the original set of equations:

\[
\begin{align*}
1x + 0y + 0z &= 60 \\
-1x + 1y + 0z &= 30 \\
0x - 1y + 1z &= -20 \\
\end{align*}
\]

One additional equation can be specified using a mass balance on the entire system, \( 0 = 100 + 30 - 40 - 20 - z \).

\[
\begin{align*}
1x + 0y + 0z &= 60 \\
-1x + 1y + 0z &= 30 \\
0x - 1y + 1z &= -20 \\
0x + 0y + 1z &= 70 \\
\end{align*}
\] (2.2)

These four linear equations are not “linearly independent.” You can test this by using row operations to make two rows identical. Simultaneously adding row 1 and row 3 to row 2 will make row 2 the same as row 4.

\[
\begin{align*}
1x + 0y + 0z &= 60 \\
0x + 0y + 1z &= 70 \\
0x - 1y + 1z &= -20 \\
0x + 0y + 1z &= 70 \\
\end{align*}
\] (2.3)

This set of equations can still be satisfied using the original solution \( x = 60, y = 90, \) and \( z = 70 \). In other cases, having more equations than unknowns may complicate the solution process a bit.
2.3.1 Reconciliation and Nonsquare Systems

For curve fitting, parameters that appear linearly can be formulated as a nonsquare solution to a linear algebraic system of equations. Given that you have some (scalar valued) measured value, \( y \), that depends on a process parameter, \( x \). Assume the model takes the form:

\[
y = mx + b
\]  

(2.4)

Technically, you only need two data points to find \( m \) and \( b \), the model parameters. Assuming that you have more than two data points, we often desire to determine the “best-fit” for the line. These parameters minimize the sum of the square of the model error. For an experiment with four data points:

\[
y(1) = m x(1) + b \\
y(2) = m x(2) + b \\
y(3) = m x(3) + b \\
y(4) = m x(4) + b
\]

(2.5)

Here, you know values of \( y \) and \( x \) but \( m \) and \( b \) are your unknown values. This can be written as a set of equations:

\[
\begin{bmatrix}
y(1) \\
y(2) \\
y(3) \\
y(4)
\end{bmatrix} = \begin{bmatrix}
x(1) & 1 \\
x(2) & 1 \\
x(3) & 1 \\
x(4) & 1
\end{bmatrix} \begin{bmatrix}
m \\
b
\end{bmatrix}
\]

You can get the “best-fit” solution to this overspecified set of equations using the psuedo-inverse of the matrix:

\[
x = (A^T A)^{-1} A^T b
\]

2.4 Vectors

A group of unknown (or known) values can be “stacked” to form a vector. In the example problem, the unknowns \( x, y, \) and \( z \) can be described by the vector \( \mathbf{x} \):

\[
\mathbf{x} = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

The solution to the problem has a known value and can be written as a vector \( \mathbf{x}_{\text{soln}} \):

\[
\mathbf{x}_{\text{soln}} = \begin{bmatrix}
60 \\
90 \\
70
\end{bmatrix}
\]

Note that the underbar is used to distinguish between \( \mathbf{x} \) (the vector) and \( x \) the unknown. A vector is NOT limited to 2 or 3 unknowns (dimension of the vector).
2.5 The Matrix

A matrix is similar to a vector, having 2 dimensions. One may think of it as a group of vectors augmented together. A Matrix has a size, \( m \times n \) representing \( m \) rows and \( n \) columns. The values for \( m \) and \( n \) are sometimes written as subscripts for the matrix. For example, the 2x3 matrix \( \mathbf{A}_{2 \times 3} \) with two rows and three columns may have values:

\[
\mathbf{A}_{2 \times 3} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}
\]

Note that each of the six elements has two indices. The first index is the row, the second is the column. For the applications in this class, a matrix will have constant coefficient values. Some example matrices:

\[
\mathbf{A}_{2 \times 3} = \begin{bmatrix} 0 & -2 & 1 \\ 5 & 1 & 0.2 \end{bmatrix}, \quad \mathbf{B}_{3 \times 3} = \begin{bmatrix} 6 & 0 & 0 \\ -2 & 0 & -1 \\ 3 & -1 & 5 \end{bmatrix}
\]

**Square Matrix** - A matrix with indices equal \( (m = n) \).

Note: A vector can be seen as a special matrix having only 1 column.

**Transpose** - The transpose operator swaps the indices of a matrix (or vector). For example, for \( \mathbf{A}_{2 \times 3} \) as before:

\[
(\mathbf{A}_{2 \times 3})^T = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ a_{1,3} & a_{2,3} \end{bmatrix}
\]

Example. For the matrix \( \mathbf{A} \)

\[
\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

\[
\mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}
\]

Finally, one can take the transpose of a vector. For \( \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \)

\[
\mathbf{x}^T = [x \ y \ z]
\]

**Row Vector** - The transpose of a vector is also known as a row vector.

**Dot Product** - The dot product of two vectors is the sum of the product of the elements taken individually. Examples:

\[
\mathbf{x} \cdot \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2 + y^2 + z^2
\]
\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\cdot
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 1x + 2y + 3z
\]

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\cdot
\begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}
= 1 \times 4 + 2 \times 5 + 3 \times 6 = 32
\]

**Matrix Multiplication** - Two matrices can be multiplied together. For example \(A_{m \times n}\) can be multiplied by \(B_{n \times j}\). Matrix \(A\) has \(m\) rows and \(n\) columns, while \(B\) has \(n\) rows and \(j\) columns.

\[
A_{m \times n} = \begin{bmatrix}
\ldots & r_1 & \ldots \\
\ldots & r_2 & \ldots \\
& \vdots & \\
\ldots & r_m & \ldots
\end{bmatrix}
\]

Here, each row up to \(r_m\) is a row vector with \(n\) elements.

\[
B_{n \times j} = \begin{bmatrix}
\vdots & \vdots & \vdots \\
c_1 & c_2 & \ldots & c_j \\
\vdots & \vdots & \vdots
\end{bmatrix}
\]

Here, each column up to column \(c_j\) is a vector (column vector) with \(n\) elements. To compute \(A_{m \times n}B_{n \times j}\) or simply \(A \times B\) or just \(AB\)

\[
A_{m \times n}B_{n \times j} = \begin{bmatrix}
r_1^T \cdot c_1 & r_1^T \cdot c_2 & \ldots & r_1^T \cdot c_j \\
r_2^T \cdot c_1 & r_2^T \cdot c_2 & \ldots & r_2^T \cdot c_j \\
& \vdots & \vdots & \\
r_m^T \cdot c_1 & r_m^T \cdot c_2 & \ldots & r_m^T \cdot c_j
\end{bmatrix}
\]

**Method** - To compute \(A_{m \times n}B_{n \times j}\), the result will have \(j\) columns. The first column of the result is computed by taking the dot product of \(B_{1 \times j}\) (first column of \(B\)) with the transpose of all the rows of \(A\). The second column of the result is computed by taking the dot product of \(B_{2 \times j}\) (second column of \(B\)) with the transpose of all the rows of \(A\). Repeat up to the \(j^{th}\) column of \(B\) which produces the \(j^{th}\) column of the result.

Note: In general, \(A \cdot B \neq B \cdot A\).

**Conformable** - In order to multiply \(A_{m \times n}B_{n \times j}\) the “inner” dimensions must be equal. In \(A_{m \times n}B_{n \times j}\), if the first matrix has \(n\) columns and the second matrix must \(n\) rows.

Matrix Multiplication Examples:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 & 6 \\
7 & 8
\end{bmatrix}
= \begin{bmatrix}
5 + 14 & 6 + 16 \\
15 + 28 & 18 + 32
\end{bmatrix}
= \begin{bmatrix}
19 & 22 \\
43 & 50
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
4 \\
5
\end{bmatrix}
= \begin{bmatrix}
-4 + 10 \\
4 + 5
\end{bmatrix}
= \begin{bmatrix}
6 \\
9
\end{bmatrix}
\]

26
\[
\begin{bmatrix}
-1 & 2 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
-x + 2y \\
x + y
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 3 \\
1 & -1 \\
5 & 0
\end{bmatrix}
\begin{bmatrix}
2 \\
0 \\
-2 \\
1
\end{bmatrix}
= \begin{bmatrix}
4 - 6 & 3 \\
2 + 2 & -1 \\
10 + 0 & 0
\end{bmatrix}
= \begin{bmatrix}
-2 & 3 \\
4 & -1 \\
10 & 0
\end{bmatrix}
\]

### 2.6 Column Example

Consider again the equations from the original distillation column example:

\[
\begin{align*}
1x + 0y + 0z &= 60 \\
-1x + 1y + 0z &= 30 \\
0x - 1y + 1z &= -20
\end{align*}
\]

Notice that the variables (with constant coefficients) are on the left side and constant values are on the right hand side. This set of linear equations can be represented in the compact notation \( A \mathbf{x} = \mathbf{b} \) where

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\]

\[
\mathbf{x} = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

\[
\mathbf{b} = \begin{bmatrix}
60 \\
30 \\
-20
\end{bmatrix}
\]

**Identity Matrix** - The identity matrix has values of one on the diagonal and zeros elsewhere. It is defined as \( I \) and for a square matrix \( A \mathbf{I} = A \) and \( \mathbf{I} A = A \).

\[
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

### 2.6.1 How to solve sets of linear equations

We need a solution to the matrix equation \( A \mathbf{x} = \mathbf{b} \). You cannot “divide” by a matrix:

\[
\mathbf{x} \neq \frac{\mathbf{b}}{A}
\]

There is no “division” operator for a matrix. Instead, an inverse is defined for some square matrices such that
\[
A \, (A)^{-1} = I
\]

Also,
\[
(A)^{-1} \, A = I
\]

Now, to solve \(Ax = b\) for \(x\)

First, multiply on the left by \((A)^{-1}\)

\[
(A)^{-1} \, Ax = (A)^{-1} \, b
\]

Realizing that \((A)^{-1} \, A = I\) replace \((A)^{-1} \, A\) with \(I\).

\[
Ix = (A)^{-1} \, b
\]

Now, realizing \(Ix\) is \(x\), the solution is

\[
x = (A)^{-1} \, b
\]

Note that multiplying on the right will not lead to a solution.

\[
Ax \, (A)^{-1} = b \, (A)^{-1}
\]

### 2.6.2 How determine a matrix inverse

To solve \(Ax = b\), you need to know \((A)^{-1}\). We are going to use row reduction to calculate \((A)^{-1}\). Start with \(A | I\), use row reduction techniques until \(A\) is \(I\). \((A)^{-1}\) if it exists will be on the right where \(I\) was originally.

**Inverse Example**

Solve the following for \(x\) using \((A)^{-1}\):

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
6
\end{bmatrix}
\]

For this procedure, one must first calculate \((A)^{-1}\). Set up \(A | I\) as:

\[
\begin{array}{c|cc}
1 & 2 & 1 \\
3 & 4 & 0
\end{array}
\]

Use row reduction to get

\[
\begin{array}{c|cc}
1 & 0 & ? \\
0 & 1 & ?
\end{array}
\]

Then verify that \(A \, (A)^{-1} = I\). Use \((A)^{-1}\) to calculate \(x\) using \(x = (A)^{-1} \, b\). Verify solution again to be safe.
START
Start by using row reduction on

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Multiply row 2 by 1/3 to get:

\[
\begin{bmatrix}
1 & 2 \\
1 & \frac{4}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & \frac{1}{3}
\end{bmatrix}
\]

Then subtract row 1 from row 2 to get:

\[
\begin{bmatrix}
1 & 2 \\
0 & \frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-1 & \frac{1}{3}
\end{bmatrix}
\]

Now, multiply row 2 by -3/2 to get:

\[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{3}{2} & -\frac{1}{2}
\end{bmatrix}
\]

To get the left side looking like the identity matrix, subtract 2 times row 2 from row 1. Note that this is a compound use of row reduction rules.

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-2 & 1 \\
\frac{3}{2} & -\frac{1}{2}
\end{bmatrix}
\]

You now have \((A)^{-1}\) = \[
\begin{bmatrix}
-\frac{2}{3} & \frac{1}{2}
\end{bmatrix}
\]

Now verify that \(A \cdot (A)^{-1} = I\)

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
-2 & 1 \\
\frac{3}{2} & -\frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
1(-2) + 2(\frac{3}{2}) & 1(1) + 2(-\frac{1}{2}) \\
3(-2) + 4(\frac{3}{2}) & 3(1) + 4(-\frac{1}{2})
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

You may also verify that \((A)^{-1} \cdot A = I\)

\[
\begin{bmatrix}
-2 & 1 \\
\frac{3}{2} & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
= \begin{bmatrix}
\frac{3}{2} & -\frac{3}{2} \\
3 & -2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Now, compute the solution, \(x = (A)^{-1} \cdot b\).

\[
\begin{bmatrix}
-2 & 1 \\
\frac{3}{2} & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
= \begin{bmatrix}
-10 + 6 \\
\frac{15}{2} - 3
\end{bmatrix}
= \begin{bmatrix}
-4 \\
\frac{9}{2}
\end{bmatrix}
\]

Again, verify the solution is the solution to the original equations:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
= \begin{bmatrix}
-4 + 9 \\
-12 + 18
\end{bmatrix}
= \begin{bmatrix}
5 \\
6
\end{bmatrix}
\]

Just as expected...
2.6.3 Steady State Control Example

Two pumps are used to fill two tanks. The pumps usually operate at 50%, keeping the tanks at levels of 75 inches and 80 inches respectively. It is known that a 1% increase in pump 1 increases the height of tank 1 by 5 inches and the height of tank 2 by 3 inches. For a 1% change in pump 2, the height of tank 2 increases by 4 inches. It is desired to change the operating levels of the tanks to 110 inches and 89 inches.

![Diagram of pump tanks](image)

**Figure 2.2: Pump / Tank example**

What do you know:

\[
5 \Delta P_1(\%) = \Delta H_1(\text{inches})
\]
\[
3 \Delta P_1(\%) + 4 \Delta P_1(\%) = \Delta H_2(\text{inches})
\]

You know the target (reference, setpoint) for \(H_1\) and \(H_2\) as 110 and 89. This translates into \(\Delta H_1 = 110 - 75 = 35\) and \(\Delta H_2 = 89 - 80 = 9\). You need to increase tank 1 by 35 inches and increase tank 2 by 9 inches. You do not know the final values of the pump speeds. You do know the original steadystate values, 50% and 50%, realizing that:

\[
P_{\text{final}} = P_{ss} + \Delta P
\]

You can now set up linear equations to solve for \(\Delta P_1\) and \(\Delta P_2\), then calculate the final values for the pump speeds.

\[
\begin{bmatrix}
5 & 0 \\
3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
\Delta P_1 \\
\Delta P_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
\Delta H_1 \\
\Delta H_2 \\
\end{bmatrix}
\]

2.7 Visualization

Each row in \(Ax = b\) is a single linear equation. For a 2D problem (\(x\) with 2 elements / unknowns) the equation defines a line in the \((x, y)\) plane. Two equations define two
lines, and the unique solution to \( A x = b \) is the point \( x \) where the lines intersect. In some cases, there may be many solutions to \( A x = b \) and in some cases there may be no solutions to \( A x = b \).

![Figure 2.3: Three 2D examples with two equations. Each equation (row) represents a line. The first case has one solution, the second case has no solution, and the third case has many solutions.](image)

For a 3D problem, each row defines the equation for a plane in 3 space. The intersection of 2 non-parallel planes is a line in 3 space, and the intersection of a line and a plane in 3 space is a point. Again, in some cases there may be a single solution, many solutions, or no solutions.

For higher dimensions, each equation defines a hyperplane in a \( n \) dimensional space, \( \mathbb{R}^n \).

### 2.7.1 Linear Transform

A vector in \( \mathbb{R}^n \) means \( x \) has \( n \) elements. Matrix multiplication of a matrix of size \( m \times n \) times a vector of size \( n \times 1 \) “maps” the vector from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).

![Figure 2.4: Matrix multiplication as a mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).](image)

### 2.7.2 Range

The range of a matrix is the space of all possible points that may be mapped to in a matrix multiplication of that matrix times an unknown vector.
Range Example 1

For example, the matrix

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

can only map to points on the line \( x + y \) in 3D as follows.

\[ A \mathbf{x} = 2x + 2y + 0z \]

The columns of the matrix define possible directions for the matrix to transform a vector. In this example, columns 1 and 2 are the same, and column 3 is the zero vector. \( A \mathbf{x} \)
where \( \mathbf{x} \) takes any real value will always be on the line defined by the direction \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \).

Range Example 2

In another example, the matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

can only map to a variety of points in 3D as follows.

\[ A \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} z \]

Again, the columns of the matrix define possible directions for the matrix to transform a vector. In this example, only points in the directions of \( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) can be reached when multiplying \( A \mathbf{x} \). These two directions form a plane in 3 dimensional space.

![Range of A](image.png)

Figure 2.5: Range of \( A \) as space in \( \mathbb{R}^m \) of all possible mappings from \( \mathbb{R}^n \) using matrix multiplication.

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Range Example 3

In another example, the matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \]

can only map to a variety of points in 3D as follows.

\[ Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} z \]

Here, column 3 is linearly dependent upon columns 1 and 2. This means that you can find some combination of columns 1 and 2 that give column 3. Column 3 lies in the plane defined by columns 1 and column 2.

**Underlying point:** For \( Ax = b \) to have a solution, the \( b \) must be in the range of \( A \).

For the last examples, if \( b = \begin{bmatrix} ? \\ ? \\ 1 \end{bmatrix} \) (if \( b \) has element in the \( z \) position) there will not be a solution to \( Ax = b \). In such a case, the possible range of \( A \) does not include \( b \).

Range Example 4

In another example, the matrix

\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \]

can map to all of the points in 3D as follows.

\[ Ax = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} z \]

Here, column 3 is NOT linearly dependent upon columns 1 and 2. This means that you can find some combination of columns 1, 2, and 3 that give any point in 3 dimensions.

**Rank** - The rank of a matrix is the number of linearly independent columns. For a square matrix of size \( n \times n \), there is a unique solution if there are \( n \) independent columns. The matrix would have rank \( n \).
Chapter 3

Laplace Transforms and Deviation Variables

The Laplace Transform is a very important mathematical tool for use in analysis of dynamic systems. Most functions of time $f(t)$ may be used in a Laplace Transform. The Laplace Transform is defined as:

$$L \{ f(t) \} = \int_0^\infty e^{-st} f(t) \, dt$$

Some common functions that are useful in analysis of dynamic systems are given in the following table. Note that all functions are assumed to be 0 for times $t < 0$.

<table>
<thead>
<tr>
<th>Name</th>
<th>$f(t)$</th>
<th>$L { f(t) }$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step (Heaviside)</td>
<td>$H(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>Exponential Decay</td>
<td>$e^{-at}$</td>
<td>$\frac{1}{s+a}$</td>
</tr>
<tr>
<td>Ramp</td>
<td>$\sigma t$</td>
<td>$\frac{\sigma}{s}$</td>
</tr>
<tr>
<td>Sinusoid</td>
<td>$\sin(\omega t)$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
</tbody>
</table>

3.1 Simple System Example

Consider a tank draining from an initial height of $h_o$ at time $t = 0$. With no flow into the tank ($F_{in} = 0$) and $F_{out} = \alpha h(t)$ the mass balance can be written:

$$A \frac{dh}{dt}(t) = 0 - \alpha h(t)$$

Moving $\alpha h(t)$ to the left half side and dividing by $\alpha$ gives:

$$A \frac{dh}{\alpha dt}(t) + h(t) = 0$$

$A$ is the tank area (constant) and $\alpha$ is the proportionality constant for flow out of the tank. These parameters can be replaced by $\tau = A/\alpha$ to give the following differential equation:

$$\tau \frac{dy}{dt}(t) + y(t) = 0 \quad (3.1)$$
The initial tank height at time $t = 0$ can be assumed to be $y(t)|_{t=0} = y_o$. Take the Laplace transform of Equation 3.1:

$$L \left\{ \tau \frac{dy}{dt}(t) \right\} + L \{y(t)\} = 0$$

$L\{y(t)\}$ is easy, $L\{y(t)\} = y(s)$ so we have:

$$L \left\{ \tau \frac{dy}{dt}(t) \right\} + y(s) = 0$$

$L \left\{ \tau \frac{dy}{dt}(t) \right\}$ is a bit more complex. First, you can realize that $\tau$ is constant. Convince yourself of this! The $L$ operator on a constant times a function is the same as a constant times the Laplace of the function:

$$L \{c f(t)\} = \int_0^\infty c e^{-st} f(t) \, dt = c \int_0^\infty e^{-st} f(t) \, dt = c L \{f(t)\}$$

So you can take the constant value outside the $L$ operator:

$$\tau L \left\{ \frac{dy}{dt}(t) \right\} + y(s) = 0$$

Now, you must remember that $L \left\{ \frac{df}{dt}(t) \right\}$ is just $s f(s) - f(t)|_{t=0}$.

$$\tau (sy(s) - y(t)|_{t=0}) + y(s) = 0$$

And we have initial conditions for the height of the tank, $y(t)|_{t=0} = y_o$

$$\tau (sy(s) - y_o) + y(s) = 0$$

Now, solving for $y(s)$:

$$\tau sy(s) - \tau y_o + y(s) = 0$$

$$\tau sy(s) + y(s) = \tau y_o$$

$$(\tau s + 1) y(s) = \tau y_o$$

$$y(s) = \frac{\tau y_o}{(\tau s + 1)}$$

You must rearrange a little bit:

$$y(s) = \tau y_o \frac{1}{(\tau s + 1)}$$

$$y(s) = \tau y_o \frac{1}{\tau (s + \frac{1}{\tau})}$$

$$y(s) = y_o \frac{1}{(s + \frac{1}{\tau})}$$
This you realize is a constant \( y_o \) times the term \( \frac{1}{s+\frac{1}{\tau}} \). To get \( y(t) \) you must use the inverse Laplace transform, \( L^{-1} \) for the \( \frac{1}{s+\frac{1}{\tau}} \) part.

\[
L^{-1}\{y(s)\} = L^{-1}\left\{y_o \frac{1}{(s+\frac{1}{\tau})}\right\}
\]

Again, \( y_o \) is a constant and can be factored out

\[
L^{-1}\{y(s)\} = y_o L^{-1}\left\{ \frac{1}{(s+\frac{1}{\tau})}\right\}
\]

And we know that \( L\{e^{-at}\} = \frac{1}{s+a} \), so in our case, \( a = \frac{1}{\tau} \).

\[
y(t) = y_o e^{-\left(\frac{1}{\tau}\right)t}
\]

This is the solution to the original differential equation! Now check your result. At time \( t = 0 \) your solution for \( y(t) \) is \( y_o e^{-\left(\frac{1}{\tau}\right)0} = y_o 1 = y_o \). This matches the initial conditions. The derivative of your result can also be found

\[
\frac{dy}{dt}(t) = \frac{dy}{dt}\left\{y_o e^{-\left(\frac{1}{\tau}\right)t}\right\} = y_o - \left(\frac{1}{\tau}\right) e^{-\left(\frac{1}{\tau}\right)t}
\]

\[
\frac{dy}{dt}(t) = -\frac{y_o}{\tau} e^{-\left(\frac{1}{\tau}\right)t}
\]

Plug that back in the original differential EQ, along with your solution for \( y(t) \):

\[
\tau \frac{dy}{dt}(t) + y(t) = 0
\]

\[
\tau \left( -\frac{y_o}{\tau} \right) e^{-\left(\frac{1}{\tau}\right)t} + y_o e^{-\left(\frac{1}{\tau}\right)t} = 0
\]

And we know we have the solution!

### 3.2 First-Order System Modeling

The first order system model is:

\[
\tau \frac{dy}{dt}(t) + y(t) = K u(t)
\]

Taking the Laplace transform:

\[
\tau sy(s) - \tau y(t)|_{t=0} + y(s) = K u(s)
\]

If we assume that \( y(t)|_{t=0} = 0 \) this simplifies the equation to

\[
\tau sy(s) + y(s) = K u(s)
\]
We can then solve for \( y(s) \)

\[
(\tau s + 1)y(s) = K u(s)
\]

\[
y(s) = \frac{K}{(\tau s + 1)} u(s)
\]

Here, \( \frac{K}{(\tau s + 1)} \) is the process model relating \( u(s) \) and \( y(s) \). This is sometimes called \( g(s) = \frac{K}{(\tau s + 1)} \). Given \( u(t) \) you can find \( u(s) \), and given a model of your system you can find \( g(s) \). Realizing that \( y(s) = g(s)u(s) \) you can then find \( y(t) \).

From a **process reaction curve** (the data for \( y(t) \) and \( u(t) \) given a step in the input \( u(t) \)) you can find the PROCESS GAIN \( K \) from the equation:

\[
K = \frac{y_{\text{fin}} - y_{\text{init}}}{u_{\text{fin}} - u_{\text{init}}} = \frac{\Delta y}{\Delta u}
\]

The time constant is a bit trickier. First, let's assume \( u(t) \) is a step at time \( t = 0 \) from a value of 0 to a new value of \( A \). The Laplace transform of the step function is:

\[
u(s) = \frac{A}{s}
\]

Now, we have enough information to get \( y(s) \) and \( y(t) \)

\[
y(s) = \frac{K}{(\tau s + 1)} u(s)
\]

\[
y(s) = \frac{K}{(\tau s + 1)} \frac{A}{s}
\]

To solve this easily, we need the partial fraction expansion:

\[
y(s) = \frac{K}{(\tau s + 1)} \frac{A}{s} = \frac{Z_1}{(\tau s + 1)} + \frac{Z_2}{s}
\]

One way to get the partial fraction expansion is: first multiply each term by the denominator of term and set that term to zero:

\[
(\tau s + 1) \frac{K}{(\tau s + 1)} \frac{A}{s} = (\tau s + 1) \frac{Z_1}{(\tau s + 1)} + (\tau s + 1) \frac{Z_2}{s}
\]

\[
(\tau s + 1)|_{s=-\frac{1}{\tau}} \frac{K}{(\tau s + 1)} \frac{A}{s} = (\tau s + 1)|_{s=-\frac{1}{\tau}} \frac{Z_1}{(\tau s + 1)} + (\tau s + 1)|_{s=-\frac{1}{\tau}} \frac{Z_2}{s}
\]

Some terms cancel, others don’t:

\[
\frac{KA}{s}|_{s=-1/\tau} = Z_1 + 0
\]

\[
\frac{KA}{-1/\tau} = Z_1 + 0
\]

\[
-KA\tau = Z_1
\]
Do this for the second term, $Z_2/s$

$$\frac{sK}{(\tau s + 1)} \frac{A}{s} = s \frac{Z_1}{(\tau s + 1)} + \frac{Z_2}{s}$$

Cancel similar terms and evaluate at $s = 0$

$$\frac{K}{(\tau s + 1)} A = s \frac{Z_1}{(\tau s + 1)} + Z_2$$

$$\frac{K}{(\tau (0) + 1)} A = 0 + Z_2$$

$$KA = Z_2$$

The result can be written:

$$y(s) = \frac{K}{(\tau s + 1)} \frac{A}{s} = \frac{Z_1}{(\tau s + 1)} + \frac{Z_2}{s}$$

Substitute in $Z_1$ and $Z_2$

$$y(s) = -KA \frac{\tau}{(\tau s + 1)} + KA \frac{1}{s}$$

Simplify terms:

$$y(s) = -KA \frac{\tau}{(\tau s + 1)} + KA \frac{1}{s}$$

We can invert each term in this expression. $L\{e^{-\alpha t}\}$ is $\frac{1}{s+\alpha}$, so $L^{-1}\left\{\frac{1}{s+\frac{1}{\tau}}\right\}$ is just $e^{-(\frac{1}{\tau})t}$. We know for the step function from 0 to 1 at time 0 the Laplace transform is $\frac{1}{s}$. The resulting solution $y(t)$ is composed of two different functions, $e^{-(\frac{1}{\tau})t}$ and a step at time 0.

$$y(s) = -KA \frac{1}{(s + \frac{1}{\tau})} + KA \frac{1}{s}$$

Again, using the argument about constants times a function, we can pull out the $KA$ terms.

$$L^{-1}\{y(s)\} = L^{-1}\left\{-KA \frac{1}{(s + \frac{1}{\tau})}\right\} + L^{-1}\left\{KA \frac{1}{s}\right\}$$
\[ y(t) = -KAe^{-(\frac{1}{\tau})t} + KA \]
\[ y(t) = KA(-e^{-(\frac{1}{\tau})t} + 1) \]
\[ y(t) = KA(1 - e^{-(\frac{1}{\tau})t}) \]

Laplace transforms assume everything is 0 before time 0. This function \( y(t) \) only is defined for \( t \geq 0 \). The two separate functions that comprise \( y(t) \) are shown in the following graph, \( e^{-(\frac{1}{\tau})t} \) and a unit step at time zero:

Graphing the actual system (the sum of the two functions):
3.3 Deviation Variables

Now we will examine a realistic First-Order system, the tank system.

\[ A \frac{dh}{dt}(t) = F_i(t) - \alpha h(t) \]

Assume the flow manipulated and has units of \( \left( \frac{m^3}{s} \right) \). The height of the tank will be measured, and the height of the tank is given in units of \( (m) \). The area of the tank is 2 \( (m^2) \). For the outlet term to be consistent with the units of other terms \( \left( \frac{m^3}{s} \right) \), \( \alpha \) must have units of \( \left( \frac{m^2}{s} \right) \). Assume \( \alpha \) has a value of 0.1 \( \left( \frac{m^2}{s} \right) \). The mass balance can be written as:

\[ 2 \frac{dh}{dt}(t) = F_i(t) - 0.1 h(t) \]

Now, assume that you normally operate this tank at a flow rate of entering the tank of 0.5 \( \left( \frac{m^3}{s} \right) \). This means we know the steady state flow rate into the tank, \( F_{iss} = 0.5 \left( \frac{m^3}{s} \right) \). This also means we can figure out the steady state height of the tank from the mass balance. At steady state, \( \frac{dh}{dt}(t) = 0 \)

\[ 2 \frac{dh}{dt}(t) = F_i(t) - 0.1 h(t) \]

\[ \frac{dh}{dt}|_{ss} = F_{iss} - 0.1 h_{ss} \]

\[ 0 = 0.5 \frac{m^3}{s} - 0.1 \frac{m^2}{s} h_{ss} \]

\[ -0.5 \frac{m^3}{s} = -0.1 \frac{m^2}{s} h_{ss} \]

\[ 5 m = h_{ss} \]

So now we know \( h_{ss} \), the steady state height of the tank. Now to make our life easier when performing a Laplace transform, we put everything in Deviation Variables. This means we subtract the steady state from the normal functions of time. The purpose of this is to make the functions all start at a value of 0. Currently, a step response for the tank system looks like:
Using the variables in deviation form, assume \( y(t) = h(t) - h_{ss} \). This means that if we start at steady state at time 0, \( y(t) \) will equal 0 at the initial steady state value, \( y(t)|_{t=0} = 0 \). The other deviation variable can be written \( u(t) = F_i(t) - F_{iss} \). This means the input \( u(t) \) equals 0 at the initial starting point, \( u(t)|_{t=0} = 0 \). Also, taking the derivative WRT time of \( y(t) = h(t) - h_{ss} \) yields

\[
\frac{dy}{dt}(t) = \frac{dh}{dt}(t) - \frac{dh_{ss}}{dt}(t)
\]

But \( h_{ss} \) does not change with time.

\[
\frac{dy}{dt}(t) = \frac{dh}{dt}(t) - 0
\]

\[
\frac{dy}{dt}(t) = \frac{dh}{dt}
\]

The dynamic mass balance is written as:

\[
2\frac{dh}{dt}(t) = F_i(t) - 0.1 h(t)
\]

The steady state mass balance is written as:

\[
0 = F_{iss} - 0.1 h_{ss}
\]

Subtracting the steady state mass balance from the dynamic mass balance gives:

\[
2\frac{dh}{dt}(t) - 0 = F_i(t) - F_{iss} - 0.1 h(t) - (-0.1 h_{ss})
\]

\[
2\frac{dh}{dt}(t) = (F_i(t) - F_{iss}) - (0.1 h(t) - 0.1 h_{ss})
\]

\[
2\frac{dh}{dt}(t) = (F_i(t) - F_{iss}) - 0.1 (h(t) - h_{ss})
\]

And replacing what we can with deviation variables:

\[
2\frac{dy}{dt}(t) = u(t) - 0.1 y(t)
\]
To put this in the “traditional” $\tau \frac{dy}{dt} + y = Ku$ form, divide by 0.1 and move $y(t)$ over.

$$\frac{2}{0.1} \frac{dy}{dt}(t) = \frac{1}{0.1} u(t) - 1 \cdot y(t)$$

$$20 \frac{dy}{dt}(t) + y(t) = 10 u(t)$$

So we know that $\tau = 20$ and $K = 10$ for this process.

Now, you can easily take the Laplace transform of this dynamic model.

$$L \left\{ 20 \frac{dy}{dt}(t) \right\} + L \{y(t)\} = L \{10 u(t)\}$$

$$20 L \left\{ \frac{dy}{dt}(t) \right\} + L \{y(t)\} = 10 L \{u(t)\}$$

$$20 \left( sy(s) - y(t)|_{t=0} \right) + y(s) = 10 u(s)$$

Since we put everything in deviation variables, $y(t)|_{t=0}$ is now 0.

$$20 \left( sy(s) - 0 \right) + y(s) = 10 u(s)$$

$$20 sy(s) + y(s) = 10 u(s)$$

Solving for $y(s)$:

$$20 sy(s) + y(s) = 10 u(s)$$

$$(20 s + 1) y(s) = 10 u(s)$$

$$y(s) = \frac{10}{(20 s + 1)} u(s)$$

Again, you see this in the form $\frac{K}{s+1}$. We want to get the expression for $y(s)$ as a function of $s$, not a function of $s$ and $u(s)$. We know the value for $u(t)$. In the original variables, $F_i(t)$ changed from 0.5 to 1.5 at time $t=0$. We do not know the Laplace transform for a step from 0.5 to 1.5 at time $t = 0$. In deviation variables, $u(t)$ changes from a value of 0 to a value of 1 at time $t = 0$. We know the Laplace transform of a step function from 0 to 1 at time $t = 0$. This value is $u(s) = \frac{1}{s}$

$$y(s) = \frac{10}{(20 s + 1)} \frac{1}{s}$$

Using partial fraction expansion:

$$y(s) = \frac{-200}{(20 s + 1)} + \frac{10}{s}$$

$$y(s) = \frac{-200}{(20 s + 1)} \frac{1/20}{1/20} + \frac{10}{s}$$

$$y(s) = \frac{-10}{(s + \frac{1}{20})} + \frac{10}{s}$$
\[ y(s) = 10 \left( \frac{1}{s} - \frac{1}{s + \frac{1}{20}} \right) \]

\[ y(t) = 10 \left( 1 - e^{-\left(\frac{1}{20}\right)t} \right) \]

This expression for \( y(t) \) can be plotted. Note that \( y(t) \) and \( u(t) \) start at zero.

\[
\begin{align*}
\text{What value does the response take when } t &= \tau? \text{ In this case, } \tau = 20. \\
y(t|_{t=20}) &= 10 \left( 1 - e^{-\left(\frac{1}{20}\right)20} \right) \\
y(t|_{t=20}) &= 10 \left( 1 - e^{-1} \right) \\
y(t|_{t=20}) &= 10 (1 - 0.3678) \\
y(t|_{t=20}) &= 10 (0.6321) \\
y(t|_{t=20}) &= 6.32
\end{align*}
\]

So at time \( t = \tau \) the response is 6.32, or 63% of the final value of 10.

This can also be simulated in Simulink:
After running the simulation, the results will be put in vectors in the Matlab workspace. These vectors are named (in this example) \( t \), \( y \), \( u \), \( h \), and \( f \). Note that the step occurs at time \( t = 0 \), so you should start the simulation at time \( t = -10 \). Also note that the “To Workspace” blocks must have the “Save Format” set to “Array”.

The following plotting command will let you plot \( u(t) \) and \( y(t) \) on the same figure:

```matlab
subplot(2,1,1)
plot(t,u)
ylabel('u(t)')
legend('u(t) = Unit step at time 0',4)
subplot(2,1,2) plot(t,y)
ylabel('y(t)')
legend('y(t), First-order step response',4)
```
Chapter 4

Basic Procedures for Common Problems

4.1 Steady State Multivariable Modeling and Control

1. Determine what variables are available to manipulate (inputs, $\Delta u$) and what variables are available to measure (outputs, $\Delta y$)

2. Note how many input and output variables you have.

3. Start to write equations for the output variables. This means write something in the form:

   $\Delta y_1 = ??$
   $\Delta y_2 = ??$
   $\vdots$
   $\Delta y_n = ??$

4. Read through the problem and establish relationships between individual inputs ($\Delta u_i$) and individual outputs ($\Delta y_j$). The relationships generally represent the gain of the individual input output relationship, for example $\Delta y_j = K \Delta u_i$. For example: “Changing input 1 by 2% decreases output 1 by 5” means $\Delta u = 2\%$ and $\Delta y = -5$ and

   $-5 = K 2$

   Or $K = -5/2$ and $\Delta y_1 = -2.5 \Delta u_1$.

5. Put all of the relationships into the equations. Keep reading through the word
expression until you relate all specified inputs and outputs:

\[ \Delta y_1 = -2.5\Delta u_1 + ??? \]
\[ \Delta y_2 = 4??? \]
\[ \vdots \]
\[ \Delta y_n = ??? \]

6. Write out the equations with all input variable in every equation, even if they have a 0 coefficient.

\[ \Delta y_1 = -2.5\Delta u_1 + 0\Delta u_2 + 3\Delta u_3 \]
\[ \Delta y_2 = 0\Delta u_1 + 4\Delta u_2 + 1\Delta u_3 \]
\[ \Delta y_3 = 5\Delta u_1 + 10\Delta u_2 + 2\Delta u_3 \]

7. Realize that this can be put in the form:

\[ \Delta y = K\Delta u \]

### 4.2 Dynamic Modeling

1. Try to figure out what is changing with time. Try to figure out what are manipulated inputs \((u_i(t))\), what are disturbances \((d_i(t))\) and what are measurements \((y_i(t))\).

2. Start to write dynamic mass and energy balances for the items that are changing.

3. Note the accumulation term

   (a) Changing volume: \(V(t) = Ah(t) \rightarrow A\frac{dh}{dt}(t)\)
   
   (b) Changing amount of species in a tank: \(VC_A(t) \rightarrow V\frac{dC_A}{dt}(t)\)
   
   (c) Changing temperature in a tank: \(V\rho C_p(T(t) - T^*) \rightarrow V\rho C_p\frac{dT}{dt}(t)\)

4. Don’t forget reaction terms for reacting systems. \(V r(t)\) where \(r(t)\) is the reaction rate, usually in the form \(r(t) = kC_A(t)\) (or more complex).

5. Write your equations and check units.

### 4.3 State Space

1. Identify \(x\) as the values that are changing with time in your accumulation term.

2. Identify your manipulated inputs \(u\).
3. Identify your measurement equations. Your measurements should be expressed as functions of the states and inputs.

4. Write your dynamic equations, including terms for every state and input (with 0 coefficient if necessary).

5. Reorder the terms in you dynamic equations such that states come first in order, then inputs. For example:

\[
\frac{dx_1}{dt} = 2x_1 + 3x_2 + 0x_3 + 2u_1 + 5u_2
\]

6. Put the dynamic equations in the form

\[
\dot{x} = Ax + Bu
\]

7. Write your measurement equations, including terms for every state and input (with 0 coefficient if necessary).

8. Put your measurement equations in the form:

\[
y = Cx + Du
\]

### 4.4 Laplace Transform of Dynamic Equations

1. If your steady state values are not all = 0, take your dynamic model equations and establish the steady state values for you inputs, states, and outputs. This is accomplished by solving for unknowns with the accumulation terms = 0.

2. If your equations are nonlinear, linearize your equations.

\[
A\frac{dh}{dt}(t) = F_{in}(t) - \sqrt{h(t)}
\]

Here, \(\sqrt{h(t)}\) is nonlinear. Near steady state, it can be approximated as

\[
\sqrt{h(t)} \simeq \sqrt{h_{ss} + \frac{1}{2}h_{ss}^{-\frac{1}{2}}(h(t) - h_{ss})}
\]

such that

\[
A\frac{dh}{dt}(t) = F_{in}(t) - \left(\sqrt{h_{ss} + \frac{1}{2}h_{ss}^{-\frac{1}{2}}(h(t) - h_{ss})}\right)
\]

3. Subtract the steady state model equations from the dynamic model equations to put everything in deviation variables. For example, \(y(t) = h(t) - h_{ss}\) and \(u(t) = F_{in}(t) - F_{inss}\).

(a) Remember to express the accumulation term with your deviation variables.

For \(y(t) = h(t) - h_{ss}\), taking the derivative, \(\frac{dy}{dt}(t) = \frac{dh}{dt}(t)\) because \(h_{ss}\) is constant.
4. Express your dynamic problem using deviation variables $u(t)$, $y(t)$, $d(t)$. These functions of time should = 0 at time $t = 0$.

5. Take the Laplace transform of your system.

6. Solve algebraically to get in the form

$$ y(s) = g(s) u(s) $$

or

$$ \frac{y(s)}{u(s)} = g(s) $$

7. If you have disturbances and inputs, your model can look like

$$ y(s) = g(s) u(s) + g_d(s) d(s) $$

Note that to get $g(s)$ you can assume $d(s) = 0$ then solve for $g(s)$. To get $g_d(s)$ you assume $u(s) = 0$ and solve for $g_d(s)$.

8. If you multiple inputs inputs, your model can look like

$$ y(s) = g_1(s) u_1(s) + g_2(s) u_2(s) $$

9. If you have multiple inputs and multiple measurements, your model can look like

$$ y_1(s) = g_{11}(s) u_1(s) + g_{12}(s) u_2(s) $$

$$ y_2(s) = g_{21}(s) u_1(s) + g_{22}(s) u_2(s) $$

10. Given the input as a function of time $u(t)$ (or input and disturbances) you can determine $u(s)$ (or $u(s)$ and $d(s)$).

11. Plug in to get an expression for $y(s)$ in terms of the variable $s$

## 4.5 Laplace of Complex Functions

1. You should be familiar with basic functions of time (step, impulse, ramp, exponential decay, sinusoid).

2. If the function is not 0 for $t < 0$ you should put the function in deviation variables. For example, a step in $F_{in}(t)$ at time 0 from 2 to 3 can be expressed as a unit step in $u(t)$ at time 0 with $u(t) = F_{in}(t) - F_{inss}$

3. You should be able to express the complex function as a single function of time. Multiply by the Heaviside function if needed. For a function that ramps from 0 with a slope of 2 until time 10 settling out at a value of 20, this can be expressed as

$$ f(t) = 2t H(t) + (-2) (t - 10) H(t - 10) $$
4. Sketch the individual terms in your function as functions of time, then add them together to check your formulation. You can plug in numbers to check your function.

5. For each term, shift it in time such that the “event” occurs at time zero and determine the Laplace transform. Use the time shift operator if necessary to express the function as some \( f(s) \). For the example:

\[
f(s) = \frac{2}{s^2} - \frac{2}{s^2} e^{-20s}
\]

### 4.6 Solving for \( y(t) \)

1. Establish \( y(s) \) as a function of \( s \). (Develop dynamic model, take Laplace of model, and determine \( u(s) \) and \( d(s) \) if needed)

2. Your response may be in the form

\[
y(s) = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)} e^{-\alpha s} + \ldots + \frac{N_3(s)}{D_3(s)} e^{-\beta s}
\]

This expression with multiple terms will be treated as multiple different responses, each shifted in time.

3. If you have a time delay, \( e^{-\alpha s} \), ignore it for now.

4. Take a term from \( y(s) \) and determine the poles, the roots of \( D_i(s) \).

5. Perform a **Partial Fraction Expansion** on the term. For expressions with unique poles \( p_i \), the result looks like:

\[
\frac{N_i(s)}{D_i(s)} = \frac{Z_1}{(s - p_1)} + \frac{Z_2}{(s - p_2)} + \ldots + \frac{Z_n}{(s - p_n)}
\]

For non-unique poles or imaginary roots, check the Appendix. Non-unique Poles will result in

\[
\frac{Z_1}{(s - p_1)} + \frac{Z_2 s}{(s - p_1)} + \frac{Z_3 s^2}{(s - p_1)}
\]

while imaginary roots result in sin or cosine in your \( y(t) \)

6. Now you should be able to determine the inverse Laplace transform of each expression to yield a function of time, \( y_1(t) \).

\[
y_1(t) = Z_1 e^{-p_1 t} + Z_2 e^{-p_2 t} + \ldots + Z_n e^{-p_n t}
\]

7. If you had a time delay in your term, shift the response by the time delay:

\[
y_1(t) = \left( Z_1 e^{-p_1 (t - \alpha)} + Z_2 e^{-p_2 (t - \alpha)} + \ldots + Z_n e^{-p_n (t - \alpha)} \right) H(t - \alpha)
\]

8. Do this procedure for all your terms in the original \( y(s) \)

9. Add up all \( y_i(t) \) to get \( y(t) \)

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Chapter 5

Lead-Lag System Modeling

A constant volume, constant flowrate mixer is used in the configuration below. Determine the unit step response for the outlet temperature $T_2(t)$.

![Diagram of a constant volume, constant flowrate mixer](image)

Procedure:
1. Dynamic Model - Develop an energy balance for the mixing tank.
2. Determine an energy balance on the mixing point.
3. Put your dynamic (and steady state) equations into deviation variables.
4. Take the Laplace transform of the equations.
5. Determine the transfer function relationship between $u(t)$ and $y(t)$
6. Show that the transfer function can be rearranged to be a pure gain in parallel with a first order system.
5.1 Solution

1. Dynamic Model - Develop an energy balance for the mixing tank:
\[
\frac{d\left(\rho V C_p(T_1(t)-T^*)\right)}{dt} = \gamma F \rho C_p (T_o(t) - T^*) - \gamma F \rho C_p (T_1(t) - T^*)
\]
\[
\rho V C_p \frac{d(T_1(t)-T^*)}{dt} = \gamma F \rho C_p (T_o(t) - T^*) - \gamma F \rho C_p (T_1(t) - T^*)
\]
\[
\frac{V}{\gamma F} \frac{d(T_1(t)-T^*)}{dt} = (T_o(t) - T^*) - (T_1(t) - T^*)
\]

2. Determine an energy balance on the mixing point:
\[
0 = (1 - \gamma) F \rho C_p (T_o(t) - T^*) + \gamma F \rho C_p (T_1(t) - T^*) - F \rho C_p (T_2(t) - T^*)
\]
\[
0 = (1 - \gamma) (T_o(t) - T^*) + \gamma (T_1(t) - T^*) - (T_2(t) - T^*)
\]

3. Put your dynamic (and steady state) equations into deviation variables. In this case, we will use the following deviation variables: \(u(t) = T_o(t) - T^*\), \(x(t) = T_1(t) - T^*\), and \(y(t) = T_2(t) - T^*\).
\[
\frac{V}{\gamma F} \frac{dx(t)}{dt} = u(t) - x(t)
\]
\[
0 = (1 - \gamma) u(t) + \gamma x(t) - y(t)
\]

4. Take the Laplace transform of the equations:
\[
\frac{V}{\gamma F} \left(sx(s) - x(t=0)\right) = u(s) - x(s)
\]
\[
0 = (1 - \gamma) u(s) + \gamma x(s) - y(s)
\]
Because of the deviation variables, \(x(t=0) = 0\)
\[
\frac{V}{\gamma F} \left(sx(s)\right) = u(s) - x(s)
\]
Rearranging the mixing tank equation:
\[
x(s) = \frac{1}{\frac{V}{\gamma F} s + 1} u(s)
\]

5. Determine the transfer function relationship between \(u(t)\) and \(y(t)\). Substitute the mixing tank equation into the mixing point equation:
\[
0 = (1 - \gamma) u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} u(s) - y(s)
\]
Rearrange to get in the form \(y(s) = g(s)u(s)\)
0 = (1 − \gamma) \ u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} \ u(s) − y(s)

\[ y(s) = (1 − \gamma) \ u(s) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} \ u(s) \]

\[ y(s) = (1 − \gamma) \ u(s) \left( \frac{\frac{V}{\gamma F} s + 1}{\frac{V}{\gamma F} s + 1} \right) + \frac{\gamma}{\frac{V}{\gamma F} s + 1} \ u(s) \]

\[ y(s) = \frac{(1 − \gamma) u(s) \left( \frac{\frac{V}{\gamma F} s + 1}{\frac{V}{\gamma F} s + 1} \right) + \gamma u(s)}{\frac{V}{\gamma F} s + 1} \]

\[ y(s) = \frac{u(s) \ ((1 − \gamma) \frac{V}{\gamma F} s + (1 − \gamma)) + \gamma u(s)}{\frac{V}{\gamma F} s + 1} \]

\[ y(s) = \frac{u(s) \ ((1 − \gamma) \frac{V}{\gamma F} s + (1 − \gamma + \gamma))}{\frac{V}{\gamma F} s + 1} \]

\[ y(s) = \frac{u(s) \left( (1 − \gamma) \frac{V}{\gamma F} s + 1 \right)}{\frac{V}{\gamma F} s + 1} \]

\[ \frac{y(s)}{u(s)} = \frac{V (1 − \gamma) s + 1}{\frac{V}{\gamma F} s + 1} \]

For \( \gamma = 1 \) this reduces to (a first order system):

\[ \frac{y(s)}{u(s)} = \frac{1}{\frac{V}{\gamma F} s + 1} \]

For \( \gamma = 0 \) this reduces to a pure gain system. The original equation

\[ \frac{y(s)}{u(s)} = \frac{\frac{V (1 − \gamma)}{\gamma F} s + 1}{\frac{V}{\gamma F} s + 1} \]

is in the form

\[ \frac{y(s)}{u(s)} = \frac{K (\xi s + 1)}{\frac{\tau}{\gamma F} s + 1} \]

with \( K = 1, \tau = \frac{V}{\gamma F} \), and \( \xi = \frac{V (1 − \gamma)}{\gamma F} \).

6. Show that the transfer function can be rearranged to be a pure gain in parallel with a first order system.

If we want this in the form:

\[ \frac{K (\xi s + 1)}{\frac{\tau}{\gamma F} s + 1} = A_0 + \frac{A_1}{\tau s + 1} \]
\[
\frac{K(\xi s + 1)}{\tau s + 1} = \frac{\tau s + A_0}{\tau s + 1} + \frac{A_1}{\tau s + 1}
\]

\[
\frac{K(\xi s + 1)}{\tau s + 1} = \frac{A_0 \tau s + A_0 + A_1}{\tau s + 1}
\]

\[
\frac{K\xi + K}{\tau s + 1} = \frac{A_0 \tau s + (A_0 + A_1)}{\tau s + 1}
\]

\[
K\xi = A_0 \tau
\]

\[
K = A_0 + A_1
\]

\[
\frac{K\xi}{\tau} = A_0
\]

\[
K = \frac{K\xi}{\tau} + A_1
\]

\[
A_1 = K - \frac{K\xi}{\tau}
\]

\[
A_1 = K \left(1 - \frac{\xi}{\tau}\right)
\]

\[
A_0 = K \frac{\xi}{\tau}
\]

Now, letting \(\rho = \frac{\xi}{\tau}\)

\[
A_0 = K\rho
\]

\[
A_1 = K(1 - \rho)
\]

\[
\frac{K(\xi s + 1)}{\tau s + 1} = K\rho + \frac{K(1 - \rho)}{\tau s + 1}
\]

This means the lead lag transfer function is really just two systems in parallel, a pure gain system and a first order system. The value \(\rho\) can be seen as a weighting value.

Back to the problem, we wanted step response. This means that \(u(s) = \frac{1}{s}\)

\[
y(s) = \frac{K(\xi s + 1)\frac{1}{s}}{\tau s + 1}
\]

Using partial fraction expansion, we need to break this down to

\[
y(s) = \frac{K(\xi s + 1)\frac{1}{s}}{\tau s + 1} = \frac{Z_1}{\tau s + 1} + \frac{Z_2}{s}
\]

Multiply by \(\tau s + 1\) and set \(s = -\frac{1}{\tau}\) to get \(Z_1\).

\[
Z_1 = \left.\frac{K(\xi s + 1)}{s}\right|_{s = -\frac{1}{\tau}}
\]

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\[ Z_1 = \frac{K \left( \xi - \frac{1}{\tau} + 1 \right)}{-\frac{1}{\tau}} \]

\[ Z_1 = -K\tau \left( \xi \left( -\frac{1}{\tau} \right) + 1 \right) \]

To get \( Z_2 \), multiply by \( s \) and set \( s = 0 \)

\[ Z_2 = \frac{1}{1} = 1 \]
Chapter 6

Transfer Functions

For the following transfer functions, determine the gain, poles, and zeros. Is the system unstable, asymptotically stable, or BIBO stable? Does the system exhibit oscillatory response? Sketch the response of each transfer function to a unit step in the input.

1. \( g(s) = \frac{20}{10s+1} \)
2. \( g(s) = \frac{-40}{10s+1}e^{-10s} \)
3. \( g(s) = \frac{50}{20s-1} \)
4. \( g(s) = \frac{-5}{(2s+1)(3s+1)} \)
5. \( g(s) = \frac{-8s-2}{7s+1} \)
6. \( g(s) = \frac{1}{s^2+2s+3} \)
7. \( g(s) = \frac{s-2}{(3s+1)(4s+1)} \)
8. \( g(s) = \frac{2}{s(4s+1)} \)
6.1 Solution

General Discussion
The gain is easily evaluated with \( g(s) \big|_{s=0} \). The poles are the roots of the denominator. The zeros are the roots of the numerator, ignoring the time delay (if any). If any pole has a positive real part, it is unstable. Otherwise, if any pole is on the Real/Imaginary border (real part=0) then it is BIBO stable and stability depends on the input to the system. If all poles have real parts <0, it is asymptotically stable. If the poles have imaginary components, the system response can be oscillatory. If the system is unstable or BIBO stable, the gain may not be well defined.

If a zero has a positive real part, it may have inverse response. You can use the initial value theorem and final value theorem along with any information from the poles and zeros to sketch the response without doing a full partial fraction expansion to get to \( y(t) \).

1. \( g(s) = \frac{20}{10s+1} \)
   The gain is 20. The system has a pole at \( s = -1/10 \). The system has no zeros. The system should be asymptotically stable. The system should not be oscillatory.

2. \( g(s) = \frac{-40}{10s+1} e^{-10s} \)
   The gain is -40. The system has a pole at \( s = -1/10 \). The system has no zeros. The system should be asymptotically stable. The system should not be oscillatory.

3. \( g(s) = \frac{50}{20s-1} \)
   The gain is 50. The system has a pole at \( s = 1/20 \). The system has no zeros. The system should be unstable. The system should not be oscillatory.

4. \( g(s) = \frac{-5}{(2s+1)(3s+1)} \)
   The gain is -5. The system has poles at \( s = -1/2, -1/3 \). The system has no zeros. The system should be asymptotically stable. This is a second order system, underdamped. The system should not be oscillatory. Since it is a second-order system, the slop is 0 at \( t = 0 \).

5. \( g(s) = \frac{(-8s-2)}{(7s+1)} \)
   The gain is -2. The system has a pole at \( s = -1/7 \). The system has a zero at \( s = -1/4 \). The system should be asymptotically stable. The system should not be oscillatory.

6. \( g(s) = \frac{1}{s^2+2s+3} \)
   The gain is 1/3. The system has a pole at \( s = -1 \pm \sqrt{2}i \). The system has no zeros. The system should be asymptotically stable. The system should not be oscillatory.

7. \( g(s) = \frac{s+2}{(3s+1)(4s+1)} \)
   The gain is -2. The system has poles at \( s = -1/3, -1/4 \). The system has a zero at \( s = 2 \). The system should be asymptotically stable. The system should not be oscillatory. The RHP zero means it should have inverse response for a step in the input.
8. \( g(s) = \frac{2}{s(4s+1)} \)

The gain is not well defined. The system has poles at \( s = 0, -1/4 \). The system has no zeros. The system should be BIBO due to the zero at the origin. The system should not be oscillatory.
Chapter 7
Modeling and Response Example

7.1 Reactor with Bypass Problem

Example Problem with Solutions

Assuming the reactor is well-mixed constant-volume with the following parameters:

- Volume $V = 20 \; (m^3)$
- Feed flow rate $F = 1 \; (m^3/min)$
- Reaction rate $k = 1 \; (m^3)$
- Feed concentration $C_{A0SS} = 1 \; (mol/m^3)$
- The input is $u(t) = (C_{A0}(t) - C_{A0SS})$
- You can measure $y(t) = (C_{A2}(t) - C_{A2SS})$

Do the following:
1. Develop a mass balance around the reactor and mixing point.

2. Determine the steady state reactor exit concentration

3. Linearize the dynamic model if necessary

4. Put the model equations in deviation form

5. Put the model in state space form and identify $A$, $B$, $C$, $D$

6. Develop a transfer function relating $y(s)$ to $u(s)$

7. For the input $C_{A0}(t)$ find $u(s)$

8. Determine $y(t = 0)$ and $y(t = \infty)$ using the Initial and Final Value Theorems

9. Find $y(t)$
7.2 Solutions

1. Mass Balance
Balance on reactor, including reaction rate term:

\[
V \frac{dC_{A1}}{dt}(t) = \frac{2}{3}FC_{A0}(t) - \frac{2}{3}FC_{A1}(t) - Vk(C_{A1}(t))^2
\]

Balance around mixing point, no accumulation:

\[
0 = \frac{1}{3}FC_{A0}(t) + \frac{2}{3}FC_{A1}(t) - FC_{A2}(t)
\]

Now, using the parameters given and canceling \( F \) in the mixing point balance:

\[
20 \frac{dC_{A1}}{dt}(t) = \frac{2}{3}C_{A0}(t) - \frac{2}{3}C_{A1}(t) - 20(C_{A1}(t))^2
\]

\[
0 = \frac{1}{3}C_{A0}(t) + \frac{2}{3}C_{A1}(t) - C_{A2}(t)
\]

2. Steady state reactor concentration
Assume the accumulation term = 0 at Steady state:

\[
0 = \frac{2}{3}C_{A0SS} - \frac{2}{3}C_{A1SS} - 20(C_{A1SS})^2
\]

You know \( C_{A0}(t) = C_{A0SS} = 1 \) at steady state from the graph of \( C_{A0} \) vs. time

\[
0 = \frac{2}{3} - \frac{2}{3}C_{A1SS}(t) - 20(C_{A1SS}(t))^2
\]

\[
0 = 1 - 1C_{A1SS}(t) - 30(C_{A1SS}(t))^2
\]

Quadratic equation form with \( x \) as the unknown (not state here)

\[
30x^2 + x - 1
\]

\[
(6x - 1)(5x + 1)
\]

Roots: \( \frac{1}{6} \) and \( -\frac{1}{5} \)

Negative roots not possible, so \( C_{A1SS} = \frac{1}{6} \)

To find the steady state output of the system, use the mass balance around the mixing point:

\[
0 = \frac{1}{3}C_{A0SS} + \frac{2}{3}C_{A1SS} - C_{A2SS}
\]

\[
0 = \frac{1}{3}(1) + \frac{2}{3}(\frac{1}{6}) - C_{A2SS}
\]
So \( C_{A2SS} = \frac{4}{9} \) for \( C_{A0SS} = 1 \) and \( C_{A1SS} = \frac{1}{6} \)

3. Linearize if necessary

Reaction rate term is nonlinear. Nonlinear ODE:

\[
20 \frac{dC_{A1}}{dt}(t) = \frac{2}{3} C_{A0}(t) - \frac{2}{3} C_{A1}(t) - 20 (C_{A1}(t))^2
\]

Linearize reaction rate term at \( C_{A1SS} \) and drop higher order terms

\[
20 \frac{dC_{A1}}{dt}(t) = \frac{2}{3} C_{A0}(t) - \frac{2}{3} C_{A1}(t) - \left[ 20 (C_{A1SS})^2 + 2 \times 20 C_{A1SS} (C_{A1}(t) - C_{A1SS}) + \ldots \right]
\]

Mixing point equation is linear:

\[
0 = \frac{1}{3} C_{A0}(t) + \frac{2}{3} C_{A1}(t) - C_{A2}(t)
\]

4. Deviation Form

Linear dynamic equation, with \( C_{A1SS} = \frac{1}{6} \)

\[
20 \frac{dC_{A1}}{dt}(t) = \frac{2}{3} C_{A0}(t) - \frac{2}{3} C_{A1}(t) - \left[ 20 (C_{A1SS})^2 + \frac{20}{3} (C_{A1}(t) - C_{A1SS}) \right]
\]

At steady state:

\[
0 = \frac{2}{3} C_{A0SS} - \frac{2}{3} C_{A1SS} - \left[ \frac{20}{3} (C_{A1SS})^2 + 0 \right]
\]

Subtract steady state from linear dynamic equation:

\[
20 \frac{dC_{A1}}{dt}(t) - 0 = \frac{2}{3} (C_{A0}(t) - C_{A0SS}) - \frac{2}{3} (C_{A1}(t) - C_{A1SS}) - \left[ 0 + \frac{20}{3} (C_{A1}(t) - C_{A1SS}) \right]
\]

Realize that if \( x(t) = C_{A1}(t) - C_{A1SS} \) then \( \frac{dx}{dt}(t) = \frac{dC_{A1}}{dt}(t) - 0 \)

Put in deviation variables

\[
20 \frac{dx}{dt}(t) = \frac{2}{3} u(t) - \frac{2}{3} x(t) - \frac{20}{3} x(t)
\]

For mixing point equation:

\[
0 = \frac{1}{3} C_{A0}(t) + \frac{2}{3} C_{A1}(t) - C_{A2}(t)
\]

\[
0 = \frac{1}{3} C_{A0SS} + \frac{2}{3} C_{A1SS} - C_{A2SS}
\]

\[
0 = \frac{1}{3} (C_{A0}(t) - C_{A0SS}) + \frac{2}{3} (C_{A1}(t) - C_{A1SS}) - (C_{A2}(t) - C_{A2SS})
\]

\[
0 = \frac{1}{3} u(t) + \frac{2}{3} x(t) - y(t)
\]
So the two equations in deviation form:

\[ 20 \frac{dx}{dt}(t) = \frac{2}{3} u(t) - \frac{2}{3} x(t) - \frac{20}{3} x(t) \]

\[ 0 = \frac{1}{3} u(t) + \frac{2}{3} x(t) - y(t) \]

5. State space form:
Dynamic equations must only have derivatives on LHS. Gather terms and put in order \(x_1, x_2\) etc. Put 0’s in if you need them.

\[ \frac{dx}{dt}(t) = -\frac{1}{30} x(t) + \frac{1}{30} u(t) \]

\[ y(t) = \frac{2}{3} x(t) + \frac{1}{3} u(t) \]

This is a little tricky. You only have 1 input, 1 state, and 1 output. All of your matrices are 1x1! \( A = \begin{bmatrix} -\frac{1}{30} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{30} \end{bmatrix}, C = \begin{bmatrix} \frac{2}{3} \end{bmatrix}, D = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \). The eigenvalue of the A matrix is just \(-\frac{13}{3}\) since it is “upper triangular” and that is the element on the “diagonal.” Normally \( A \) and \( C \) have the same number of columns as the number of states, \( C \) and \( D \) have the same number of columns as the number of inputs, \( A \) and \( B \) have the same number of rows as the number of states, \( B \) and \( D \) have the same number of rows as the inputs. If you “squish” them together, they should fit nicely like the following:

\[
\begin{array}{ccc}
A & || & B \\
\hline \\
& - & - \\
C & || & D
\end{array}
\]

6. Transfer function
You have the following differential and algebraic equations in deviation form:

\[ \frac{dx}{dt}(t) = -\frac{1}{30} x(t) + \frac{1}{30} u(t) \]

\[ y(t) = \frac{2}{3} x(t) + \frac{1}{3} u(t) \]

Taking the Laplace transform:

\[ s x(s) - x(t=0) = -\frac{11}{30} x(s) + \frac{1}{30} u(s) \]

\[ y(s) = \frac{2}{3} x(s) + \frac{1}{3} u(s) \]

But you know since \( x(t) \) is in deviation form, \( x(t=0) = 0 \). So

\[ s x(s) = -\frac{11}{30} x(s) + \frac{1}{30} u(s) \]
\[ y(s) = \frac{2}{3} x(s) + \frac{1}{3} u(s) \]

Solving a bit:

\[ sx(s) + \frac{11}{30} x(s) = \frac{1}{30} u(s) \]

\[ x(s) = \frac{\frac{1}{30}}{s + \frac{11}{30}} u(s) \]

Plug that into the \( y(s) \) equation

\[ y(s) = \frac{2}{3} \left( \frac{1}{s + \frac{11}{30}} u(s) \right) + \frac{1}{3} u(s) \]

\[ y(s) = \left( \frac{\frac{1}{35} + \frac{1}{3} \left( s + \frac{11}{30} \right)}{s + \frac{11}{30}} \right) u(s) \]

\[ y(s) = \left( \frac{\frac{1}{35} s + \frac{11}{90} + \frac{1}{45}}{s + \frac{11}{30}} \right) u(s) \]

You can put this in Lead-Lag form:

\[ g(s) = \frac{K (\xi s + 1)}{(\tau s + 1)} \]

\[ y(s) = \left( \frac{\frac{1}{3} s + \frac{11}{90} + \frac{2}{35}}{s + \frac{11}{30}} \right) u(s) \]

\[ y(s) = \left( \frac{\frac{1}{3} s + \frac{13}{90}}{s + \frac{11}{30}} \right) u(s) \]

\[ y(s) = \left( \frac{\frac{1}{3} s + \frac{13}{90}}{s + \frac{11}{30}} \right) \frac{30}{11} u(s) \]

\[ y(s) = \left( \frac{\frac{20}{11} \left( \frac{1}{3} s + \frac{13}{90} \right)}{\left( \frac{30}{11} s + 1 \right)} \right) u(s) \]
\[ y(s) = \left( \frac{10}{11}s + \frac{13}{33} \right) u(s) \]

\[ y(s) = \left( \frac{13}{33} \frac{33}{13} \frac{10}{11}s + 1 \right) u(s) \]

\[ y(s) = \left( \frac{30}{33} \frac{30}{11}s + 1 \right) u(s) \]

7. What is \( u(s) \)?

From the graph
\[ C_{A_0}(t) = 1 + H(t) - H(t - 10) \]

That means the input deviation is just:

\[ u(t) = H(t) - H(t - 10) \]

A positive step at time \( t = 0 \) of size 1 and a step at time \( t = 10 \) of size -1.

Laplace of these two functions gives:

\[ u(s) = \frac{1}{s} + (-1) \frac{1}{s} e^{-10s} \]

8. What are \( y(t = 0) \) and \( y(t = \infty) \) using IVT and FVT.

We know the following:

\[ y(s) = g(s)u(s) \]

We now know

\[ y(s) = \left( \frac{13}{33} \frac{30}{13} \frac{10}{11}s + 1 \right) u(s) \]

and

\[ u(s) = \frac{1}{s} + (-1) \frac{1}{s} e^{-10s} \]

So that

\[ y(s) = \left( \frac{13}{33} \frac{30}{13} \frac{10}{11}s + 1 \right) \left( \frac{1}{s} + (-1) \frac{1}{s} e^{-10s} \right) \]

For IVT, \( y(t = 0) \) is \( sy(s) \) evaluated at \( s = \infty \)

\[ y(t = 0) = sy(s) |_{s = \infty} = s \left( \frac{13}{33} \frac{30}{13} \frac{10}{11}s + 1 \right) \left( \frac{1}{s} + (-1) \frac{1}{s} e^{-10s} \right) |_{s = \infty} \]

Cancel \( s \):

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\[ y(t = 0) = \left( \frac{13}{33} \left( \frac{30}{13} s + 1 \right) \right) \left( 1 - e^{-10s} \right) \bigg|_{s=\infty} \]

\[ y(t = 0) = \left( \frac{13}{33} \left( \frac{30}{13} \infty + 1 \right) \right) \left( 1 - e^{-10\infty} \right) \bigg|_{s=\infty} \]

\[ y(t = 0) = \left( \frac{13}{33} \left( \frac{30}{13} \infty + 1 \right) \right) (1 - 0) \bigg|_{s=\infty} \]

Must use L’Hôpital’s or the multiply by \((1/s)^N\) trick. In this case, they should give you

\[ y(t = 0) = \left( \frac{13}{33} \left( \frac{30}{13} s + 1 \right) \right) = \left( \frac{13}{33} \right) \left( \frac{30}{13} \right) \left( \frac{11}{30} \right) = \frac{1}{3} \]

This makes sense. A lead lag system should have some initial “jump” at time 0 for a step input, the direct effect. Note this is for \(y(t)\), not the actual value \(C_{A2}(t)\) due to deviation.

For final value, \(y(t = \infty)\) is \(sy(s)\) evaluate at \(s = 0\):

\[ y(t = \infty) = sy(s) \bigg|_{s=0} = s \left( \frac{13}{33} \left( \frac{30}{13} s + 1 \right) \right) \left( \frac{1}{s} + (-1) \frac{1}{s} e^{-10s} \right) \bigg|_{s=0} \]

\[ y(t = \infty) = sy(s) \bigg|_{s=0} = \left( \frac{13}{33} \left( \frac{30}{13} s + 1 \right) \right) \left( 1 - e^{-10s} \right) \bigg|_{s=0} \]

\[ y(t = \infty) = sy(s) \bigg|_{s=0} = \left( \frac{13}{33} \left( \frac{30}{13} \infty + 1 \right) \right) \left( 1 - e^{-10\infty} \right) \bigg|_{s=0} \]

\[ y(t = \infty) = sy(s) \bigg|_{s=0} = \left( \frac{13}{33} \right) \left( 1 - e^{-10\times0} \right) \bigg|_{s=0} \]

\[ y(t = \infty) = sy(s) \bigg|_{s=0} = \left( \frac{13}{33} \right) (1 - 1) \bigg|_{s=0} \]

\[ y(t = \infty) = sy(s) \bigg|_{s=0} = \left( \frac{13}{33} \right) (0) \bigg|_{s=0} = 0 \]

This also makes sense. The input returns to the normal value of \(u = 0\). For a lead-lag system, it should also return to 0. Note this is for \(y(t)\), not the actual value \(C_{A2}(t)\) due to deviation.

9. Find \(y(t)\) given \(u(t)\)

\[ y(s) = \left( \frac{K(\xi s + 1)}{(\tau s + 1)} \right) \left( \frac{1}{s} + (-1) \frac{1}{s} e^{-10s} \right) \]
\[ y(s) = \left( \frac{K(\xi s + 1)}{\tau s + 1} \right) \left( \frac{1}{s} \right) - \left( \frac{K(\xi s + 1)}{\tau s + 1} \right) \left( \frac{1}{s} \right) e^{-10s} \]

\[ y(s) = \left( \frac{K(\xi s + 1)}{\tau s + 1} \right) \left( \frac{1}{s} \right) - \left( \frac{K(\xi s + 1)}{\tau s + 1} \right) \left( \frac{1}{s} \right) e^{-10s} \]

So we know

\[ y(t) = y_1(t) - y_1(t - 10) \]

The total response is just the step response of a Lead-Lag system MINUS the step response of a Lead-Lag system at time \( t = 10 \). So you just need to find \( y_1(t) \)

\[ y_1(s) = \left( \frac{K(\xi s + 1)}{\tau s + 1} \right) \left( \frac{1}{s} \right) = \frac{A_1}{s + \frac{1}{\tau}} + \frac{A_2}{s} \]

\[ y_1(s) = \left( \frac{K(\xi s + 1)}{\tau s + 1} \right) \left( \frac{1}{s} \right) = -\left( \frac{1 - \xi}{s + \frac{1}{\tau}} \right) + \frac{K}{s} \]

\[ y_1(t) = K\mathcal{H}(t) - \left( 1 - \frac{\xi}{\tau} \right) e^{-\frac{t}{\tau}} \mathcal{H}(t) \]

So that \( y_1(t - 10) \) is

\[ y_1(t) = K\mathcal{H}(t - 10) - \left( 1 - \frac{\xi}{\tau} \right) e^{-\frac{t-10}{\tau}} \mathcal{H}(t - 10) \]

So that

\[ y(t) = \left( K\mathcal{H}(t) - \left( 1 - \frac{\xi}{\tau} \right) e^{-\frac{t}{\tau}} \mathcal{H}(t) \right) - \left( K\mathcal{H}(t - 10) - \left( 1 - \frac{\xi}{\tau} \right) e^{-\frac{t-10}{\tau}} \mathcal{H}(t - 10) \right) \]

And \( C_{A2}(t) \) (which is found from \( y(t) = C_{A2}(t) - C_{A2ss} \)) becomes:

\[ C_{A2}(t) = \frac{4}{9} + \left( K\mathcal{H}(t) - \left( 1 - \frac{\xi}{\tau} \right) e^{-\frac{t}{\tau}} \mathcal{H}(t) \right) - \left( K\mathcal{H}(t - 10) - \left( 1 - \frac{\xi}{\tau} \right) e^{-\frac{t-10}{\tau}} \mathcal{H}(t - 10) \right) \]
But that is in deviation, so negative $y(t)$ is ok. The real values for the input and output are:

![Graph](image)

This also makes sense. You increase the feed concentration at time $t = 0$ so some immediately goes around the bypass to the sensor (the immediate jump). This material is not reacted. The reactor is consuming $A$, so the concentration in the outlet should not stay that high once the product from the reactor reaches the mixing point. The steady state value matches the gain of the system and is higher for an increase in the input concentration.

### 7.3 Extruder Problem

Your job at GameCock Co. Inc. requires that you improve the operation of an extruder used to produce thermoset plastic. This process takes a feed of plastic pellets and mixes them to produce a final product. The amount of crosslinked polymer determines the final product quality.

Assume the following:

- The mass flow rate of pellets entering is constant, $M$
- The feed temperature is $T_o(t)$
- The percent of crosslinked polymer in the feed is $W_o(t)$
- The length of the extruder is $L$ with cross-section $A$
- The screw extruder can be assumed to be well-mixed
• The volumetric crosslinking rate (mass based) is \( r(t) = kT(t) \)

• The heat of reaction for the crosslinking is negligible

• The extruder is poorly insulated and the heat loss rate to the environment is given as \( hA(T(t) - T_{atm}) \)

• The atmospheric temperature \( T_{atm} \) is constant

• The deviation feed percentage and deviation feed temperature can be treated as inputs

• The deviation temperature of the product is measured, \( y(t) = T(t) - T_{ss} \)

• Physical properties are constant

### 7.4 Solution

• Assume weight percent \( W(t) \) is mass % and total mass of polymer \( V\rho = LA\rho (kg) \)
  
  - Total mass of crosslinked polymer is \( LA\rho W(t)(kg \text{ cl polymer}) \)

• Mass balance on crosslinked polymer and Steady State (SS) eqn

\[
LA\rho \frac{dW}{dt}(t) = MW_o(t) - MW(t) + LAkT(t)
\]

\[
0 = MW_{oss} - MW_{ss} - LAkT_{ss}
\]

  - \( M \) is mass flow of all polymer rate, \((kg/s)\), so units on each term are \((kg \text{ cl polymer} / s)\)

  - Reaction rate for creating crosslinked polymer is mass based and volumetric, \( r(t) = kT(t) \ (kg \text{ cl polymer} / m^3s) \)

  - All units on all terms now should be the same, including accumulation: \((kg \text{ cl polymer} / s)\)

• Energy balance on extruder and Steady State (SS) eqn

\[
(LA\rho C_p) \frac{dT}{dt}(t) = (MC_p)T_o(t) - (MC_p)T(t) - (hA)(T(t) - T_{atm})
\]

\[
0 = (MC_p)T_{oss} - (MC_p)T_{ss} - (hA)(T_{ss} - T_{atm})
\]

  - \( M \) is total polymer flowing in.out \((kg/s)\), both crosslinked and non have same \( C_p \)

  - All terms should now be energy / time, \((J/s)\)

  - No heat of reaction term, assume heat loss is total rate, \((J/s)\)
• Try to go to deviation form: use dynamic eqn and subtract the steady state equation while matching terms:

\[ \frac{d}{dt} \left( \frac{L \rho}{d} \right)(t) = M \left( W(t) - W_{ss} \right) - M \left( W(t) - W_{ss} \right) + \left( L \alpha k \right) (T(t) - T_{ss}) \]

\[ (L \rho C_p) \frac{dT}{dt}(t) = (MC_p) (T(t) - T_{ss}) - (MC_p)(T(t) - T_{ss}) - (hA)(T(t) - T_{ss}) \]

• Identify state variables. Anything with a \( \frac{d}{dt} \) is a state. In this case, \( W(t) \) and \( T(t) \).

- \( x_1(t) = W(t) - W_{ss} \)
- \( x_2(t) = T(t) - T_{ss} \)

- Derivatives are the same because the steady state value does not change with time, \( \frac{dx_1}{dt}(t) = \frac{dW}{dt}(t), \frac{dx_2}{dt}(t) = \frac{dT}{dt}(t) \)

\[ \frac{d}{dt} \left( x_1(t) \right) = \frac{d}{dt} \left( W(t) - W_{ss} \right) \]

\[ \frac{d}{dt} \left( x_2(t) \right) = \frac{d}{dt} \left( W(t) \right) + \frac{d}{dt} \left( -W_{ss} \right) = \frac{dW}{dt}(t) \]

• Use state / input variables, collect terms, include measurement \( y(t) \) equation

\[ L \rho \frac{dx_1}{dt}(t) = Mu_2(t) - Mx_1(t) + (L \alpha k) x_2(t) \]

\[ (L \rho C_p) \frac{dx_2}{dt}(t) = (MC_p) u_1(t) - (MC_p) x_2(t) - (hA) x_2(t) \]

\[ y(t) = T(t) - T_{ss} = x_2(t) \]

• Algebra to get in matrix / state space form (get \( d/dt \) on LHS, put in 0s) and identify \( A, B, C, \) and \( D \):

\[ \frac{dx_1}{dt}(t) = \left( \frac{-M}{L \rho} \right) x_1(t) + \left( \frac{L \alpha k}{L \rho} \right) x_2(t) + 0 \quad u_1(t) + \left( \frac{M}{L \rho} \right) u_2(t) \]

\[ \frac{dx_2}{dt}(t) = 0 \quad x_1(t) + \left( \frac{-MC_p + hA}{L \rho C_p} \right) x_2(t) + \left( \frac{MC_p}{L \rho C_p} \right) u_1(t) + 0 \quad u_2(t) \]

\[ y(t) = 0 \quad x_1(t) + 1 \quad x_2(t) + 0 \quad u_1(t) + 0 \quad u_2(t) \]

• So in the state space form \( \frac{dx}{dt} = Ax + Bu \) and \( y = Cx + Du \)

\[ A = \begin{bmatrix} \frac{M}{L \rho} & \left( \frac{L \alpha k}{L \rho} \right) \\ 0 & \left( \frac{-MC_p + hA}{L \rho C_p} \right) \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 \\ \frac{MC_p}{L \rho C_p} \end{bmatrix} \]

\[ C = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

\[ D = \begin{bmatrix} 0 & 0 \end{bmatrix} \]
• Eigenvalues of $A$ tell you stability of open-loop system. In this case, the $A$ matrix is diagonal, so the eigenvalues are easy to obtain. They are negative, so the system is stable.

• You may notice that measurement temperature is also the second state. It only depends on $x_2$ and $u_1$.

• You could take the Laplace of the system to get:

$$sx_1(s) = \left(\frac{-M}{LA\rho}\right)x_1(s) + \left(\frac{LAk}{LA\rho}\right)x_2(s) + 0 \quad u_1(s) + \left(\frac{M}{LA\rho}\right)u_2(s)$$

$$sx_2(s) = 0 \quad x_1(s) + \left(\frac{- (MC_p + hA)}{LA\rho C_p}\right)x_2(s) + \left(\frac{MC_p}{LA\rho C_p}\right)u_1(s) + 0 \quad u_2(s)$$

$$y(s) = 0 \quad x_1(s) + 1 \quad x_2(s) + 0 \quad u_1(s) + 0 \quad u_2(s)$$

– These equations could be manipulated to end up with the following transfer function representation:

$$y(s) = (g_1(s))u_1(s) + (g_2(s))u_2(s)$$

* Since $u_2$ has no effect on $x_2$ ($y$) this means $g_2(s) = 0$
Chapter 8

Frequency Analysis

8.1 Basic Bode Plot Rules

8.1.1 Amplitude Ratio

1. Amplitude Ratios for systems in series multiply.
2. The gain of the system is usually the low frequency AR value.
3. The high-frequency slope is the relative degree (how many more zeros than poles there are).
4. The AR will “bend” down at a pole. The frequency corresponds to $\frac{1}{\tau}$.
5. The AR will “bend” up at a zero. The frequency corresponds to $\frac{1}{\xi}$.
6. The AR for a second-under damped system may exhibit a “bump.”
7. A time delay is not seen in the AR, $AR(\omega) = 1$.
8. For a pure-capacity system (pole at $s = 0$) the low frequency AR has a slope of $-1 \frac{\text{decade}}{\text{decade}}$ for a single integrator.

8.1.2 Phase Angle

1. Phase angles for systems in series add.
2. $\phi$ starts at $0^\circ$ and goes to $-90^\circ$ at high frequency for a pole.
3. $\phi$ starts at $0^\circ$ and goes to $90^\circ$ at high frequency for a negative (LHP) zero ($\xi > 0$).
4. $\phi$ starts at $0^\circ$ and goes to $-90^\circ$ at high frequency for a positive (RHP) zero ($\xi < 0$).
5. $\phi$ starts at $-90^\circ$ and does not change for a pure integrator.
6. For individual poles and zeros which go from 0 to $\pm 90^\circ$ the frequency where $\phi$ is around $45^\circ$ is about where the AR bend occurs, $\frac{1}{\tau}$ or $\frac{1}{\xi}$.
7. $\phi$ goes to $-\infty$ at high frequency for a time delay.
8.2 Bode Plots of Simple Systems

\[ \frac{K}{(\tau s + 1)}, K=100, \tau = 10 \]

\[ \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}, K=70, \tau_1 = 10, \tau_2 = 7 \]
8.3 Derivations of Frequency Response for Simple Systems

8.3.1 First-Order System

\[ g(s) = \frac{K}{\tau s + 1} \]
\[ g(j\omega) = \frac{K}{\tau j\omega + 1} \]
\[ g(j\omega) = \frac{K}{1 + \tau \omega j} \]
\[ g(j\omega) = \frac{K}{1 + \tau \omega j} \frac{1 - \tau \omega j}{1 - \tau \omega j} \]
\[ g(j\omega) = \frac{K(1 - \tau \omega j)}{(1 + \tau \omega j)(1 - \tau \omega j)} \]
\[ g(j\omega) = \frac{K - K\tau j}{1 + \tau^2 \omega^2 j^2} \]
\[ g(j\omega) = \frac{K - K\tau j}{1 + \tau^2 \omega^2 (-1)} \]
\[ g(j\omega) = \frac{K - K\tau j}{1 - \tau^2\omega^2} \]

\[ AR(\omega) = |g(j\omega)| = \sqrt{\left(\frac{K}{1 - \tau^2\omega^2}\right)^2 + \left(\frac{-K\tau}{1 - \tau^2\omega^2}\right)^2} \]

For phase angle as a function of frequency \( \omega \)

\[ \phi(\omega) = \angle g(j\omega) = \arctan \left( \frac{b}{a} \right) = \arctan \left( \frac{-K\tau}{1 - \tau^2\omega^2} \right) \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan (-\tau\omega) \]

### 8.3.2 Second-Order System

\[ g(s) = \frac{K}{\tau^2 s^2 + 2\tau\zeta s + 1} \]

\[ g(j\omega) = \frac{K}{\tau^2(j\omega)^2 + 2\tau\zeta(j\omega) + 1} \]

\[ g(j\omega) = \frac{K}{\tau^2(-1)\omega^2 + 2\tau\zeta j\omega + 1} \]

\[ g(j\omega) = \frac{K}{1 - \tau^2\omega^2 + 2\tau\zeta j\omega} \]

\[ g(j\omega) = \frac{K}{(1 - \tau^2\omega^2) + 2\tau\zeta j\omega} \]

\[ g(j\omega) = \frac{K((1 - \tau^2\omega^2) - 2\tau\zeta j\omega)}{(1 - \tau^2\omega^2) - (2\tau\zeta j\omega)^2} \]
\[ g(j\omega) = \frac{K((1 - \tau^2\omega^2) - 2\tau\zeta j\omega)}{(1 - \tau^2\omega^2)^2 - (-1)(2\tau\zeta\omega)^2} \]

\[ g(j\omega) = \frac{K((1 - \tau^2\omega^2) - 2\tau\zeta j\omega)}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2} \]

\[ AR(\omega) = |g(j\omega)| = \sqrt{\left(\frac{K(1 - \tau^2\omega^2)}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2}\right)^2 + \left(\frac{2\tau\zeta\omega}{(1 - \tau^2\omega^2)^2 + (2\tau\zeta\omega)^2}\right)^2} \]

For phase angle as a function of frequency \( \omega \)

\[ \phi(\omega) = \angle g(j\omega) = \arctan \left( \frac{b}{a} \right) = \arctan \left( \frac{-2\tau\zeta\omega}{(1 - \tau^2\omega^2)} \right) \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan \left( \frac{-2\tau\zeta\omega}{(1 - \tau^2\omega^2)} \right) \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan \left( -\tau\omega \right) \]

### 8.3.3 Time Delay System

\[ g(s) = e^{-\alpha s} \]

\[ g(j\omega) = e^{-\alpha j\omega} \]

\[ g(j\omega) = e^{-\alpha \omega j} \]

Using the Euler identity:

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]
\[ g(j\omega) = e^{(-\alpha\omega)}j = \cos(-\alpha\omega) + j\sin(-\alpha\omega) \]

\[ g(j\omega) = \cos(-\alpha\omega) + \sin(-\alpha\omega) j \]

\[ AR(\omega) = |g(j\omega)| = \sqrt{(\cos(-\alpha\omega))^2 + (\sin(-\alpha\omega))^2} \]

\[ AR(\omega) = |g(j\omega)| = \sqrt{1} = 1 \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{\sin(-\alpha\omega)}{\cos(-\alpha\omega)}\right) \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan(\tan(-\alpha\omega)) \]

\[ \phi(\omega) = \angle g(j\omega) = -\alpha\omega \]

### 8.4 Frequency Response Questions

1. The Bode Plot for a first order system is given below. Identify the transfer function for the system.

![Bode Plot](image-url)

2. Sketch the Bode plot for the following transfer function. Label any distinguishing characteristics.

\[ g(s) = \frac{100e^{-2s}}{10s + 1} \]

3. You are in charge of operating the sludge furnace at the local Ideal Gas company plant. You must design a holding tank with limited level variation, given that the supply
flow of sludge varies beyond your control. The flow rate from the upstream process varies with a period of 45 min and an amplitude of $\pm 1 \text{ m}^3/\text{hr}$. Your goal is to calculate the cross sectional area of a buffer tank that will vary in height by $\pm 0.1 \text{ m}$. The flow rate from the tank is given as $F = kh$ where $k = 1 \text{ m}^2/\text{hr}$.

a. What is the frequency of upstream oscillation in $\text{rad}/\text{hr}$?

b. What is the transfer function for the system in the form $\frac{K}{\tau s + 1}$ relating the upstream input flow rate to the tank liquid level?

c. For this system, what is the expression for the Amplitude Ratio as a function of $\omega$?

d. What is the area of the tank in $\text{m}^2$ that will limit level variation to $\pm 0.1 \text{ m}$?

4. Your boss at the Ideal Gas Company put you in charge of analyzing two tanks, each with cross sectional area of $2 \text{ m}^2$. The tanks are arranged in series. The flow from tank 1 to tank 2 is $F_1 = kh_1$ and the flow from tank 2 is $F_2 = kh_2$. The flow into the first tank is known to vary with a frequency of $0.5 \text{ rad}/\text{hr}$. You are told that $k = 2 \text{ m}^2/\text{hr}$.

a. What is the transfer function for the process relating the flow into tank 1 to the flow out of tank 2?

b. For this system, what is the expression for the Amplitude Ratio and Phase Angle as a function of $\omega$?

c. What is amplitude of the variation in the flow out of tank 2 as a function of $\omega$?

d. For a frequency of oscillation of $10 \text{ rad}/\text{hr}$, what is amplitude of the variation in the flow out of tank 2?
8.5 Bode Plot Modeling

For the following Bode Plots, determine the SS gain for the transfer functions (if possible). What is the relative degree? Does the system have time delay? Are there any obvious poles or zeros? If so, what are approximate $\tau$ and $\xi$ values? Is there a pole at $s = 0$? Could the system be second order underdamped? Why?

8.5.1 Solution

The steady state gain is found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^1$ and $10^2$, approximately 80. The high frequency slope of the Amplitude Ratio is 0, so there are as many poles as zeros. The plot “bends” down at approximately $\omega = 10^{-2}$ rad/s so there is a pole with $\tau = \frac{1}{0.1} = 100$. The amplitude ratio bends up around $\omega = 10^0$ rad/s so there is a zero with $\xi = 1$. The phase angle $\phi$ shows that the zero is negative, since the high frequency $\phi$ is 0. A pole always adds $-90^\circ$ at high frequency. A LHP zero adds $90^\circ$ at high frequency.
The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is not well-defined. That means there is a pole at $s = 0$, a pure-integrating system. The initial slope of the Amplitude Ratio is $-1 \ \text{decade}^{-1}$ so there is only one integrator. The high frequency slope of the Amplitude Ratio is $-2 \ \text{decade}^{-1}$, so there are two more poles than zeros. The plot “bends” down at approximately $\omega = 10^{-2} \ \text{rad/s}$ so there is a pole with $\tau = \frac{1}{\omega} = 100$. The phase angle $\phi$ shows that there is a pure capacity pole since it starts at $-90^\circ$ at low frequency. A pole adds $-90^\circ$ at high frequency.
The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^0$ and $10^1$, approximately 8. The initial slope of the Amplitude Ratio is $0 \frac{\text{decades}}{\text{decade}}$, so there are no integrators. The high frequency slope of the Amplitude Ratio is $-2 \frac{\text{decades}}{\text{decade}}$, so there are two more poles than zeros. The plot “bends” down at approximately $\omega = 10^{-1} \frac{\text{rad}}{s}$ so there is a pole with $\tau = \frac{1}{1} = 10$. The phase angle $\phi$ shows that there are probably two poles since it goes to $-180^\circ$ at high frequency. The bump at $\omega = 10^{-1} \frac{\text{rad}}{s}$ means it is a second-order underdamped system that can oscillate at this frequency. Technically, you could have a zero and three poles to get a similar response (bend up, down, down, down) but it probably would not be as “crisp.”

The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^0$ and $10^1$, approximately 8. The initial slope of the Amplitude Ratio is $0 \frac{\text{decades}}{\text{decade}}$, so there are no integrators. The high frequency slope of the Amplitude Ratio is $-2 \frac{\text{decades}}{\text{decade}}$, so there are two more poles than zeros. The plot “bends” down at approximately $\omega = 10^{-1} \frac{\text{rad}}{s}$ so there is a pole with $\tau = \frac{1}{1} = 10$. The phase angle $\phi$ shows that there are probably two poles since it goes to $-180^\circ$ at high frequency. The bump at $\omega = 10^{-1} \frac{\text{rad}}{s}$ means it is a second-order underdamped system that can oscillate at this frequency. Technically, you could have a zero and three poles to get a similar response (bend up, down, down, down) but it probably would not be as “crisp.”

The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^0$ and $10^1$, approximately 8. The initial slope of the Amplitude Ratio is $0 \frac{\text{decades}}{\text{decade}}$, so there are no integrators. The high frequency slope of the Amplitude Ratio is $-2 \frac{\text{decades}}{\text{decade}}$, so there are two more poles than zeros. The plot “bends” down at approximately $\omega = 10^{-1} \frac{\text{rad}}{s}$ so there is a pole with $\tau = \frac{1}{1} = 10$. The phase angle $\phi$ shows that there are probably two poles since it goes to $-180^\circ$ at high frequency. The bump at $\omega = 10^{-1} \frac{\text{rad}}{s}$ means it is a second-order underdamped system that can oscillate at this frequency. Technically, you could have a zero and three poles to get a similar response (bend up, down, down, down) but it probably would not be as “crisp.”
The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^2$ and $10^3$, approximately 200. The initial slope of the Amplitude Ratio is $0 \frac{{\text{decades}}}{{\text{decade}}}$ so there are no integrators. The high frequency slope of the Amplitude Ratio is $-2 \frac{{\text{decades}}}{{\text{decade}}}$, so there are two more poles than zeros. The plot “bends” down at approximately $\omega = 10^{-3} \frac{{\text{rad}}}{{\text{s}}}$ so there is a pole with $\tau = \frac{1}{0.001} = 1000$. The plot also “bends” down at approximately $\omega = 10^0 \frac{{\text{rad}}}{{\text{s}}}$ so there is a pole with $\tau = 1$. The phase angle $\phi$ shows that there are probably two poles since it goes to $-180^\circ$ at high frequency.
The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^{-2}$ and $10^{-1}$, approximately 0.05. The initial slope of the Amplitude Ratio is $0 \, \text{decades/decade}$, so there are no integrators. The high frequency slope of the Amplitude Ratio is $-1 \, \text{decades/decade}$, so there is one more pole than zeros. The plot “bends” down at approximately $\omega = 10^{-1} \, \text{rad/s}$ so there is a pole with $\tau = \frac{1}{0.1} = 10$. The phase angle $\phi$ shows that there is a time delay since it head to $-\infty$.

The steady state gain is usually found from the low frequency slope of the Amplitude Ratio. In this case, it is somewhere between $10^{-1}$ and $10^{0}$, approximately 0.2. The initial slope of the Amplitude Ratio is $0 \, \text{decades/decade}$, so there are no integrators. The high frequency slope of the Amplitude Ratio is $-2 \, \text{decades/decade}$, so there are two more pole than zeros. The plot “bends” down at approximately $\omega = 10^{-1} \, \text{rad/s}$ so there is a pole with $\tau = \frac{1}{0.1} = 10$. 
Since you cannot see another bend, we can assume both poles are at about the same value. The phase angle $\phi$ shows that there is a time delay since it head to $-\infty$. 
Consider the following system:

Hot and cold feed streams are mixed in a tank before being sent to a reactor. The reactor cooling jacket flow may periodically change and this flow rate is measured, $F_c(t)$. Temperature measurements are available after the mixing tank ($T_1(t)$) and well after the reaction system, $T_2(t)$. The following diagram shows the Process Flow Diagram:

1. Draw a feedback control system above to regulate the product temperature.

2. Draw a feedforward control system above to regulate the product temperature.
3. Draw a cascade feedback control systems to regulate the product temperature.

4. Draw a combined feedforward and cascade system above to regulate the product temperature.
5. Given the following:

(a) The transfer function relating the control valve position to $T_0(s)$

$$T_0(s) = \frac{5}{s + 1} u(s) = g_1(s)u(s)$$

(b) The transfer function relating temperature $T_0(s)$ to $T_1(s)$

$$T_1(s) = \frac{s - 4}{-8s - 1} T_0(s) = g_2(s)T_0(s)$$

(c) The product stream takes 2 minutes to flow from $T_1$ to $T_2$

(d) The effect of $F_c$ on $T_2$ is modeled as

$$T_2(s) = \frac{-2}{6s + 1} e^{-s} F_c(s) = g_d(s)d(s)$$

(e) The local controller for $T_0$ is a P controller, $g_{c1} = K_c = 2$

(f) The controller for $T_2$ is $g_{c2}(s)$.

Draw the block diagram for the cascade control system.

6. For the inner loop controlling $T_0$, how much measurement delay would be possible before the control loop became unstable?

7. Assuming no delay in the inner control loop, calculate the ultimate gain for the master controller.

8. Sketch the Bode plot for the overall control system.

9. What is the closed-loop transfer functions for the system assuming the outer controller is a P controller, $K_c = 0.5 K_{CU}$?

10. How much offset is there in the control system for a disturbance of magnitude -3?

11. Draw the block diagram for a feedforward system assuming no feedback control.

12. Determine a feedforward controller assuming no feedback control.

13. Using no feedback control for the value from $u$ to $T_2$, develop an IMC control system and draw the block diagram.

14. Assuming that the cooling water flow is now adjustable $u_2$, sketch the 2x2 open-loop block diagram for the system assuming $y_1 = T_0$ and $y_2 = T_2$ and $u_1 = u$.

15. What is the 2x2 system gain? What are the poles of the multivariable system? Is there a system zero?

16. What is the 2x2 system RGA? What pairing does that suggest? Does that make sense? Why?
9.1 Solution

Overall Control System Example

Consider the following system:

Hot and cold feed streams are mixed in a tank before being sent to a reactor. The reactor cooling jacket flow may periodically change and this flow rate is measured, $F_c(t)$. Temperature measurements are available after the mixing tank ($T_0(t)$) and well after the reaction system, $T_2(t)$. The following diagram shows the Process Flow Diagram:

1. Draw a feedback control system above to regulate the product temperature.

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4. Draw a combined feedforward and cascade system above to regulate the product temperature.
5. Given the following:

(a) The transfer function relating the control valve position to $T_0(s)$

$$T_0(s) = \frac{5}{s + 1} u(s) = g_1(s)u(s)$$

(b) The transfer function relating temperature $T_0(s)$ to $T_1(s)$

$$T_1(s) = \frac{s - 4}{-8s - 1} T_0(s) = g_2(s)T_0(s)$$

(c) The product stream takes 2 minutes to flow from $T_1$ to $T_2$

(d) The effect of $F_c$ on $T_2$ is modeled as

$$T_2(s) = \frac{-2}{6s + 1} e^{-s} F_c(s) = g_d(s)d(s)$$

(e) The local controller for $T_0$ is a P controller, $g_{c1} = K_c = 2$

(f) The controller for $T_2$ is $g_{c2}(s)$.

Draw the block diagram for the cascade control system.

6. For the inner loop controlling $T_0$, how much measurement delay would be possible before the control loop became unstable?

Remember that the closed-loop transfer function from $T_{osp}$ to $T_0$ is

$$T_0 = \frac{g_{c1}g_1}{1 + g_{c1}g_1} T_{osp} = \frac{2 \frac{5}{s + 1}}{1 + 2 \frac{5}{s + 1}} T_{osp} = \frac{2 \times 5}{s + 1 + 2 \times 5} T_{osp}$$

But we aren’t really interested in the closed-loop transfer function. If you now introduce time delay in the process, so that the open-loop control + process is

$$g_{c1}g_1 = \frac{10}{s + 1} e^{-\alpha s}$$

Now, you must find the Amplitude Ratio and Phase Angle formulas for this $gg_c$. Since you have a first-order system ($K = 5, \tau = 1$) in series with a time delay and the P controller, the functions $AR$ and $\phi$ are:

$$AR(\omega) = \frac{10}{\sqrt{1 + \omega^2(1)^2}} \times 1$$

$$\phi(\omega) = atan(-\omega) - \omega \alpha$$

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Setting $\phi = -180^\circ = -\pi$ you cannot solve the phase angle equation. However, you can solve the AR equation setting $AR = 1$. You should get $\omega = 9.89 \text{ rad min}$. Now, you can solve the phase angle equation:

$$-\pi = \tan(-\omega) - \omega \alpha$$

$$\omega \alpha = \tan(-\omega) + \pi$$

$$\alpha = \frac{(\tan(-\omega) + \pi)}{\omega}$$

and the maximum delay with this controller gain is then $\alpha = 0.17\text{ minutes}$.

7. **Assuming no delay in the inner control loop, calculate the ultimate gain for the master controller.**

The inner control loop from $T_{0sp}$ to $T_0$ is given by

$$\frac{2}{s+1} \frac{5}{s+1} = \frac{10}{s+1} = \frac{10}{s+1} = \frac{10}{s+1} = \frac{10}{s+1}$$

The

$$T_1(s) = g_2(s)T_0(s) = \frac{s-4}{-8s-1} T_0(s) = \frac{4}{(8s+1)} T_0(s)$$

The whole process from $T_{0sp}$ to $T_2$ is:

$$\frac{10}{(1/11 s+1)} \frac{4}{(8s+1)} e^{-2s}$$

Which is two poles, a lead (with $\xi = -\frac{1}{4}$) and a time delay. Assuming $K_c = 1$

$$AR = \frac{1}{\sqrt{1 + \omega^2 (\frac{1}{11})^2}} \sqrt{1 + \omega^2 (\frac{1}{11})^2}$$

$$\phi = \tan(-\tau_1 \omega) + \tan(-\tau_2 \omega) + \tan(\xi \omega) - \omega \alpha$$

$$\phi = \tan(-\frac{1}{11} \omega) + \tan(-8 \omega) + \tan(-\frac{1}{4} \omega) - 2\omega$$

Solving for $\phi(\omega) = -\pi$ gives you a critical frequency of $\omega_c = 0.745 \text{ rad min}$ and $AR(\omega_c) = 0.611$. So the maximum gain for the controller is then $1/AR(\omega_c) = 1.63$.

8. **Sketch the Bode plot for the overall control system.**
You should note the gain at low $\omega$ for AR is 4. With two poles ($\tau$ at $\frac{1}{11}$ and 8) and a zero ($\zeta = -\frac{1}{3}$) the “bends” occur at 11, 4, and $\frac{1}{8}$, so they are not distinguishable. The high frequency slope in terms of decade/decade should be 1 since the relative degree is 1. The phase angle starts at 0 at low $\omega$ and rolls off to $-\infty$ due to the time delay.

9. **What is the closed-loop transfer functions for the system assuming the outer controller is a P controller, $K_c = 0.5 K_{CU}$?**

So $K_{CU} = 1.63$ so $g_{c2} = 0.8$. Using $g_{c1} = 2$ and the other transfer functions in the look ($g_1, g_2$, and time delay) the whole cascade system closed-loop block diagram for setpoint tracking is then:

$$T_2 = \left( \frac{\frac{80}{(\frac{80}{11}s + 1)}e^{-2s}}{1 + \left( \frac{\frac{80}{(\frac{80}{11}s + 1)}e^{-2s}}{40(\frac{-\frac{1}{4}s + 1}{8s + 1})e^{-2s}} \right)} \right) T_{2sp}$$

Since the disturbance only affects $T_2$, the overall relationship can be written as

$$T_2 = \left( \frac{\frac{80}{(\frac{80}{11}s + 1)}e^{-2s}}{1 + \left( \frac{\frac{80}{(\frac{80}{11}s + 1)}e^{-2s}}{4(\frac{-\frac{1}{4}s + 1}{8s + 1})e^{-2s}} \right)} \right) T_{2sp} + \left( \frac{-\frac{2}{(6s + 1)}e^{-s}}{1 + \left( \frac{\frac{80}{(\frac{80}{11}s + 1)}e^{-2s}}{40(\frac{-\frac{1}{4}s + 1}{8s + 1})e^{-2s}} \right)} \right) F_c$$

10. **How much offset is there in the control system for a disturbance of magnitude $-3$?**

We can assume that the setpoint does not change, so $T_{2sp} = 0$. The disturbance is then $F_c(t) = -3H(t)$ or $F_c(s) = \frac{-3}{s}$ so that

$$T_2(s) = \left( \frac{-\frac{2}{(6s + 1)}e^{-s}}{1 + \left( \frac{\frac{80}{(\frac{80}{11}s + 1)}e^{-2s}}{40(\frac{-\frac{1}{4}s + 1}{8s + 1})e^{-2s}} \right)} \right) \left( \frac{-3}{s} \right)$$
Using the final value theorem, we can find $T_2(t = \infty)$.

$$sT_2(s)|_{s=0} = \left( s \left( \frac{-2}{6s+1} e^{-s} \right) \right) \left( \frac{3}{s} \right) \bigg|_{s=0}$$

$$T_2(t = \infty) = sT_2(s)|_{s=0} = \left( \frac{-2}{(0+1)} \right) - 3 \bigg|_{s=0}$$

$$T_2(t = \infty) = \left( \frac{-2}{1 + \frac{20}{110}} (-3) \right) = 1.53$$

Remember, the setpoint value was 0, so for this disturbance using a P controller the system ends up at $T_2 = 1.53$. Without control, the $g_d$ gain is $-2$ and the size of the disturbance is $-3$ so $T_2$ would have ended up at 6, so P control helps a bit.

11. Draw the block diagram for a feedforward system assuming no feedback control.

![Block Diagram](image)

12. Determine a feedforward controller assuming no feedback control.
   The feedforward controller assumes a measured disturbance value. The feedforward controller is

$$g_{ff} = \frac{u}{d} = -\frac{g_d}{g}$$

The disturbance transfer function is

$$T_2(s) = \frac{-2}{6s+1} e^{-s} F_c(s) = g_d(s)d(s)$$

The process model is from $u$ to $T_2$ so

$$g(s) = g_1 g_2 e^{-2s} = \frac{5}{(s+1)} \left( \frac{(s-4)}{(-8s-1)} \right) e^{-2s}$$

$$g(s) = \frac{5}{(s+1)} \left( -\frac{1}{4} s + 1 \right) e^{-2s}$$

$$g(s) = \frac{20 \left( -\frac{1}{4} s + 1 \right) e^{-2s}}{(s+1) (8s+1)}$$
So the feedforward controller is then:

\[
g_{ff} = \frac{u}{d} = \frac{-g_d}{g} = \frac{-2}{(6s + 1)} e^{-s} \frac{(s + 1)(8s + 1)}{20 \left(-\frac{1}{4}s + 1\right) e^{-2s}}
\]

\[
g_{ff} = \frac{-2}{(6s + 1)} e^{-s} \frac{(s + 1)(8s + 1) e^{2s}}{20 \left(-\frac{1}{4}s + 1\right)}
\]

But you have a RHP pole in the transfer function, and time delay since \(e^{-s} e^{2s} = e^s\). We have set up the transfer function so that the RHP pole can be dropped and the gain does not change.

\[
g_{ff} = \frac{-2 (s + 1)(8s + 1)}{20 (6s + 1)}
\]

But the transfer function is improper. Include a filter to make it at least semi-proper.

\[
g_{ff} = \frac{-\frac{1}{10} (s + 1)(8s + 1)}{(6s + 1)(\lambda s + 1)}
\]

13. **Using no feedback control for the value from \(u\) to \(T_2\), develop an IMC control system and draw the block diagram.**

We know the process transfer function in the nominal case is given as

\[
g(s) = g_m(s) = \frac{20 \left(-\frac{1}{4}s + 1\right) e^{-2s}}{(s + 1)(8s + 1)}
\]

The IMC controller requires you find \(g_I = 1/g_m\)

\[
g_I(s) = \frac{1}{g_m(s)} = \frac{(s + 1)(8s + 1)}{20 \left(-\frac{1}{4}s + 1\right) e^{-2s}}
\]

\[
g_I(s) = \frac{1}{g_m(s)} = \frac{\frac{1}{20} (s + 1)(8s + 1) e^{2s}}{(-\frac{1}{4}s + 1)}
\]

But we still have a RHP pole in the denominator and “time prediction”. Plus you need a filter to make the transfer function semi-proper.

\[
g_I(s) = \frac{\frac{1}{20} (s + 1)(8s + 1)}{(\lambda s + 1)^2}
\]
14. Assuming that the cooling water flow is now adjustable $u_2$, sketch the 2x2 open-loop block diagram for the system assuming $y_1 = T_0$ and $y_2 = T_2$ and $u_1 = u$.

![Diagram](https://via.placeholder.com/150)

15. What is the 2x2 system gain? What are the poles of the multivariable system? Is there a system zero?

The system gain is found from the individual transfer function gains.

$$
K = \left| G(s) \right|_{s=0} = \left| \begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array} \right|_{s=0}
$$

$$
K = \left| \begin{array}{cc} g_1 & 0 \\ g_1g_2e^{-2s} & g_d \end{array} \right|_{s=0}
$$

$$
K = \left| \begin{array}{cc} 5 & 0 \\ 5 & -2 \end{array} \right|
$$

The poles are found from the individual system poles. Poles are then located at $s = -\frac{1}{6}$, $-\frac{1}{8}$, and $-1$. Zeros are found from

$$
\det G(s) = 0
$$

$$
\det \left| \begin{array}{cc} g_1 & 0 \\ g_1g_2e^{-2s} & g_d \end{array} \right| = 0
$$

$$
g_1g_d = 0
$$
\[ \frac{5}{s+1} \frac{-2}{(6s+1)} e^{-s} = 0 \]

The LHS only takes a value of 0 when \( s = \infty \), and even then you would have to use L’hopital’s rule to make sure. So no zeros to worry about.

16. **What is the 2x2 system RGA? What pairing does that suggest? Does that make sense? Why?**

The RGA is calculated for a 2x2 system gain matrix:

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\]

\[
\zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}}
\]

\[ RGA = \Lambda = \begin{bmatrix}
\frac{1}{1-\zeta} & -\zeta \\
-\zeta & \frac{1}{1-\zeta}
\end{bmatrix}
\]

In the case of the problem above,

\[
\begin{bmatrix}
5 & 0 \\
5 & -2
\end{bmatrix}
\]

So that \( \zeta = 0 \) and

\[ RGA = \Lambda = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

which suggests a pairing of \( u_1 \leftrightarrow y_1 \) and \( u_2 \leftrightarrow y_2 \). This makes sense because \( u_2 \) has no effect on \( y_1 \) so it would be totally useless to control \( y_1 \) using \( u_2 \).
Chapter 10

Multivariable Systems

Multivariable System Modeling

Multivariable systems can be modeled as dynamic systems using transfer functions the same way that SISO systems can be modeled using transfer functions. In multivariable systems, a vector of inputs goes into the transfer function and a vector of outputs comes out:

\[ y(s) = G(s) \ u(s) \]

Just like in multivariable steady-state modeling:

\[ \Delta y = K \Delta u \]

Where the multivariable system of equations represent steady-state relationships, in the dynamic case the multivariable transfer function represents dynamic relationships between the inputs and outputs. In the case of a 2 × 2 system, \( G(s) \) will be a 2 × 2 matrix with four transfer functions, \( g_{11}(s) \), \( g_{12}(s) \), \( g_{21}(s) \), and \( g_{22}(s) \). The first row is for the first set of equations relating the first output to the rest of the inputs.
The multiple transfer functions can be developed in the usual manner. Open-loop step tests for each process input could be used to determine gain, time-constant and time delay for simplified FOTD models, or fundamental mass and energy balances could be used to develop dynamic equations that can then be linearized and transformed into the Laplace domain.

### 10.1 Relative Gain Array

The Relative Gain Array (RGA) is a tool that can be used to help analyze multivariable systems. When considering control of multivariable control systems, one must consider interaction. In a $2 \times 2$ MIMO system, changing $u_1$ will usually affect both $y_1$ and $y_2$. Likewise, changing $u_2$ will usually affect both $y_1$ and $y_2$. Using our traditional SISO PID controllers, this can lead to problematic situations where two controllers “fight” each other significantly. The RGA can be used to help determine loop pairings for SISO controllers in a MIMO process.

For example, in the $2 \times 2$ system there are only two options: Option 1, pair $u_1 \leftrightarrow y_1$, $u_2 \leftrightarrow y_2$ OR Option 2, $u_1 \leftrightarrow y_2$, $u_2 \leftrightarrow y_1$. 

\[
y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s)
\]
\[
y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s)
\]
In a $3 \times 3$ MIMO system, there would be six options for loop pairing, this grows as $n!$.

The RGA can be calculated for a $2 \times 2$ system as follows. First, calculate the steadystate gain matrix, $K = G(s = 0)$. Next, determine $\zeta$ where

$$\zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}}$$

Then determine the RGA matrix, $\Lambda$

$$\Lambda = \begin{bmatrix} \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \\ \frac{-\zeta}{1-\zeta} & \frac{1}{1-\zeta} \end{bmatrix}$$

For a general $n \times n$ system, the RGA is given as follows:
\[ \Lambda = K \times (K^{-1})^T \]

The \( \times \) operator represents element by element multiplication of the two \( n \times n \) matrices. In Matlab, this can be done as follows:

\[ R = K \times \text{inv}(K) \]

Note that in the general \( n \times n \) case you are taking the inverse of the steady-state gain matrix. If the square system has no inverse, you cannot calculate the inverse. This also means that your equations are linearly dependent, implying that a linear combination of your inputs can be equivalent. For example, increasing \( u_1 \) and \( u_2 \) have the same effect on the outputs. This type of system cannot be controlled in all output directions.

### 10.1.1 RGA Rules

These are approximate rules for loop pairing. The RGA is a steady-state analysis tool and may not hold true in all situations. These are guidelines for first considerations in multivariable systems.

1. If the \( \lambda_{ij} \) element is less than or equal to zero, avoid pairing output \( i \) with input \( j \). This is the worst case for pairing and should be avoided.
2. If the \( \lambda_{ij} \) element is equal to one, pair output \( i \) with input \( j \).
3. If possible avoid cases of \( 0 < \lambda_{ij} < 0.5 \).
4. In all other cases, there will be interaction, but the quality of the closed-loop response depends on the controller tuning, the amount of nonlinearity, the magnitude of disturbances, and the process measurement noise.

### 10.1.2 Examples

**Example 1**

\[ K = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} \]

\[ \zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}} = \frac{2 \cdot 4}{-1 \cdot 3} = \frac{-8}{3} \]

Then determine the RGA matrix, \( \Lambda \)

\[ \Lambda = \begin{bmatrix} \frac{1}{1-\zeta} & -\zeta \\ \frac{1}{1-\zeta} & \frac{1}{1-\zeta} \end{bmatrix} \]

\[ \Lambda = \begin{bmatrix} \frac{1}{1+\frac{8}{3}} & \frac{8}{1+\frac{8}{3}} \\ \frac{1}{1+\frac{8}{3}} & \frac{1}{1+\frac{8}{3}} \end{bmatrix} \]
\[ \Lambda = \begin{bmatrix} \frac{3}{11} & \frac{8}{11} \\ \frac{2}{11} & \frac{3}{11} \end{bmatrix} \]

Implying that you should pair \( u_1 \leftrightarrow y_2, \ u_2 \leftrightarrow y_1 \) since the (1,2) element (row 1, column 2) and (2,1) elements are \( \frac{8}{11} \), close to \( 1 \).

**Example 2**

\[ K = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} \]

\[ \zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}} = \frac{2(4)}{-1(-3)} = \frac{8}{3} \]

Then determine the RGA matrix, \( \Lambda \)

\[ \Lambda = \begin{bmatrix} -1 & \frac{8}{3} \\ 4 & -1 \frac{8}{3} \end{bmatrix} \]

\[ \Lambda = \begin{bmatrix} \frac{3}{5} & \frac{8}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} \]

Here, the (1,1) and (2,2) elements are negative. Avoid the \( u_1 \leftrightarrow y_1, \ u_2 \leftrightarrow y_2 \) pairing in this case, so you should use the \( u_1 \leftrightarrow y_2, \ u_2 \leftrightarrow y_1 \) pairing.

**Example 3**

\[ K = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ -3 & 1 & 2 \end{bmatrix} \]

\[ \Lambda = \begin{bmatrix} 0.25 & -2.25 & 3 \\ 0 & 3.5 & -2.5 \\ 0.75 & -0.25 & 0.5 \end{bmatrix} \]

In row 2, the only good option appears to be pair \( y_2 \) with \( u_2 \). There will be interaction on this loop, as the value of 3.5 predicts. There are now two different ways to consider the problem. If you consider column 1 first, you would pair \( y_3 \) with \( u_1 \) as a value of 0.75 is better than 0.25, then end up with \( y_1 \) paired with \( u_3 \) for a value of 3. The alternative that would also be valid is pair \( y_1 \) with \( u_1 \) for a value of 0.25 and \( y_3 \) with \( u_3 \) for a value of 0.5. Either option is valid.
Example 4

\[ K = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ -3 & -3 & 2 \end{bmatrix} \]

\[ \Lambda = \begin{bmatrix} -1 & 2 & 0 \\ 3.2 & -2.8 & 0.6 \\ -1.2 & 1.8 & 0.4 \end{bmatrix} \]

First, consider column 1. Row elements (1,1) and (3,1) are both negative, implying that you should pair \( y_2 \) with \( u_1 \). Now, examine row 1. \( u_1 \) is already paired with \( y_2 \), so \( y_1 \) should be paired with \( u_2 \) since the (1,3) element is 0. This leave \( y_3 \) to be paired with \( u_3 \) for a value of 0.4. Every pairing will have interaction. This could be foreseen to some extent. Examine the “direction” of columns 1 and 2. Increasing either \( u_1 \) or \( u_2 \) will force the output measurements in almost the same direction.
Chapter 11
Numerical Optimization

Introduction

Optimization methods attempt to find the best solution to a problem. If the best solution can be expressed mathematically in terms of design variables, numerical methods may be available to help solve the problem. In many cases, the unknown variables that determine the quality of the solution may be limited by additional mathematical expressions.

Optimization plays a vital role in many situations. Everyday tasks such as walking across campus can be seen as optimization problems: minimize the distance traveled while staying within the bounds of the sidewalks. For engineers working in industry, each company expects employees to help maximize the profit for the company, within legal and ethical constraints. For many specific engineering tasks, numerical optimization methods become very useful for finding the best solution to a problem without resorting to trial-and-error methods.

This work attempts to provide an introduction to some basic concepts in the area of numerical optimization for algebraic problems. The reader is expected to have some experience with vector calculus and linear algebra. Examples are provided using Solver in Microsoft Excel. Topics considered include the objective function, algebraic constraints, optimality conditions, convexity analysis, classification of solution methods, problem relaxations, and deterministic global solutions.

The Objective Function

Numerical optimization methods typically assume that one can calculate a scalar value that is to be maximized or minimized. This is considered the cost function or the objective function. Generally, the cost function is a mathematical function of decision variables or unknowns. In many introductory calculus classes, a function of a single variable is minimized or maximized by finding the critical points where the first derivative is equal to zero. In more advanced calculus classes, a function of two variables can be minimized by finding points that make the gradient equal to zero. These concepts extend directly to more complicated cases.

In many engineering problems, you could have numerous decision variables. In order
to simplify things, these unknowns can be stacked together using vector notation. You
are not limited to two or three unknowns in a vector value. Typically, our vector with \( n \)
unknown values will be written simple as \( \mathbf{x} \), with \( \mathbf{x} \subset \mathbb{R}^n \). This means \( \mathbf{x} \) is some point in
a \( n \)-dimensional space. Usually we will know bounds on \( \mathbf{x} \), so this means our variables \( \mathbf{x} \)
belong to the set of points \( X \), which can also be written as \( x \in X \subset \mathbb{R}^n \).

For example, in a chemical plant design problem, the size of four reactors could
dictate the overall plant cost:

\[
\mathbf{x} = \begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
\]

Let us define the cost function as a function of the decision variables \( \mathbf{x} \) as \( f(\mathbf{x}) \). We
will assume that we only have a single cost function, so this function \( f \) can be thought
of as a mapping from any point in the set \( X \) to a number, \( f : X \to \mathbb{R} \). An optimization
routine must search the allowable solution space of the decision variables to find the
best value of the objective function that satisfies the problem constraints. The general
mathematical form of the problem could be written:

\[
\min f(\mathbf{x})
\]

subject to constraints on \( \mathbf{x} \)

There are many different objective functions one could seek to minimize or maximize.
Some examples include:

- The distance from a point in 2-D or 3-D space with variables \( x \), \( y \), and \( z \). \( \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \)

- The sum square error for a model as compared to actual measurements with model
parameters as variables

- The total distance traveled between two points, with variables indicating whether
or not a path was taken.

- The total cost for a chemical plant with the number and size of reactors as variables.

- The value of a stock portfolio with different investment options as variables.

- The overall strength of a composite material with individual material quantities
as variables.

Variables can be continuous like the size and temperature of a reactor or the position
in space. In other cases, variables can take binary values to indicate whether or not a
certain action is performed, 0 =no, 1 =yes. In other cases, variables take only integer
values, like the number of reactors in a chemical plant. Binary and integer variables
complicate things in many cases. This will be considered in more detail later.

Additionally, some problems require minimization while others require maximization.
Minimizing \( f(\mathbf{x}) \) is the same as maximizing \( -f(\mathbf{x}) \). We will only discuss minimization
problems in the current work.
Unconstrained Optimization

The general form for an unconstrained optimization problem can be written as:

$$\min f(x)$$

The critical points for this function occur when the gradient of $f(x)$ is equal to 0, \( \frac{\partial f}{\partial x}(x) = 0 \). The gradient of the objective function, \( \frac{\partial f}{\partial x}(x) = \nabla f(x) \), is a \( n \)-dimensional vector function of \( x \). For example, say your cost function maps points in a 2 dimensional space to a cost:

$$f(x) = x^3y^2 + x^2y^4 + 5x + 7y$$

The gradient of this function

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2y^2 + 2xy^4 + 5 \\ 2x^3y + 4x^2y^3 + 7 \end{bmatrix}$$

To find a point that minimizes this function, one would have to solve a fairly complex set of nonlinear algebraic equations. Obviously, this becomes difficult for even simple functions involving only two variables. You would have \( n \) different functions to solve in order to make the \( n \) different partial derivatives equal zero for a problem with \( n \) variables.

![Objective function surface for \( f(x) = x^3y^2 + x^2y^4 + 5x + 7y \).](image)

Figure 11.1: Objective function surface for \( f(x) = x^3y^2 + x^2y^4 + 5x + 7y \).

The gradient vector for \( f(x) \) is obviously very important for determination of the final solution. Given any point \( x_0 \) in \( X \), the gradient vector points in the direction of steepest increasing value of \( f(x) \) from the point \( x_0 \). One could imagine a hill-climbing method for maximization problems or a steepest descent method for minimization. For minimization, from a starting point \( x_0 \), one could perform iterative search looking for
\[ \nabla f(x) = 0 \] using the formula \[ x_{\text{new}} = x_{\text{old}} - K \frac{\partial f}{\partial x}(x_{\text{old}}). \] For some values of \( K \) this numerical method may be unstable. Remember that the result may only be a local optima, there may be better solutions in other parts of the solution space. Additionally, care should be taken to avoid saddle points. Second derivative information can provide additional information about the shape of the objective function at a point in the parameter space.

**Constraints**

In many problems, additional functions are used to limit the solution space. Constraints arise from the model equations of the problem. Variable bounds can be seen as constraints that limit the variable values to a limited region in the \( n \) dimensional space, \( \mathbb{R}^n \). For example, the variable for product inventory level during a given time period could be a related to the previous inventory level, the number of deliveries received over the period, and the number of orders sent during that period, resulting in the constraint:

\[ x_{t+1} = x_t + d_t - o_t \]

A general form for the constrained optimization problem is:

\[
\begin{align*}
\max & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0
\end{align*}
\]

with the constraint functions \( g : X \to \mathbb{R}^m \) for \( m \) separate constraints. This means that the function \( g \) maps the points in the set \( X \) to a \( m \) dimensional space. Inequality constraints are like fences that keep the variable values inside a specified region.

Consider the case where \( g(x) \) is a single constraint, a scalar function of the vector \( x \). The problem can be written using a Lagrangian relaxation to form an unconstrained problem. The new problem becomes:

\[
\max f(x) - \lambda g(x)
\]

where \( \lambda \) is a non-negative value, \( \lambda \geq 0 \). This can be seen as a penalization for violating the constraint \( g(x) \leq 0 \). When \( g(x) \) is positive, the objective function increases, so it is desirable to have \( g(x) \) be negative. The unconstrained optimization problem can be solved iteratively, changing the values of \( \lambda \) until the minimum value for \( \lambda \) is found that still keep \( g(x) \leq 0 \).

**Smallville Example**

You have decided to take a job in Smallville. You are searching for a house inside the city limits, but you would like to find a house close to your job in order to minimize your commute distance and time. You can assume that house locations in Smallville are denoted by an \( x \times y \) position with \( 0 \leq x \leq 3 \) and \( 0 \leq y \leq 2 \). Your business location is at \( x = 4, y = 3 \). There is a dump at the \( x = 0, y = 1 \) location and you can smell it for at least 1.5 miles.
The solution to this problem is pretty obvious, but we should try to formulate the problem in the standard form. Our variables are \( x \) and \( y \) so our vector of unknowns is \( \mathbf{x} = [x \ y]^T \). The distance from work is \( \sqrt{(x - 4)^2 + (y - 3)^2} \). There are limits on \( x \) and \( y \) based on the city limits. The dump poses an interesting constraint, such that \( x \) and \( y \) must be at least 1.5 units away from the \((0, 1)\) position. This constraint can be written in the form \( (x - 0)^2 + (y - 1)^2 \geq 1.5 \).

\[
\begin{align*}
\min & \quad \sqrt{(x - 4)^2 + (y - 3)^2} \\
\text{s.t.} & \quad 0 \leq x \leq 3 \\
& \quad 0 \leq y \leq 2 \\
& \quad (x - 0)^2 + (y - 1)^2 \geq 1.5
\end{align*}
\]

We can now identify \( f(\mathbf{x}) \) and \( g(\mathbf{x}) \) for this problem. The distance from work is:

\[
f(\mathbf{x}) = \sqrt{(x - 4)^2 + (y - 3)^2}
\]

To simplify this objective function a bit, you can try to minimize the distance from work, squared:

\[
f(\mathbf{x}) = (x - 4)^2 + (y - 3)^2
\]

There are five constraints for this problem that help limit our solution space:

\[
g(\mathbf{x}) = \begin{cases} 
-x & \leq 0 \\
x - 3 & \leq 0 \\
-y & \leq 0 \\
y - 2 & \leq 0 \\
1.5 - \sqrt{(x - 0)^2 + (y - 1)^2} & \leq 0
\end{cases}
\]

The overall problem can now be formulated in a standard form with five constraints:

\[
\begin{align*}
\min & \quad (x - 4)^2 + (y - 3)^2 \\
& \quad -x \quad \leq 0 \\
& \quad x - 3 \quad \leq 0 \\
& \quad -y \quad \leq 0 \\
& \quad y - 2 \quad \leq 0 \\
& \quad 1.5 - \sqrt{(x - 0)^2 + (y - 1)^2} \leq 0
\end{align*}
\]

The solution in this case is pretty clear: the point \( x = 3, \ y = 2 \) is the closest point to work that is in the city limits and outside the range of the dump. This point satisfies exactly constraints 2 and 4. To be satisfied exactly means that \( g_i(\hat{x}) = 0 \). The other constraints have a value less than 0, \( g_i(\hat{x}) < 0 \).

**Optimality Conditions**

Once you have a feasible point that does not violate any of your constraints, you can check that point to see if it may be a locally optimum solution. Usually for constrained
optimization problems, local solutions occur where some of the inequality constraints are exactly satisfied. For constraint \( g_i(\hat{x}) \leq 0 \), the constraint is exactly satisfied at the point \( \hat{x} \) if \( g_i(\hat{x}) = 0 \). These constraints are called “active” or “binding” constraints. Equality constraints must always be satisfied, so they should always be active. Once you have a point in your feasible space, you can check for optimality using conditions developed by Karush, Kuhn, and Tucker, the KKT conditions. Note that these conditions do not say anything about how to find a KKT point, they just give you a test for a given point.

For many years these conditions were just KT conditions, as they were originally published by Kuhn and Tucker. Eventually, it was discovered that an obscure Indian mathematician had already discovered them, Karush, so now we just call them KKT conditions.

**KKT Conditions**

For a potential solution \( \hat{x} \), the following conditions hold. The set \( I \) specifies the binding constraints at the point \( \hat{x} \), \( I = \{i : g_i(\hat{x}) = 0\} \). Binding constraints are satisfied exactly at \( \hat{x} \). Additionally, \( \nabla g_i(\hat{x}) \) should be linearly independent. If the following conditions hold at \( \hat{x} \), then \( \hat{x} \) is a KKT point and a local solution.

\[
-\nabla f(\hat{x}) = \sum_{i \in I} \lambda_i \nabla g_i(\hat{x})
\]

\( \lambda_i \geq 0 \)

Note that this does not specify how to find a KKT point. Also note that a KKT point is not necessary to minimize a convex problem.
Example 1

From our Smallville problem, we think the point (3, 2) is the solution. You know that constraints 2 and 4 are satisfied at this point. You can find the gradient of these constraints and the gradient of the objective function at this point, (3, 2). The KKT conditions basically say that you must be able to find positive multipliers for the gradient directions of the active constraints that will add the active constraint directions together to get the gradient of the objective function improving direction. These active constraints constraints at (3, 2) are:

\[
\begin{align*}
x - 3 & \leq 0 \\
y - 3 & \leq 0
\end{align*}
\]

The gradients of these two constraints are

\[
\nabla g_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nabla g_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

With the negative gradient of the objective function at (3, 2) being \([-2 -2]^T\), the KKT conditions at (3, 2) are:

\[
\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

\[
\lambda_1 \geq 0 \\
\lambda_1 \geq 0
\]

In the case of the point (3, 2), \(\lambda_i = 2\) for both constraints 2 and 4. The \(\lambda\) values are positive, so the point (3, 2) is a KKT point or a local solution.

Now consider optimality conditions at the point (3, 0). This point also has two active constraints, constraints 2 and 3. The KKT conditions at the point (3, 0) with two active constraints are:

\[
\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}
\]

\[
\lambda_1 \geq 0 \\
\lambda_1 \geq 0
\]

There are no positive values for \(\lambda\) that will resolve the gradients of the active constraints in the direction of \(-\nabla f\). The gradient of the two active constraints dictates two directions, and assuming \(\lambda_i\) must all be positive multipliers for these two directions you end up with a cone of points expanding out to the right and down in the figure. The the improving direction \(-\nabla f\) must lie in this cone of directions, but at the point (3, 0) the improving direction does not lie in that cone. Therefore, the point (3, 0) will not be a KKT point. Solving the set of two linear algebraic equations from the KKT conditions, \(\lambda_1 = 2\) for constraint number 2 and \(\lambda_2 = -6\) for constraint number 3.
\[
-\nabla f(\hat{x}) = - \begin{bmatrix}
2(x - 4) \\
2(y - 3)
\end{bmatrix}
= - \begin{bmatrix}
2(3 - 4) \\
2(2 - 3)
\end{bmatrix}
= \begin{bmatrix}
2 \\
2
\end{bmatrix}
\]

Workplace: (4, 3)

\[
\nabla g_2(\hat{x}) = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[
\nabla g_3(\hat{x}) = \begin{bmatrix}
0 \\
-1
\end{bmatrix}
\]

\[
\nabla g_4(\hat{x}) = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[
-\nabla f(\hat{x}) = \begin{bmatrix}
2 \\
6
\end{bmatrix}
\]

\[
\nabla g_2(\hat{x}) = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Figure 11.3: Active constraint gradients and objective function improving direction for Smallville example at solution (3, 2) and the point (3, 0).
Equality Constraints and Infeasibility

Equality constraints may restrict the feasible region. Equality constraints can be written as two inequality constraints.

\[
g_i(x) = 0 \\
\downarrow \\
g_i(x) \leq 0 \\
g_i(x) \geq 0
\]

In the Smallville example, you may want to locate your house on Main St. In this problem, the locations on Main St. correspond to:

\[
y = x^2 - 1.5
\]

This can introduce two new constraints to the problem:

\[
g_6(x) = y - x^2 + 1.5 \leq 0 \\
g_7(x) = -y + x^2 - 1.5 \leq 0
\]

This means the feasible set of points are only the ones along the equality constraint. Instead of a fence boundary as in inequality constraints, equality constraints dictate a single “street” that points can move along when searching for optimality. The resulting solution would be at the point \((\sqrt{3.5}, 2)\). This point is inside the rectangular box of constraints dictating the size of the town, more than 1.5 miles from the dump location, and exactly main street. Constraints 4, 6, and 7 would be active. At the point \((\sqrt{3.5}, 2)\) the following KKT conditions could be considered

\[
\begin{bmatrix}
-2(\sqrt{3.5} - 4) \\
2
\end{bmatrix}
= \lambda_1 \begin{bmatrix} 0 \\
1
\end{bmatrix} + \lambda_2 \begin{bmatrix} -2\sqrt{3.5} \\
1
\end{bmatrix} + \lambda_3 \begin{bmatrix} 2\sqrt{3.5} \\
-1
\end{bmatrix}
\]

\[
\lambda_1 \geq 0 \\
\lambda_2 \geq 0 \\
\lambda_3 \geq 0
\]

Constraint 4 and 6 define a cone of directions that contains the direction \(-\nabla f\), so positive multipliers for \(\lambda_1\) and \(\lambda_2\) can be found with \(\lambda_3 = 0\) to satisfy the KKT conditions.

Perhaps we would like to limit our search to locations within half a mile. The resulting inequality constraint is:

\[
\sqrt{(x - 4)^2 + (y - 3)^2} \leq 0.5
\]

Obviously, we now have too many constraints and no feasible points exist. The problem can be infeasible if no points can be found that satisfy the problem constraints.
Figure 11.4: Addition of an equality constraint restricts the feasible region to only the points that exactly satisfy the equality constraint. In this case, $y = x^2 - 1.5$. Addition of the constraint $\sqrt{(x - 4)^2 + (y - 3)^2} \leq 0.5$ makes the problem infeasible, as no points can satisfy all the constraints.

Convexity

At this point, issues involving convexity of sets and multivariable functions should be addressed. A convex set $X$ satisfies the relationship $\lambda x_1 + (1 - \lambda)x_2 \in X$ for all $0 \leq \lambda \leq 1$, $\forall x_1, x_2 \in X$. This just means that given a set of points $X$, you can draw a line between any two points in the set and all points on the line will still be in the set. This definition is not especially useful, since we usually deal with functions instead of sets.

A convex set can be constructed from a convex function by evaluating the epigraph of a convex function. If $x \in X \subset \mathbb{R}^n$, $f : X \rightarrow \mathbb{R}$, $epi(f) \in \mathbb{R}^{n+1}$. These are all the points “above” the function. In our example, you can imagine the objective function mapping values of $x$ and $y$ to a surface in 3D space. The points above this surface make up a set of points. If the function is convex, the set of points in the epigraph will be convex.

As in convexity results from calculus, convexity of a function requires analysis of a second-order condition. The Hessian matrix $H$ can be calculated for $f(x)$ as:

$$
H(x) =
\begin{bmatrix}
\frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}
\end{bmatrix}
$$
A given function \( f(x) \) is convex if the Hessian of the function is positive semi-definite. This means that all the eigenvalues of \( H(x) \) are \( \geq 0 \). Note that the Hessian matrix may be a function of \( x \), making calculation of the eigenvalues more difficult. You could evaluate the Hessian at a single point to determine convexity of the function at that single point. For optimization problems, \( x \) belongs to a set of possible feasible points. Some methods do exist to bound the smallest eigenvalue for a general Hessian matrix using interval analysis methods. In general, determining if a general nonlinear function is convex over a region \( x \) can be quite difficult.

Why is convexity of a function important? If all the constraint functions and the objective function are convex and there is a feasible point in the solution space, there is single solution to the problem. This single solution could be degenerate, meaning multiple points in \( X \) result in the same objective function value. If a problem involves nonconvex constraint or objective functions, simple solution methods can only guarantee local solutions.

Given the function \( f(x) = (x_1)^3 + x_2 \), the gradient is determined by \( \nabla f = \begin{bmatrix} 2x_1^2 \\ 1 \end{bmatrix} \),

and the \( 2 \times 2 \) Hessian matrix is given by \( H(x) = \begin{bmatrix} 4x_1 & 0 \\ 0 & 0 \end{bmatrix} \). Since the Hessian in this case is a diagonal matrix, the eigenvalues are known to be \( 4x_1 \) and 0. Note that the minimum eigenvalue value depends on the range of values for \( x_1 \). If the lower bound on \( x_1 \) is positive, the minimum eigenvalue is 0. This makes the Hessian positive semidefinite and the function convex over the range of \( x \). If the lower bound on \( x_1 \) is \( < 0 \), the Hessian is not semidefinite and the function is nonconvex over the range of \( x \).

Linear equality and inequality constraints are convex. Nonlinear inequality constraints may be convex. Nonlinear equality constraints are always nonconvex. A nonlinear function \( f(x) = 0 \) can be written as two inequality constraints:

\[
0 \leq f(x) \leq 0
\]

This implies that if \( f(x) \) is nonlinear and convex at some point. One of the two following inequality constraints would be nonconvex at the same point as

\[
0 \leq f(x) \\
f(x) \leq 0
\]

Since this is the same as:

\[
-f(x) \leq 0 \\
f(x) \leq 0
\]

Therefore one constraint must be nonconvex at the point in question. Think of the equation of a line, \( y = x^2 \). This is show in Figure 11.5. The set of points that satisfy this equality constraint are only the points on the line. Now consider only the points defined by \( y \geq x^2 \), the points “above” the line. This single inequality would define a
convex set of points. Putting the inequality constraint in our general form \( x^2 - y \leq 0 \), our constraint becomes \( g(x) = x^2 - y \leq 0 \). This constraint generates a convex set of points in the \( x \times y \) space. The function \( x^2 \) is convex for all values \( x, x \in \mathbb{R}^1 \), since the Hessian is the \( 1 \times 1 \) matrix \( [2] \). The epigraph of \( x^2 \) defines the convex set of points for \( y \) such that \( y \geq x^2 \), the convex set of points in \( \mathbb{R}^2, x \times y \).

Consider the constraint for all points in a sphere of radius \( r \) where \( r \) is some constant value:

\[
x^2 + y^2 + z^2 \leq r^2
\]

The Hessian of the constraint function is:

\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

This implies the eigenvalues are all positive at a value of 2. We know intuitively the points in a sphere should be a convex set. Flipping the sign on the inequality would examine the points outside the sphere of radius \( r \). The “Swiss cheese” set is nonconvex, as it is easy to imagine line between two points containing points inside the empty sphere.

![Line including points not included in the set \( y=x^2 \)](image)

Figure 11.5: The nonconvex set \( y = x^2 \).
Special Types of Optimization

Linear Programming (LP) In some cases, the general optimization form has linear objective functions and linear constraints. An optimization problem can be found in the form:

$$\min \ Cx$$

subject to $Ax \leq b, \ lb \leq x \leq ub$

This is a special linear constrained case where the objective function is a linear function of the decision variable vector $x$ and the constraints are also linear. This can readily be solved using the $lp$ command in Matlab, even for large scale problems (hundred or thousands of variables and constraints). Generally the simplex method is used, but interior point methods are gaining popularity and speed of calculation.

Quadratic Programming (QP) For cases with a convex quadratic objective of the form:

$$\min \ \frac{1}{2} x^T H x + C x$$

subject to $Ax \leq b, \ lb \leq x \leq ub$

The problem is termed a Quadratic Program (QP). The Matlab command $qp$ can be used to solve this type of problem.

Nonlinear Programming (NLP) For general cases nonlinear functions for constraints and objective functions:

$$\min \ f(x)$$

subject to $g(x) = 0, \ h(x) < 0, \ lb \leq x \leq ub$

The problem is termed a Nonlinear Program, NLP. The Matlab command $fmincon$ can be used to solve this type of problem. If nonlinear equality constraints are included, the problem is definitely nonconvex and local solution may be encountered. In special cases:

$$\min \ f(x)$$

subject to $Ax \leq b, \ h(x) < 0, \ lb \leq x \leq ub$

if the functions $f(x)$ and $h_i(x)$ are all convex over the range of $x$, the problem is a convex NLP and has a single solution.

Mixed Integer Linear Programming For problems where some variables can only take binary values, the problem is considered a Mixed Integer problem. A common mixed integer problem is the Mixed Integer Linear Programming (MILP) problem of the form

$$\min \ C x$$

subject to $Ax \leq b, \ lb \leq x \leq ub, \ x_i \in \{0, 1\}$

There exist specialized methods for solving problems where the decision variables are only allowed to take values of 0 or 1 rather than values between 0 and 1 (inclusive).
A two tank system is arranged in series as shown in the above figure. The molar gas flow rate into tank 1 can be changed by the operator. The tanks are constant volume and isothermal. The volumetric flow of gas across a valve is usually written \( F = k \sqrt{\Delta P} \). In this case, using ideal gas law \( PV = nRT \) you should realize that the molar amount of gas in each tank is proportional to the pressure in the tank, \( n = \frac{V}{RT} \) where \( V \), \( R \), and \( T \) are constant. As a result, the molar flow rate between the two tanks across a valve can be written as \( F_1(t) = k_1 \sqrt{n_1(t) - n_2(t)} \) and the flow across a valve to the atmosphere can be written \( F_2(t) = k_2 \sqrt{n_2(t) - c} \)

1. Develop a dynamic mass balance for the two tank system.
2. Linearize any nonlinear terms.
3. Develop linear dynamic approximation for the system.
4. Take the Laplace transform of your linear ordinary differential equations.
5. Derive the transfer function relating the input flow to the number of moles in tank 2.
6. Determine the analytical response for the number of moles in tank 2 for a unit step change in \( F_o(t) \) at time \( t = 0 \).
7. Sketch the bode plot for this system.
8. Assuming a feedback controller of the form

\[ g_c = K_c + \frac{K_c}{\tau_I s} \]

derive the closed-loop transfer function.
Chapter 13

Dynamics and Control Topics

General Control Configurations
Jargon: MV, CV, DV
Feedback on PFD
Feedforward on PFD
Cascade on PFD

Linear Algebra
Steady state modeling \( \Delta y = K \Delta u \)
Solving \( A\mathbf{x} = \mathbf{b} \) by row reduction
Solving \( \mathbf{A} \mathbf{x} = \mathbf{b} \) by calculating \( \mathbf{A}^{-1} \)
Matrix multiplication
Determinant / Eigenvalues of \( \mathbf{A} \)

Dynamic Modeling (Open-loop)
Dynamic mass and energy balances
State Space Representation for ODEs

Laplace Transforms
step, delayed step, impulse
ramp, sinusoid, exponential
time delay and Heavyside function
derivative, integral of function
Solving Ordinary Differential Equations (ODEs)
Step response of First-Order system

Partial Fraction Expansion
Linearity applied to complex functions
\( f(t) = f_1(t) + f_2(t) \Rightarrow f_1(s) + f_2(s) = f(s) \)
Compound / Composite functions

Dynamic Modeling (Open-loop)
Dynamic mass and energy balances
CSTR, Mixing Tank, Tank Level

Transfer Function Representation
\( y(s) = g(s)u(s) + g_d(s)d(s) \)
Block diagrams
Poles and Zeros of transfer functions
Low Order Systems
  First Order
  Pure Gain
  Pure Capacity
  Lead Lag
High Order systems
  Two first order in series
  Interacting tanks
  General 2nd order
  Higher order
Inverse Response (RHP zero)
Time Delay
Stability
  poles and eigenvalues for stability
  BIBO stab. of oscillatory systems (pole at $s = 0$)
Poles and Zeros of state space representation
Frequency Response
  Amplitude Ratio and Phase Angle for $g(s)$
  Basic Bode Plots given $g(s)$
  Complex Bode Plots for $g_1(s)g_2(s)...g_n(s)$
  Developing models from frequency response
Linearization of nonlinear ODEs
Model Identification
Feedback Control
  Process Reaction Curve ($K, \tau, \alpha$)
  Basic PID Controller Tuning ($K_c, \tau_I, \tau_D$)
  PID Transfer Function for $g_c(s)$
  Internal Model Control
  Direct Synthesis
Feedforward Control
Cascade Control
Multivariable Open-loop Modeling
  Transfer function based
  State space
  Multivariable system poles and zeros
Multivariable Control Issues
  Relative Gain Array and loop pairing
  Decoupling control
  Actuator constraints
  Moving horizon control
  Optimization
Chapter 14

Overall Problem

After an your embarrassing karaoke performance at the annual Christmas party, your boss moved you to take charge of the plant human sewage treatment plant at your facility.

1. Draw a process flow diagram for the system from the following information:

   (a) Raw sewage enters a mixing / aeration tank
   (b) The mixing tank empties into a holding pond
   (c) Chlorine is added in the holding pond to further reduce the concentration
   (d) The holding pond flows down a long pipe to the river
   (e) Some of the holding pond sewage is recycled to the mixing / aeration tank
   (f) You can adjust the speed of mixing (this affects the mixing tank removal rate)
   (g) You can adjust the rate of chlorine addition
   (h) The feed raw sewage is diluted with clean water and mixed in a pipe before it flows to the mixing / aeration tank
   (i) You can adjust the flow rate of dilution water
   (j) You can measure the concentration of sewage in the feed stream. This may vary with time, especially on “Burrito Friday”
   (k) You can measure the total flow rate of feed sewage and dilution water entering the mixing / aeration tank, \( Q(t) \)
   (l) You can measure the concentration in the mixing / aeration tank and the concentration in the pond
   (m) You can measure the product concentration before it flows into the river
   (n) The sewage concentration fed to the system is \( C_s(t) \), the concentration entering the tank is \( C_0(t) \), the tank concentration is \( C_1(t) \), the pond is \( C_2(t) \), and the final product stream is \( C_3(t) \).

2. Draw a simple feedback control scheme to maintain the flow rate into the mixing / aeration tank.
3. Draw a cascade system to control the final concentration entering the river by adjusting the mixing rate in the aeration tank.

4. Draw a feedforward control scheme assuming the feed concentration varies.

5. Develop a dynamic model of the sewage system assuming the following information:
   (a) The volume of the mixing tank $V_1$ is 10,000 L and the volume of the pond $V_2$ is 100,000 L
   (b) The flow from the mixing tank to the pond is 2,000 L/hr
   (c) The flow from the pond into the mixing tank is 1,000 L/hr
   (d) The tank and pond are constant volume and well-mixed
   (e) The nominal feed flow of entering sewage is 100 L/hr with a concentration is $C_s(t)$
   (f) The mixing tank concentration is $C_1(t)$ and the pond concentration is $C_2(t)$
   (g) The concentration entering the river is $C_3(t)$
   (h) The volumetric reduction rate for the mixing tank is given as:
       $$ r_1(t) = \frac{1}{10} C_1(t) m(t) $$
       where $m(t)$ is the normalized mixing rate, $0 \leq m(t) \leq 1$
   (i) The volumetric reduction rate for the pond is given as:
       $$ r_2(t) = \frac{2}{25} C_2(t) (F(t))^2 $$
       where $F(t)$ is the normalized chlorine flow rate, $0 \leq F(t) \leq 1$
   (j) It takes 10 hours for the flow to leave the pond and reach the river at concentration $C_3(t)$
   (k) The feed flow (dilution + sewage) is constant, $Q(t) = Q_{H2O} + Q_{Sewage} = Q$

6. Determine the steady state tank and pond concentrations, assuming the following:
   (a) The feed concentration $C_s(t)$ is 40 units per L
   (b) The nominal value for $m(t)$ and $F(t)$ are both 0.5

7. Linearize your dynamic model at the nominal steady state values. Identify your deviation variables.

8. Put your linear dynamic model in state space form assuming the following:
   (a) Only $C_1(t)$ and $C_2(t)$ deviation values are measured as $y_1(t)$ and $y_2(t)$
   (b) The mixing rate deviation is $u_1(t)$
(c) The chlorine flow rate deviation is \( u_2(t) \)

(d) The feed concentration deviation is \( u_3(t) = d(t) \)

9. What are the eigenvalues of your state space system? Should your system be open-loop stable? **HINT:** \( \lambda = -0.259, -0.0309 \)

10. Take the Laplace transform of your linear model from problem 7. Develop SISO transfer function models relating the three inputs to \( y_1(t) \) (\( C_1(t) \) deviation) and \( y_3(t) \) (\( C_3(t) \) deviation). **HINT:** Make sure your poles match your eigenvalues from above.

11. Determine the poles, gains, and zeros of your individual transfer functions. Are the models stable? Is there inverse response? Do the gains make sense? Is there underdamped response?

12. For a sustained unit increase in the feed concentration, determine the steady state effect on the concentration entering the river using your linear model and the Final Value Theorem.

13. For a sustained unit increase in the initial sewage concentration, determine the analytical response for the concentration entering the river. For simplicity, approximate this transfer function as

\[
y_3(s) = \frac{\frac{1}{40} e^{-10s}}{(4s + 1)(32s + 1)} u_3(s)
\]

14. For an instantaneous delivery of chlorine (\( u_2(t) = \delta(t) \)), determine the analytical response for the concentration entering the river. For simplicity, approximate this transfer function as

\[
y_3(s) = \frac{-\frac{5}{7}(5s + 1) e^{-10s}}{(4s + 1)(32s + 1)} u_2(s)
\]

15. For a .1 sustained increase in the mixing rate \( u_1 \), determine the analytical response for the concentration entering the river \( y_3 \). Plot this response and determine approximate values for a First-Order-Time-Delay model. For

\[
y_3(s) = \frac{-\frac{1}{2} e^{-10s}}{(4s + 1)(32s + 1)} u_1(s)
\]

16. Determine controller tuning parameters for a PI controller regulating \( y_3 \) using input \( u_1 \) using Cohen-Coon tuning parameters.

17. Make a Bode plot between \( u_1(t) \) and \( y_2(t) \) (Find \( AR(\omega) \) and \( \phi(\omega) \), sketch the plot) using the approximate model

\[
y_2(s) = \frac{-\frac{1}{2}}{(4s + 1)(32s + 1)} u_1(s)
\]
18. Make a Bode plot between $u_1(t)$ and $y_3(t)$ (Find $AR(\omega)$ and $\phi(\omega)$, sketch the plot) using the approximate model from problem 15.

19. Find the ultimate gain for a P controller regulating $y_3$ using input $u_1$ using the approximate model.

20. Develop a feedforward controller for $y_3$ affected by $d(u_3)$ using $u_1$ as the manipulated variable. Use models from problems 13 and 15. Sketch the feedforward block diagram.

21. Develop an IMC controller for $y_3$ using $u_1$ as the manipulated variable and sketch the block diagram both in traditional IMC formulation and traditional feedback configuration.

22. Draw the cascade block diagram for $u_1 \rightarrow y_1 \rightarrow y_3$ and determine the CLTF for the system assuming $y_3(s) = g_2(s)y_1(s)$ and $y_1(s) = g_1(s)u_1(s)$.

23. Draw the 2x2 block diagram for the control system using only $y_1$ and $y_3$ along with inputs $u_1$ and $u_2$.

24. Determine the steady state gain matrix for the 2x2 system.

25. Using your steady state gain matrix, determine the change in the inputs for a desired decrease in the outputs $y_1$ and $y_2$ of $[-1 -1]$

26. Using your steady state gain matrix, determine the RGA and suggest a input pairing. Does this make sense?

27. Develop a decoupling control system for the 2x2 system and draw the block diagram.
15.1 Tank Modeling Problem With Explanation

1. (25 pts.) A system consists of three tanks as shown below. The flow rate $F_0$ can be manipulated. A fraction of the flow rate $F_0$ into the system goes into tank 1 and the rest of the flow enters into tank 3 as shown. The fraction of flow $F_0$ into tank 1 is $\gamma$, with $0 \leq \gamma \leq 1$ and $\gamma$ remaining constant. The flow rate from tank 1 to tank 2 is given as $F_1 = k_1 h_1$. The flow rate into tank 3 from tank 2 is $F_2 = k_2 (h_2 - h_3)$. The flow rate out of tank 3 is $F_3 = k_3 h_3$. The constant cross sectional tank areas are $A_1$, $A_2$, and $A_3$, respectively.

![Diagram of the system](image)

a. Derive the differential equation model for the system.

b. Put your differential equation model into State Space form ($\dot{x} = Ax + bu$, $y = c^T x$) for the system, given that $u = F_0$, $y = h_3$, and $z$ with $z$:

$$z = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$
15.1.1 Solution

First of all, you must realize that you need to perform a dynamic mass balance for this system. Dynamic balances include a nonzero accumulation term and can result in a differential equation model of your process.

To start the mass balance, remember that you should perform balances around individual systems. In this case, you will need three balances, one over each tank system. The amount of “stuff” in the first tank is \( \rho V_1(t) \) or more simply \( \rho A_1 h_1(t) \). As the density and cross-sectional area are not functions of time, the accumulation term (or rate of change of “stuff” in the tank”) can be written as:

\[
\rho A_1 \frac{dh_1}{dt}(t)
\]

Assuming the flow rates are all in volumetric terms, the mass balance on the first system can be written as:

\[
\rho A_1 \frac{dh_1}{dt}(t) = \rho \gamma F_0(t) - \rho k_1 h_1(t)
\]

Note that for this tank, you have a flow rate in term and a flow rate out term. Also note that only a portion of the flow into the system goes into tank 1. There is no reaction taking place in this system. Similarly, for the other two tanks you can write similar mass balances:

\[
\rho A_2 \frac{dh_2}{dt}(t) = \rho k_1 h_1(t) - \rho k_2 (h_2(t) - h_3(t))
\]

\[
\rho A_3 \frac{dh_3}{dt}(t) = \rho (1 - \gamma) F_0(t) + \rho k_2 (h_2(t) - h_3(t)) - \rho k_3 h_3(t)
\]

Such that the overall model is in the form:

\[
\rho A_1 \frac{dh_1}{dt}(t) = \rho \gamma F_0(t) - \rho k_1 h_1(t)
\]

\[
\rho A_2 \frac{dh_2}{dt}(t) = \rho k_1 h_1(t) - \rho k_2 (h_2(t) - h_3(t))
\]

\[
\rho A_3 \frac{dh_3}{dt}(t) = \rho (1 - \gamma) F_0(t) + \rho k_2 (h_2(t) - h_3(t)) - \rho k_3 h_3(t)
\]

Note the sign difference in terms. If something is assumed to flow out of one tank and into another, the same term should appear in both mass balances, only with a different sign in each. Also note that the flow from tank 2 to tank 3 is assumed to be positive (so long as \( h_2 > h_3 \)). The term appears with a negative sign in the second balance and with a positive sign in the third balance. In some cases, \( h_3 \) may exceed \( h_2 \). In such a case, the sign of the term would automatically change, taking care of the reverse flow in the model. The negative term for flow out: \(-\rho k_2 (h_2(t) - h_3(t))\) for the tank 2 balance would become positive value if \( h_3 \) exceeds \( h_2 \) and the flow out term would actually become a
flow in term. Nothing special must be done in these cases, except making sure the terms have different signs if they appear in different balances.

You now have a full differential mass balance. Now we would like to get our equations in state space form. You have three accumulation terms, so you should have three states:

\[
x(t) = \begin{bmatrix}
h_1(t) \\
h_2(t) \\
h_3(t)
\end{bmatrix}
\]

Next, simplify the equations. First, divide out the density from all the terms.

\[
\begin{align*}
A_1 \frac{dh_1}{dt}(t) &= \gamma F_0(t) - k_1 h_1(t) \\
A_2 \frac{dh_2}{dt}(t) &= k_1 h_1(t) - k_2 (h_2(t) - h_3(t)) \\
A_3 \frac{dh_3}{dt}(t) &= (1 - \gamma) F_0(t) + k_2 (h_2(t) - h_3(t)) - k_3 h_3(t)
\end{align*}
\]

Next, get the accumulation terms to all have 1 as the leading coefficient. This means divide each equation by the cross sectional area in this case:

\[
\begin{align*}
\frac{dh_1}{dt}(t) &= \frac{\gamma}{A_1} F_0(t) - \frac{k_1}{A_1} h_1(t) \\
\frac{dh_2}{dt}(t) &= \frac{k_1}{A_2} h_1(t) - \frac{k_2}{A_2} (h_2(t) - h_3(t)) \\
\frac{dh_3}{dt}(t) &= \frac{(1 - \gamma)}{A_3} F_0(t) + \frac{k_2}{A_3} (h_2(t) - h_3(t)) - \frac{k_3}{A_3} h_3(t)
\end{align*}
\]
Now, write all the the equations in terms of all the states and the inputs, including the measurement equation, \( y(t) = h_3(t) \)

\[
\frac{dh_1}{dt}(t) = -\frac{k_1}{A_1}h_1(t) + 0h_2(t) + 0h_3(t) + \frac{\gamma}{A_1}F_0(t)
\]

\[
\frac{dh_2}{dt}(t) = \frac{k_1}{A_2}h_1(t) - \frac{k_2}{A_2}h_2(t) + \frac{k_2}{A_2}h_3(t) + 0F_0(t)
\]

\[
\frac{dh_3}{dt}(t) = 0h_1(t) + \frac{k_2}{A_3}h_2(t) - \frac{k_2}{A_3}h_3(t) - \frac{k_3}{A_3}h_3(t) + (1 - \gamma)\frac{A_3}{A_3}F_0(t)
\]

\[
y(t) = 0h_1(t) + 0h_2(t) + 1h_3(t) + 0F_0(t)
\]

Now, it is easier to pick out your state space matrices, \( A, B, C, D \).

\[
A = \begin{bmatrix}
-\frac{k_1}{A_1} & 0 & 0 \\
-\frac{k_2}{A_2} & -\frac{k_3}{A_3} & -\frac{k_2}{A_3} \\
0 & -\frac{k_2}{A_3} & -\frac{k_3}{A_3}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\frac{\gamma}{A_1} \\
0 \\
(1 - \gamma)\frac{A_3}{A_3}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 1
\end{bmatrix}
\]

\[
D = [0]
\]

And the matrices should fit together nicely in the form:

\[
A \parallel B \\
== \parallel = \\
C \parallel D
\]

Which means \( A \) and \( C \) should have the same number of columns (= # states), while \( A \) and \( B \) should have the same number of rows (= # states). \( B \) and \( D \) should have the same number of columns (= the number of inputs) while \( C \) and \( D \) should have the same number of rows (= the number of output measurements).

Now, after you finish the mass / energy balance but before you put your equations in state space, I could have asked you to take the Laplace transform and get a transfer function for the relationship between \( F_0(t) \) and \( h_3(t) \). When you take the Laplace transform of your differential equations and measurement equations, you will have a number of different equations in the \( s \) domain, like:

\[
sh_1(s) = \frac{\gamma}{A_1}F_0(s) - \frac{k_1}{A_1}h_1(s)
\]

And a couple of other equations. You would have to use the equations to eliminate the variables you don’t want. In this case, you only want \( F_0(s) \) and \( h_3(s) \), so you would have to eliminate \( h_1(s) \)and \( h_2(s) \). This would be a mess on this problem, but you could feasibly do it.
15.2 CSTR Modeling Problem With Explanation

1. At the Ideal Gas Company, you are in charge of operating a reactant mixing system. Your boss wants a dynamic model of the system to be used for process control and process optimization. The constant volume mixing tank has two feed streams with constant volumetric flowrates of $F_1$ and $F_2$. Feed stream 1 contains both species A and species B, while stream 2 only contains species A. You can modify the initial concentrations of the two species coming into the tank system, $u_1(t) = C_{A10}(t)$, $u_2(t) = C_{B10}(t)$, $u_3(t) = C_{A20}(t)$. At the exit stream, due to instrumentation limitations, you can only measure the total concentration of both components, $y(t) = C_A(t) + C_B(t)$.

![Diagram of a CSTR model with flow rates and concentrations](image)

a. **(4 points)** What is the dynamic mass balance describing the concentrations of species A and species B at the exit of the mixing tank?

b. **(4 points)** Put your model in state space form. Clearly identify $x$, $A$, $B$, $C$, and $D$. Example state space form:

$$
\dot{x} = Ax + Bu \\
y = Cx + Du
$$
15.2.1 Solution

Ok, one tank, two species, three inputs, one measurement. Two dynamic mass balances should work. $F_1$, $F_2$, $F_3$, and $V$ are all constant. No reaction, this is a mixing tank. One balance will consider the amount of species A in the system, while the other will model the amount of species B. The total amount of A in the system is:

$$VC_A(t)$$

And the accumulation term for A will be:

$$V\frac{dC_A}{dt}(t)$$

So the mass balances become:

$$V\frac{dC_A(t)}{dt} = F_1C_{A10}(t) + F_2C_{A20}(t) - F_3C_A(t)$$

$$V\frac{dC_B(t)}{dt} = F_1C_{B0}(t) - F_3C_B(t)$$
The states (concentrations in the reactor) and inputs (inlet concentrations for inlet flows) in this problem are:

\[ \mathbf{x}(t) = \begin{bmatrix} C_A(t) \\ C_B(t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} C_{A10}(t) \\ C_{B10}(t) \\ C_{A10}(t) \end{bmatrix} \]

The measurement equation is a little tricky... You can measure the total concentration of both components:

\[ y(t) = C_A(t) + C_B(t) \]

For state space, divide both equations by \( V \), write all equations in terms of all states \( x \) and all inputs \( u \).

\[
\begin{align*}
\frac{dC_A(t)}{dt} &= -F_3/V C_A(t) + 0C_B(t) + F_1/V C_{A10}(t) + 0C_{B10}(t) + F_2/V C_{A20}(t) \\
\frac{dC_B(t)}{dt} &= 0C_B(t) - F_3/V C_B(t) + 0C_{A10}(t) + F_1/V C_{B0}(t) + 0C_{A20}(t)
\end{align*}
\]

And the matrices should fit together nicely in the form:

\[
\begin{array}{c|c}
A & B \\
\hline
C & D
\end{array}
\]

so that:

\[
\begin{array}{ccc|ccc}
-\frac{F_3}{V} & 0 & \frac{F_1}{V} & 0 & \frac{F_2}{V} \\
0 & -\frac{F_3}{V} & 0 & \frac{F_1}{V} & 0 \\
1 & 1 & 0 & 0 & 0
\end{array}
\]

Note the eigenvalues of the \( A \) matrix. These are the poles of your system. Note they are identical. \( C_A(t) \) has no effect on \( C_B(t) \).
1. (15 pts.) The Ideal Gas Company is attempting to develop a dynamic process model for a combustion chamber which burns a stream of aqueous liquid waste. The process output and the process input are shown for a input step change. What is the transfer function for this system, assuming it is a first order process?

![Graph 1]

2. (15 pts.) What is the Laplace transform \( u(s) \) of the following function?

\[
u(t) = \begin{cases} 
0 & t < 0 \\
\sigma t & 0 \leq t < b \\
-2A + \sigma t & b \leq t < c \\
0 & c \leq t
\end{cases}
\]

![Graph 2]
3. (25 pts.) A system consists of three tanks as shown below. The flow rate $F_0$ can be manipulated. A fraction of the flow rate $F_0$ into the system goes into tank 1 and the rest of the flow enters into tank 3 as shown. The fraction of flow $F_0$ into tank 1 is $\gamma$, with $0 \leq \gamma \leq 1$ and $\gamma$ remaining constant. The flow rate from tank 1 to tank 2 is given as $F_1 = k_1 h_1$. The flow rate into tank 3 from tank 2 is $F_2 = k_2 (h_2 - h_3)$. The flow rate out of tank 3 is $F_3 = k_3 h_3$. The constant cross sectional tank areas are $A_1$, $A_2$, and $A_3$, respectively.

![Diagram of three tanks with flow rates and heights]

a. Derive the differential equation model for the system.

b. Put your differential equation model into State Space form \( \dot{x} = Ax + Bu, \quad y = c^T x \) for the system, given that $u = F_0$, $y = h_3$, and $x$ with $x = [h_1 \ h_2 \ h_3]$.

4. (20 pts.) For the following system, steam is used to heat the liquid in a constant volume tank. The available measurements include the temperature of the liquid in the tank, the temperature of the feed flowing into the tank, and the steam flow rate. The steam valve can be manipulated. It is desired to regulate the temperature of the exit flow from the tank at a constant value.
a. In the figure above, draw a feedback control loop for the system

b. In the figure above, draw a feed forward control loop, assuming the feed temperature acts as the disturbance.
c. In the figure above, assuming the steam flow rate varies unpredictably, draw a cascade configuration using two feedback controllers.
1. (4 pts.) A preheater furnace is used to increase the temperature of crude oil from $T_i$ to $T$, the target value. The preheated hot crude oil is then sent downstream to a reactor. The crude oil enters the furnace at the flow rate $F$ and leaves at the same rate. Fuel and air are mixed and burned in the furnace to heat the crude oil. See diagram below.

Construct two different feedback control configurations. Also, construct two different feedforward control configurations. Clearly label what is measured and what is manipulated.

3. (2 pts.) What are the eigenvalues of the following matrix?

$$
\begin{bmatrix}
-1 & 3 \\
2 & 5
\end{bmatrix}
$$
1. (4 pts.) An agricultural process requires that trays of plants be maintained at specified temperatures. Three lamps are used to warm three plant trays as seen below.

A 3x3 steady state model is desired relating the change in voltages $\Delta V_i$ (for each lamp $i$) to the change in plant temperature $\Delta T_j$ (for each plant $j$). It is known that increasing the voltage for Lamp A by 1 volt increases the temperature of Plant A by 3.3 degrees and increases the temperature of Plant B by 2.1 degrees. Increasing Lamp B voltage by 1 volt increases both Plant B and Plant C by 2 degrees. Increasing Lamp C voltage by 1 volt increases the temperature of Plant C by 4 degrees.

Develop a model in the form $Ax = b$ and identify $A$, $x$, and $b$.

You may want to check your model by assuming arbitrary values for the change in lamp voltages, then verifying the expected change in plant temperatures.

3. (2 pts.) a. What is the determinant of the following matrix?

$$
\begin{bmatrix}
1 & 1 & 0 \\
0 & -6 & 7 \\
-1 & -2 & 3
\end{bmatrix}
$$
15.6 Fall 2002 Quiz 1

You must develop a model of paper machine sheet forming process. A simple schematic is shown below. A feed stream of pulp (wood fibers and water) is sprayed onto a moving screen (conveyor belt). As the screen moves, water drains out of the pulp, through the screen. At the product end of the paper machine, the pulp is effectively just wet paper.

![Diagram of a sheet forming process.](image)

Three valves are available to adjust the flowrate of pulp when the pulp concentrations change. Three sensors measure the thickness of the wet paper. Increasing the valves on the edge by 1% ($v_1$ and $v_3$) increases the thickness in the corresponding paper location by 2mm. A 1% increase in $v_1$ and $v_3$ will also decrease the thickness in the center position by 1 mm. A 1% increase in $v_2$ will increase the thickness in the center by 3 mm and reduce the edge thickness by 0.5mm.

1. (1pt) What are the controlled variables, manipulated variables, and disturbances for this paper making process?

2. (3pts) Develop a model of this process relating $\Delta s$ and $\Delta v$.

3. (2pts) Put your model in the form $Ax = b$ and clearly identify $A$, $x$, and $b$.

4. (2pts) What is the determinant of the following matrix?

$$
\begin{bmatrix}
0 & 2 & -1 \\
1 & 1 & 4 \\
1 & 3 & 1
\end{bmatrix}
$$

5. (2pts) What are the eigenvalues of the following matrix?

$$
\begin{bmatrix}
-5 & -2 \\
3 & -10
\end{bmatrix}
$$
1. At the Ideal Gas Company, you are in charge of operating a reactant mixing system. Your boss wants a dynamic model of the system to be used for process control and process optimization. The constant volume mixing tank has two feed streams with constant volumetric flowrates of $F_1$ and $F_2$. Feed stream 1 contains both species A and species B, while stream 2 only contains species A. You can modify the initial concentrations of the two species coming into the tank system, $u_1(t) = C_{A10}(t)$, $u_2(t) = C_{B10}(t)$, $u_3(t) = C_{A20}(t)$. At the exit stream, due to instrumentation limitations, you can only measure the total concentration of both components, $y(t) = C_A(t) + C_B(t)$.

![Diagram of the mixing tank](image)

a. **(4 points)** What is the dynamic mass balance describing the concentrations of species A and species B at the exit of the mixing tank?

b. **(4 points)** Put your model in state space form. Clearly identify $x$, $A$, $B$, $C$, and $D$. Example state space form:

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

2. **(2 points)** After running some step tests for your system varying $u_1(t)$ and measuring the output $y(t)$ you have the following process data. Identify the approximate process gain for this Single-Input-Single-Output system.
3. Bonus - Dr. Gatzke has a flower bed with three sprinkler heads. In one minute, sprinkler 1 delivers 2 mm of water to its coverage area, sprinkler 2 delivers 0.5 mm of water to its coverage area, and sprinkler 3 delivers 4 mm of water to its coverage area. Plant A is covered by sprinkler 1 and 3, plant B is covered by all sprinkler, and plant C is covered by sprinkler 2 and 3. The system is currently set to operate at normal operating program times. Develop a steady state model relating possible changes in sprinkler operating times to changes in amount of water delivered to each plant. Put your model in the form:

\[ Ax = b \]

3b. Assume that plant A needs an additional 2 mm of water, plant B needs 1 mm less, and Plant C is fine the way it is. How does the problem change? What changes to the sprinkler operating times would make this change? Solve using Row Reduction Methods.
15.8 Fall 2002 Exam 1 Practice Problems

1. A series of tanks are shown below. You can manipulate $F_0(t)$ and you can measure the flow rate out of tank 3, $F_3(t)$.

```

F0(t) = u(t)
F1(t) = k1 h1(t)
F2(t) = k2 h2(t)
F3(t) = k3 h3(t) = y(t)
```

a. Assuming constant density, develop a mass balance for the system.
b. Put your model in state space form. Clearly identify $x$, $A$, $B$, $C$, and $D$. Example state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
c. What are the eigenvalues of your $A$ matrix from your system?
d. From part (a.) take the Laplace transform of your dynamic model assuming the tanks are empty initially. Solve the three equations for the relationship between $y(s)$ and $u(s)$.

2a. Express the following function as a simple function of time (You may need to use the heaviside function multiplied by other functions.)

```

u(t)
```

0 5 15

-2
b. Establish the Laplace transform of the function, $u(s)$.

3a. Express the following function as a simple function of time.

b. Establish the Laplace transform of the function, $u(s)$.

c. Assuming this function $u(t)$ is the input to a first-order system, $g(s) = \frac{5}{10s+1}$, $y(s) = g(s)u(s)$. Establish $y(s)$ and $y(t)$.

4. Assuming a constant volume mixing tank for two species, A and B. Assuming you can change the inlet concentrations of A and B and measure the outlet concentrations of A and B, develop a dynamic mass balance and put your equations in state space form.
1. (15 pts.) The Ideal Gas Company is attempting to develop a dynamic process model for a chemical reactor. The process output and the process input are shown below for a input step change.

a. Determine the process gain (K), the process time constant (τ) and the process dead time (α) for the system.

b. What is the transfer function for this system, g(s), assuming it is a first order process?

2. (15 pts.) What is the the time domain expression for the following function expressed using the Heaviside function? What is the Laplace transform u(s) of the following function?

\[
u(t) = \begin{cases} 
0 & t < 0 \\
20 + 3t & 0 \leq t < 20 \\
50 & 20 \leq t \leq \infty 
\end{cases}
\]
3. **(25 pts. total)** A system consists of two mixing tanks in series, as pictured below. Two manipulated inputs are available: the initial concentration of species A entering tank 1 and the initial temperature of the liquid entering tank 1. You can measure the temperature at the exit of tank 1 and the concentration of A at the exit of tank 2. You may assume well-mixed tanks, constant volumetric flow rates, constant volume tanks, constant density, and constant heat capacity. You may use the reference temperature $T^*$. 

- **F₁**
  - $u_1(t) = C_A0(t) - C_A0ss(t)$
  - $u_2(t) = T_0(t) - T_0ss$

- **F₂**
  - $C_A1(t)$
  - $T_1(t)$
  - $V_1, \rho, C_p$

- **F₃**

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<td>$y_2(t)$</td>
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**Figure 15.1:** Two mixing tanks in series.

a. **(12 pts.)** Derive the differential equation model for the system.

b. **(6 pts.)** Put your differential equation model into deviation variables by subtracting the steady state equations. Use the following variables:

- $u_1(t) = C_A0(t) - C_A0ss$, $u_2(t) = T_0(t) - T_0ss$, $x_1(t) = C_A1(t) - C_A1ss$, $x_2(t) = T_1(t) - T_{1ss}$, and $x_3(t) = C_A2(t) - C_A2ss$.

c. **(7 pts.)** Put your deviation differential equation model into State Space form ($\dot{z} = Az + Bu$, $y = Cz + Du$) for the system, given $y_1(t) = T_1(t) - T_{1ss}$, and $y_2(t) = C_A2(t) - C_A2ss$. $z$:

$$ z = \begin{bmatrix} C_A1(t) - C_A1ss \\ T_1(t) - T_{1ss} \\ C_A2(t) - C_A2ss \end{bmatrix} $$
4. (15 pts.) For the following system, a stream containing radioactive solids is passed through a crossflow filter. The clean filtrate is separated from the radioactive slurry. The available online measurements include the concentration of the slurry entering the filter, the flow rate of the slurry entering the filter, and the flow rate of the filtrate exiting the filter. The inlet valve can be manipulated. It is desired to regulate the filtrate flow, keeping it at a constant value.

![Diagram](attachment:/path/to/diagram.png)

a. In the figure above, draw a feedback control loop for the system

b. In the figure above, draw a feed forward control loop, assuming the feed concentration changes unpredictably.

c. In the figure above, assuming the feed stream pressure changes and will affect the flow through the valve, draw a cascade configuration using two feedback controllers.
5. (30 pts. total) A constant volume salt mixing tank system can be modeled by a first order transfer function relating the measurement $C_A(t)$ to the manipulated input $C_{Ao}(t)$:

$$\frac{V}{F} \frac{dC_A}{dt}(t) + C_A(t) - C_{Ao}(t) = 0$$

a. (4 pts.) From this differential equation, what are the values for the process time constant, $\tau$, and process gain, $K$ in terms of tank volume $V$ and flow rate $F$ ($F_{in} = F_{out} = F$) ?

b. (5 pts.) Justify using physical arguments the value of the steady state process gain.

c. (5 pts.) Assuming $C_A(t = 0) = 0$, $V = 2m^3$ and $F = 0.05 \frac{m^3}{sec}$, express $C_A(s)$ in terms of $s$ and $C_{Ao}(s)$ by taking the Laplace transform of the differential equation.

d. (6 pts.) When salt is not being added to the system, the inlet flow to the tank gets clogged with dried salt. When the salt is first added to the system, the salt plug flows into the tank and the inlet concentration spikes to a value of 110 for 1 second, then returns to the desired value of 10. Assume that the short time, high level rectangular pulse in $C_{Ao}(t)$ can be expressed as an impulse. For this start up procedure, sketch $C_{Ao}(t)$, determine $C_{Ao}(t)$ in terms of $H(t)$ and $\delta(t)$, and determine the Laplace transform of this function, $C_{Ao}(s)$. Assume the impulse occurs at time $t = 0$.

e. (4 pts.) What is the exit concentration response, $y(s)$, realizing $y(s) = g(s)u(s)$?

f. (6 pts.) This response can be broken into two portions, $y_1(t)$ and $y_2(t)$. Determine $y_1(t)$ and $y_2(t)$ then sketch $y_1(t)$, $y_2(t)$, and the overall response $y(t)$.
1. **(15 pts. total)** The Ideal Gas Company is attempting to develop a dynamic process model for a chemical reactor. The process output and the process input are shown below for a input step change.

   a. **(12 pts.)** Determine the process gain \((K)\), the process time constant \((\tau)\) and the process dead time \((\alpha)\) for the system. Time units on the graph are in minutes.

   b. **(3 pts.)** What is the transfer function for this system, \(g(s)\), assuming it is a first order process with time delay?
2. (15 pts.) What is the Laplace transform \( u(s) \) of the following function, given that 
\( u(t) = 0 \) for \( t < 0 \)?

\[
u(t) = \begin{cases} 
3e^{-2t} & 0 \leq t < 5 \\
3e^{-2t} - 4e^{-2(t-5)} & 5 \leq t < 10 \\
3e^{-2t} - 4e^{-2(t-5)} + \frac{1}{2}(t - 10) & 10 \leq t < \infty 
\end{cases}
\]
3. (30 pts. total) A reactor is to be used to produce a new product. The reactor is a constant volume system, with constant volume $V$ and constant flow rate in / out $F$. Three species are present. Species $A$ can react to form species $B$ at a rate of $r_{AB} = k_1 C_A$. Species $A$ can ALSO react to form species $C$ at a rate of $r_{AC} = k_2 C_A$. Species $B$ will react to form species $C$ at a rate of $r_{BC} = k_3 C_B$. Reaction rates are volumetric, $(\text{mol} \text{L}^{-1} \text{min}^{-1})$. Only species $A$ is entering the system. You can adjust the concentration of species $A$ entering the system, $C_{Ao}(t)$. You can measure the concentration of $C$ leaving the system. You do not need to carry unit throughout the problem, just make sure you have the correct terms in each balance.

- $A \rightarrow B$ with reaction rate $r_{AB} = k_1 C_A$
- $A \rightarrow C$ with reaction rate $r_{AC} = k_2 C_A$
- $B \rightarrow C$ with reaction rate $r_{BC} = k_3 C_B$
- Constant volume $V$ and flows $F$
- Single input, $u(t) = C_{Ao}(t)$
- Single measurement, $y(t) = C_C(t)$

a. (3 pts.) What are some additional assumptions you will use to model this system?
b. (15 pts.) Derive a differential equation model for the system.
c. (12 pts.) Put your deviation differential equation model into State Space form ($\dot{x} = Ax + Bu, y = Cx + Du$) for the system, given that concentrations are all = 0 initially (no deviation variables needed in this case).
4. (40 pts. total) At the Ideal Gas Company, a model of a simple chemical reactor system was developed by a previous employee. You are expected to verify the model and determine the time domain response of the model for changes in the input value.

a. (10 pts.) For the following differential equation:

\[ \frac{d^2 y}{dt^2}(t) + 7 \frac{dy}{dt}(t) + 12y(t) = \frac{du}{dt}(t) + u(t) \]

show that for the initial conditions \( y(t = 0) = 0, \frac{dy}{dt}(t = 0) = 0 \) \( u(t = 0) = 0 \), and \( \frac{du}{dt}(t = 0) = 0 \), the following transfer function relationship holds:

\[ y(s) = \frac{s + 1}{s^2 + 7s + 12} u(s) \]

b. (5 pts.) Given that the input to the system model is a unit impulse at time \( t = 1 \) (NOT at \( t=0 \)) show that \( y(t = 0) = 0 \) using the Initial Value Theorem.

c. (5 pts.) Given that the input to the system model is a unit impulse at time \( t = 1 \) (NOT at \( t=0 \)) show that \( y(t = \infty) = 0 \) using the Final Value Theorem.

d. (20 pts.) Given that the input to the system is a unit impulse at time \( t = 1 \) (NOT at \( t=0 \)) find the analytical response \( y(t) \) of the system to the unit impulse as an explicit function of time.

**BONUS**, sketch \( u(t) \) and \( y(t) \).
1. *(4 pts.)* A continuous polymerization reactor has two feed streams. Four species are measured at the exit of the reactor. The temperature of the reactor can be modified using a cooling jacket. Additionally, the mixing speed can be modified.

A 4x4 steady state model is desired relating the manipulated variables changes to the change in the output product concentrations, $\Delta x_A$, $\Delta x_B$, $\Delta x_C$, $\Delta x_D$. Input flows ($\Delta F_1$ and $\Delta F_2$), the change in the reactor jacket temperature ($\Delta T$), and the change in mixing speed ($\Delta M$) affect product quality in the following manner:

- A +10 GPH change in $F_1$ increases $x_C$ by 2%
- A +10 GPH change in $F_1$ increases $x_D$ by 4%
- A +10 GPH change in $F_1$ decreases $x_B$ by 3%
- A +10 GPH change in $F_2$ increases $x_A$ by 1%
- A +10 GPH change in $F_2$ increases $x_C$ by 1%
- A +1 change in $T$ increases all concentrations by 0.2%
- A +2 RPM increase in $M$ increases $x_B$ by 5%

Develop a model in the form $K \Delta u = \Delta y$ and identify $K$, $\Delta u$, and $\Delta y$.

You may want to check your model by assuming arbitrary values for the change in inputs, then verify the expected change in concentrations.
2. (2 pts.) Given the following system, draw a simple feedback control scheme to control the product quality $x_C$ by manipulating the cooling water flow.

3. (2 pts.) Given the following system, draw a simple feedforward control scheme to control the product quality $x_C$ by manipulating the cooling water flow given variations in the inlet cooling water temperature.

4. (2 pts.) Given the following system, draw a cascade control scheme to control the product quality $x_C$. 


15.12 Fall 2004 Quiz 1

Chemical Process Dynamics and Control

Quiz #1

September 3, 2004

1. (3 pts.) Given the following process system, draw two separate simple feedback control loops to control tank levels in tanks 1 and 2. Be sure to use control valves in your loops.
2. (2 pts.) Given the following system, draw a simple feedforward control schemes to help minimize variation in tank levels given changes in $F_2(t)$. Be sure to use control valves in your loops.

3. (2 pts.) Given the following system, draw a cascade control scheme to control the level in tank 3, noting that the level in tanks 1 or 2 would have some effect on the level of tank 3.
4. (2 pts.) What are the eigenvalues of the following matrix? Please show your work.

\[
\begin{bmatrix}
3 & 1 \\
-4 & 1
\end{bmatrix}
\]

5. (1 pts.) What is the determinant of the following matrix? Please show your work.

\[
\begin{bmatrix}
0 & -2 & -1 \\
0 & 2 & 3 \\
-1 & 0 & 2
\end{bmatrix}
\]
1. At the Ideal Gas Company, you are expected to develop a steady state model of the following process:

Given the following information:
- A 1% change in $V_1$ increases the level in Tank 1 by 3 inches
- A 1% change in $V_1$ increases the level in Tank 3 by 1 inch
- A 3% change in $V_2$ increases the level in Tank 1 by 4 inches
- A 3% change in $V_2$ increases the level in Tank 2 by 5 inches
- A 3% change in $V_2$ increases the level in Tank 3 by 6 inches
- A 1% change in $V_3$ decreases the level in Tank 3 by 2 inches

a. (1 point) What are $\Delta u$ and $\Delta y$?

b. (4 points) Develop a steady state multivariable model relating the inputs to the outputs.

c. (1 point) Put your model in the form $\Delta y = K \Delta u$ and clearly identify the $K$ matrix.
2. *(4 points)* What is the Laplace transform of the following input sequence, \( u(t) \).

\[
\begin{align*}
12 & \\
6 & \\
0 & \\
0 & \\
\end{align*}
\]

\( u(t) \)

\[
\begin{align*}
t = 0 & \quad t = 10 \\
0 & \quad \frac{6}{10}t \\
0 & \quad 10 \leq t
\end{align*}
\]

\[
u(t) = \begin{cases} 
0 & t < 0 \\
6 + \frac{6}{10}t & 0 \leq t < 10 \\
0 & 10 \leq t
\end{cases}
\]

3. **BONUS** In *five words or less*, why can we analyze dynamic systems with complex composite forcing functions by treating each part separately?
15.14 Fall 2004 Exam 1

ECHE 550, Fall 2004
Chemical Process Dynamics and Control

Exam #1
September 27, 2004

1. (20 pts. total) The Ideal Gas Company is attempting to develop a dynamic process model for a continuous processing nylon production system. Data from the process output and the process input are shown below for a step change and a simulated impulse.

a. (15 pts.) Determine the

- process gain ($K$)
- process time constant ($\tau$)
- process dead time (time delay) ($\alpha$)

for the system. Time units on the graph are in minutes.

b. (5 pts.) Given a unit step change increase in the process input at time $t = 0$, what is the expected response of your model as a function of time?
2. (20 pts. total) Determine the eigenvalues of the following matrix. Note, you probably should not use row reduction methods in the solution of this problem.

a. (10 pts.) Set up the problem to be solved.

b. (10 pts.) Find simplified numerical values for the eigenvalues.

\[
\begin{bmatrix}
0 & -4 & 1 \\
0 & -3 & 5 \\
0 & -4 & 1 \\
\end{bmatrix}
\]
3. **(25 pts. total)** You must develop a dynamic model based on fundamental principles for a pressure tank system as pictured below. You may assume that the total number of moles of gas in tank \( i \), \( n_i(t) \), may also be expressed as \( \frac{V_i}{R} P_i(t) \) using the ideal gas law. All flow rates are molar flow rates. You can change the valve position on the inlet stream, \( u(t) \). You do not need to carry units throughout the problem, just try to make sure you have the correct terms in each balance.

\[
\begin{align*}
F_1(t) &= k_1 (P_1(t) - P_3(t)) \\
F_2(t) &= k_2 (P_2(t) - P_3(t)) \\
F_3(t) &= k_3 (P_3(t))
\end{align*}
\]

- The total molar flow rate into the system is \( k_0 u(t) \)
- The molar flow rate into tank 1 is \( \gamma k_0 u(t), 0 \leq \gamma \leq 1 \)
- The molar flow rate into tank 2 is \( (1 - \gamma) k_0 u(t) \)
- The molar flow from tank 1 into tank 3 is \( k_1 (P_1(t) - P_3(t)) \)
- The molar flow from tank 2 into tank 3 is \( k_2 (P_2(t) - P_3(t)) \)
- The molar flow from tank 3 into the atmosphere is \( k_3 P_3(t) \)
- The tanks are constant volume, \( V_1, V_2, V_3 \).
- The gas in the tanks is at a constant temperature.
- You can measure the pressure in tanks 2 and 3.

a. **(15 pts.)** Derive a differential equation model for the system.

b. **(10 pts.)** Put your deviation differential equation model into State Space form \( \dot{x} = Ax + Bu, \ y = Cx + Du \) for the system, given that all pressures are \( = 0 \) initially (no deviation variables needed in this case).
4. (35 pts. total) At the Ideal Gas Company, a model of a simple chemical reactor system was developed by a previous employee. You are expected to verify the model and determine the time domain response of the model for changes in the input value.

a. (7 pts.) For the following differential equation:

\[ 6 \frac{dy}{dt}(t) + 2y(t) = 4 \frac{du}{dt}(t) + u(t) \]

show that for the initial conditions \( y(t = 0) = 0 \) and \( u(t = 0) = 0 \), the following transfer function relationship holds:

\[ y(s) = \frac{4s + 1}{6s + 2} u(s) \]

b. (6 pts.) What are the poles of your transfer function? What are the zeros?

c. (4 pts.) What is the gain of this model?

d. (5 pts.) Given that you implement a negative step change of magnitude 3 at time zero in the input \( u(t) \), what is the ultimate response? Use the Final Value Theorem to find \( y(t = \infty) \).

d. (13 pts.) Given that you implement a negative step change of magnitude 3 at time zero in the input \( u(t) \), find the analytical response \( y(t) \) of the system.

**BONUS**, sketch \( u(t) \) and \( y(t) \).
1. (4 pts.) Your first assignment for GameCockCo is in the silicon wafer production facility. Each wafer must be maintained at a high temperature during the etching process. Since very high temperatures are required, radiative heat transfer using high temperature lamps will be used to heat the chamber. The triangular chemical vapor decomposition chamber has three variable intensity lamps, one in each corner of the chamber, with intensities $I_1$, $I_2$, and $I_3$. The chamber also has three temperature sensors, denoted by $S_1$, $S_2$, and $S_3$. These sensors are also located in the corners of the chamber. You are told by a senior engineer in your department that a 5% increase in the intensity of any one of the three lamps results in a 8 degree increase in the corresponding temperature sensor location and a 3 degree increase in the temperature in the sensors in both of the opposite corners of the chamber.

Develop a model in the form $K \Delta u = \Delta y$ and clearly identify $K$, $\Delta u$, and $\Delta y$. 
2. (2 pts.) During a production run, temperature sensor 1 is 6 degrees below optimal, sensor 2 is 1 degree above optimal, and sensor 3 is 9 degrees below optimal. What problem would you solve to get the chamber back to the nominal operating temperature (what is your $\Delta y$ value?). How would you solve this problem? (What formula or method would you use?)

3. (2 pts.) What are the eigenvalues of the following matrix? Please show your work.

$$ \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} $$

4. (2 pts.) What is the determinant of the following matrix? Please show your work.

$$ \begin{pmatrix} 1 & -4 & 0 \\ 0 & 3 & 0 \\ -1 & 3 & 2 \end{pmatrix} $$

BONUS, why are eigenvalues and determinants useful?
BONUS, solve problem 2 above for a real answer.
1. (4 pts.) A continuous bio-reactor for yeast fermentation has a single glucose feed stream. The growth reaction takes place in a jacketed CSTR. It is assumed that low-level control systems are in place such that the inlet glucose concentration and temperature can be specified (the mixing of pure cold water, pure hot water, and high concentration glucose is not to be considered here). Additionally, the temperature of water entering the jacket can be specified. Given these three manipulated input values, you are expected to develop a state-space model of the system.

The following is known about the system:

- The reactor is well-mixed with volume $V$
- The liquid in the jacket is well-mixed with volume $V_j$
- The reaction is a first-order reaction with volumetric reaction rate $kC_A(t)$
- The heat of reaction is $-\Delta H$
- The reactor volumetric flow rate in (and out) is $F$
- The jacket volumetric flow rate in (and out) is $F_j$
- The heat transfer from the reactor to the jacket is $hA(T(t) - T_j(t))$
- The liquids all have constant physical properties, $C_p$, $\rho$, $\mu$
• The steady state input values for $u_1$, $u_2$, and $u_3$ are $C_{AinSS}$, $T_{inSS}$, and $T_{jinSS}$ respectively

• The steady state state values are $C_{ASS}$, $T_{SS}$, and $T_{jSS}$

• The jacket temperature deviation value $T_j(t) - T_{jSS}$ and the reactor temperature deviation value $T(t) - T_{SS}$ are measured

a. Develop a dynamic mass and energy balance for the system.

b. Put your system in deviation variable form using the steady state values.

c. Identify your states and put your system in state-space form and identify $A$, $B$, $C$, and $D$. 
1. (30 pts. total) The Ideal Gas Company has a simple mixing system for preparation of a reactor feed. You must develop a dynamic model of this system.

- No reaction is taking place in either tank, both well-mixed
- The mixing tanks are constant volume, $V_1$ and $V_2$
- The volumetric flow rate $F$ is fixed
- The inlet flow is equally split between the first tank and the side stream
- Initially all concentrations are 0

a. (15 pts.) Develop a dynamic model of the system. Note that instantaneous mixing occurs at the mixing point shown in the dotted box (no accumulation at the mixing point). State any additional assumptions you make for your system.

b. (15 pts.) Assuming that $F = 2\frac{m^3}{min}$, $V_1 = 20m^3$, $V_2 = 10m^3$, $C_0(t) = u(t)$ and $C_3(t) = y(t)$ take the Laplace transform of your model equations and show that the following transfer function holds:

$$\frac{y(s)}{u(s)} = g(s) = \frac{(10s + 1)}{(5s + 1)(20s + 1)}$$
2. (30 pts. total) Assume that the inlet concentration momentarily changes, allowing some of the reactant to flow into the system. You can assume that \( u(t) = \delta(t) \), with the system modeled as:

\[
y(s) = \frac{(10s + 1)}{(5s + 1)(20s + 1)}u(s)
\]

a. (2 pts.) What is \( u(s) \)?

b. (6 pts.) Given this input, what is the initial value for \( y \), \( y(t = 0) \)?

c. (6 pts.) Given this input, what is the final value for \( y \), \( y(t = \infty) \)?

d. (10 pts.) What is the actual response of the outlet concentration as an analytical expression, \( y(t) \)? Note that:

\[
L^{-1}\left\{\frac{1}{\tau s + 1}\right\} = \frac{1}{\tau} e^{-\frac{t}{\tau}}
\]

e. (6 pts.) Sketch \( y(t) \).

3. (20 pts. total) The inlet concentration for your system can be manipulated, but there are limits to the response of the inlet concentration value. What is the Laplace transform of the following function of time, \( C_0(s) \)?

\[
C_0(t) = \begin{cases} 
0 & t \leq 0 \\
0.5 + 0.5t & 0 \leq t \leq 1 \\
1.0 & 1 \leq t \leq 3 \\
0 & 3 \leq t 
\end{cases}
\]
4. **(20 pts. total)** From the following dynamic response data, determine the values for the gain, time constant and time delay for the real experimental system \((K, \tau, \alpha)\). The inlet concentration is manipulated using a ratio valve and the exit concentration measurement is reported as a signal voltage. The valve is limited in the ability to open, resulting in the abnormal \(u(t)\) value.

b. **(5 points)** What is the transfer function, \(g(s)\), for this system?

*BONUS*, Derive a state space model for your dynamic system from problem 2.
15.18 Fall 2006 Quiz 1

ECHE 550, Fall 2006

Chemical Process Dynamics and Control

Quiz #1

September 8, 2005

1. (5 pts.) As an intern at GameCockCo, you get stuck in the warehouse. The warehouse has had problems with product loss due to poor heating in the winter. The complex HVAC system is not maintaining a uniform temperature in the warehouse due to poor mixing (channeling in the warehouse ventilation flow). Rather than buy fans to force improved convection in the warehouse, you suggest an improved control system, since the current system runs all furnaces at the same rate. Four temperature sensors are available, $T_1$, $T_2$, $T_3$, and $T_4$. Three furnaces are available, $F_1$, $F_2$, and $F_3$. The furnaces run on a 0-100 scale and are controlled by a centralized computer system.

- Increasing Furnace 1 by 10 units increases $T_1$ by 3 degrees, $T_2$ by 2 degrees, and $T_3$ by 1 degree
- Increasing Furnace 2 by 10 units increases $T_1$ by 1.5 degrees and $T_2$ by 5 degrees
- Increasing Furnace 2 by 10 units increases $T_3$ by 4 degrees, and $T_4$ by 2.5 degrees
- Increasing Furnace 3 by 10 units increases $T_2$ by 1.5 degrees, $T_3$ by 2.1 degrees, and $T_4$ by 3.2 degrees

![Diagram of Warehouse with Sensors and Furnaces]

Develop a model in the form $K \Delta u = \Delta y$ and clearly identify $K$, $\Delta u$, and $\Delta y$.

2. (1 pt) During a production run, temperature sensor 1 is 2 degrees below optimal, sensor 2 is 1 degree above optimal, sensor 3 is 5 degrees below optimal, and sensor 4 is 2 degrees above optimal. What problem would you solve to get the chamber back to the nominal operating temperature (what is your $\Delta y$ value?).
3. \textbf{(2 pts.)} For the furnace system, it is desired to regulate the outlet temperature $T_o$. Both air and oil are fed to the furnace, and both flow rates strongly influence the outlet temperature. Draw a simple feedback control system for the furnace below.

\begin{center}
\includegraphics[width=0.7\textwidth]{furnace_feedback.png}
\end{center}

4. \textbf{(2 pts.)} For the furnace system, the inlet air temperature has some influence on the outlet temperature. Draw a feedforward control system to mitigate the effects of the incoming air temperature on the furnace outlet temperature.

\begin{center}
\includegraphics[width=0.7\textwidth]{furnace_feedforward.png}
\end{center}
1. (5 pts.) For the following function of time:

\[ u(t) = \begin{cases} 
5 & \text{if } t = 0 \\
-5 & \text{if } t = 4 \\
5 & \text{if } t = 9 
\end{cases} \]

a). Express \( u(t) \) as a sum of simple functions of time.
b). Find \( u(s) \), the Laplace transform of \( u(t) \).

2. (4 pts) For the following process data, determine the gain, time constant, and time delay.
b). (1 pt) What is the first-order-plus-time-delay transfer function for this system?
15.20 Fall 2006 Exam 1

ECHE 550, Fall 2006

Chemical Process Dynamics and Control

Exam #1 - October 4, 2005

1. (15 pts. total) Your nice window office at GameCock Co. is on the west side of the building. Due to mistakes when installing the HVAC system, the temperature in your office is poorly regulated, since the cooling takes place only in the office next door and the windows are not insulated. Given the following steady state data:

<table>
<thead>
<tr>
<th></th>
<th>Office 1</th>
<th>Office 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T_1$</td>
<td>$T_2$</td>
</tr>
<tr>
<td>Windows</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Insulated walls and doors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Q$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_o$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- A 3 degree increase in the external temperature $T_o$ results in a 2 degree increase in $T_1$
- A 3 degree increase in the external temperature $T_o$ results in a 1 degree increase in $T_2$
- A increase of 5 units in the chiller $Q$ results in a 4 degree decrease in $T_1$
- A increase of 5 units in the chiller $Q$ results in a 2.5 degree decrease in $T_2$

a. (8 pts.) Develop a model in the form $K \Delta u = \Delta y$ and clearly identify $K$, $\Delta u$, and $\Delta y$.

b. (7 pts.) Given that you want $T_1$ to remain constant and you want $T_2$ to decrease by 1 degree, what would have to happen to $Q$ and $T_o$?
2. (30 pts. total) Develop a dynamic model of your office. Assume the following:

- The offices have no air moving in or out, but the air in the offices is well-mixed (fans)
- The volume of air in each office is $V_1$ and $V_2$ respectively
- The heat capacity and density of the air in each office is $C_p$ and $\rho$
- The rate of energy entering the each office from the outside is equal: $Q_{in}(t) = c_1(T_o(t))^4$
- The rate of energy transferred across the thin wall is: $Q_a(t) = hA(T_1(t) - T_2(t))$
- Physical properties and parameters do not change with time
- Deviation values are:
  - $x_1(t) = T_1(t) - T_{1ss}$
  - $x_2(t) = T_2(t) - T_{2ss} = y(t)$
  - $u_1(t) = T_o(t) - T_{oss}$
  - $u_2(t) = Q(t) - Q_{ss}$

a. (10pts.) Develop a dynamic model for this system.

b. (10pts.) Develop a linear dynamic model for this system in deviation form.

c. (10pts.) Put your linear model in state space form and clearly identify $A$, $B$, $C$, and $D$. 

- Insulated walls and doors
- Windows
- Fans
3. **(15 pts. total)** The external temperature follows the following trajectory during the day. What is the Laplace transform of the following function of time, \( u_1(t) \)?

\[
u_1(t) = \begin{cases} 
0 & 0 \leq t \leq 9 \\
(t - 9) & 9 \leq t \leq 12 \\
3 & 12 \leq t \leq 14 \\
6 & 14 \leq t \leq 21 \\
0 & 21 \leq t 
\end{cases}
\]

4. **(15 pts. total)** From the following dynamic response data, determine an empirical transfer function for the system.
5. (25 pts. total) The previous employee that sat in your office developed the following model for office temperature as a function of external temperature.

a. (5 pts.) For the following differential equation:

\[
2 \frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 5 y(t) = 5 \frac{du(t)}{dt} + 10 u(t)
\]

show that for the initial conditions \( y(t = 0) = \frac{dy}{dt}(t = 0) = 0 \) and \( u(t = 0) = 0 \), the following transfer function relationship holds:

\[
y(s) = \frac{5s + 10}{2s^2 + 3s + 5} u(s)
\]

b. (5 pts.) What are the poles of your transfer function? What are the zeros?

c. (5 pts.) What is the gain of this model?

d. (5 pts.) Given that you implement a negative step change of magnitude 5 at time zero in the input \( u(t) \), what is the initial value for \( y(t) \)? Use the Initial Value Theorem to find \( y(t = 0) \).

e. (5 pts.) Given that you implement a negative step change of magnitude 5 at time zero in the input \( u(t) \), what is the ultimate response? Use the Final Value Theorem to find \( y(t = \infty) \).

Bonus: Does this model make sense for the system?
1. (5 pts.) You have recently been promoted to junior process engineering at GameCockCo. This gives you responsibility over the polymer blending and extrusion process. Two jacketed reactors feed polymer into two separate extruders. In the extruder section, the polymer is blended, extruded, and sent to chip-out to make pellets. Online viscosity measurements are available for the polymer entering the extruders, $V_1$ and $V_2$. It is known that:

- Increasing reactor 1 temperature by 10 degrees increases $V_1$ by 100 cp and $V_2$ by 25 cp
- Increasing reactor 2 temperature by 4 degrees increases $V_1$ by 30 cp and $V_2$ by 85 cp

Develop a model in the form $K \Delta u = \Delta y$ and clearly identify $K$, $\Delta u$, and $\Delta y$.

2. (1 pt) During a production run, $V_1$ is 20 cp above the desired value and $V_2$ is 15 cp below the desired value. What problem would you solve to get the chamber back to the nominal operating temperature (what is your $\Delta y$ value?).
3. (4 pts.) To make changes in the reactor temperature, simple feedback control of the cooling system is required. In the diagram below do the following. For reactor 1, propose a cascade control system to control reactor temperature. For reactor 2, propose a simple feedback control system to control reactor temperature.

**Bonus:** Solve part 2 using Gaussian elimination. Show work.
1. (5 pts.) For the following function of time:

\[ y(t) = \begin{cases} 
0 & t < 5 \\
-(t - 5) & 5 \leq t < 10 \\
-5 & 10 \leq t < 20 \\
-5 \exp\left(-\frac{1}{2}(t - 20)\right) & 20 \leq t 
\end{cases} \]

Find \( u(s) \), the Laplace transform of \( u(t) \).

2. (4 pts) For the following process data, determine the gain, time constant, and time delay.

b). (1 pt) What is the first-order-plus-time-delay transfer function for this system?
1. (30 pts. total) Consider ethanol metabolism in the body. Liquids are consumed, entering the gut. Ethanol transport into the blood stream can be modeled as a diffusion process. Ethanol is removed from the body due to urination and metabolism. You must develop a dynamic model of this process relating the inlet concentration of ethanol to the resulting concentration in the blood.

Assume the following:

- Liquid enters and leaves the body at a constant volumetric flow rate, $F$
- Entering liquid contains a time-varying concentration of ethanol, $C_o(t)$
- The gut can be assumed to be well-mixed volume $V$ at concentration $C_G(t)$
- The blood stream can be assumed to be well-mixed with volume $V_b$ at concentration $C_B(t)$
- The total rate of ethanol transferred from the gut to the blood is: $j(t) = DA(C_G(t) - C_B(t))$
- The total rate of ethanol removed from the blood is $r(t) = kC_B(t)$
- Physical properties and parameters do not change with time
- All concentrations are normally zero.

a. (10pts.) Develop a dynamic model for this system, assuming the blood concentration is measured.

b. (10pts.) Put your linear model from part a in state space form and clearly identify $x(t)$, $A$, $B$, $C$, and $D$.

c. (10pts.) Derive a transfer function model from your linear model from part a:

$$y(s) = g(s)u(s)$$
2. **(20 pts. total)** For a given person, the initial feed concentration follows the following trajectory during the day. What is the Laplace transform of the following function of time, \( u(s) \)?

\[
    u(t) = \begin{cases} 
        0 & t \leq 6 \\
        3 & 6 \leq t < 7 \\
        3 - (t - 7) & 7 \leq t < 9 \\
        1 & 9 \leq t < 18 \\
        0 & 18 \leq t 
    \end{cases}
\]
3. (20 pts. total) From the following dynamic response data, determine an empirical transfer function for the response to changes in the input concentration.
4. **(30 pts. total)** You are given the following model for blood stream concentration as a function of input concentration.

a. **(5 pts.)** For the following differential equation:

\[
4 \frac{d^2 y(t)}{dt^2} + 8 \frac{dy(t)}{dt} + 3y(t) = 5 \frac{du(t)}{dt} - 9u(t)
\]

show that for the initial conditions \(y(t = 0) = \frac{du}{dt}(t = 0) = 0\) and \(u(t = 0) = 0\), the following transfer function relationship holds:

\[
y(s) = \frac{5s - 9}{4s^2 + 8s + 3} u(s)
\]

b. **(5 pts.)** What are the poles of your transfer function? What are the zeros?

c. **(3 pts.)** What is the gain of this model?

d. **(5 pts.)** Given that you implement a positive step change of magnitude 2 at time zero in the input \(u(t)\), what is the initial value for \(y(t)\)? Use the Initial Value Theorem to find \(y(t = 0)\).

e. **(5 pts.)** Given that you implement a positive step change of magnitude 2 at time zero in the input \(u(t)\), what is the ultimate response? Use the Final Value Theorem to find \(y(t = \infty)\).

f. **(7 pts.)** Given that you implement a positive step change of magnitude 2 at time zero in the input \(u(t)\), what is analytical value of the response? Find \(y(t)\).

Bonus: Does this model make sense for the body system? Why or why not?
15.24 Fall 2009 Quiz 1

ECHE 550, Fall 2009

Chemical Process Dynamics and Control

Quiz #1

September 4, 2009

1. (5 pts.) You just started as a process intern at GameCockCo. You are working on the power generation process where wood chips are burned in a two-step process to make steam and electricity. In the first step, three reactors are used to partially combust the wood chips, producing syn gas. The gas is combined and burned cleanly in the main furnace.

Online flow and temperature measurements are available at various locations in the process. Computer-controlled continuously adjustable blowers are available to modulate the inlet air-flow from the main air feed manifold to the partial combustion reactors. You must develop a steady-state gain relationship model for the partial combustion reactors. It is known that:

- Increasing blower 1 speed $S_1$ by 10% increases the temperature reactor 1 by 2 degrees and reduces the temperature in the other two reactors by 1 degree

- Increasing blower 2 speed $S_2$ by 10% increases the temperature reactor 2 by 4 degrees and reduces the temperature in the other two reactors by 2 degrees

- Increasing blower 3 speed $S_3$ by 10% increases the temperature reactor 3 by 1 degree and reduces the temperature in reactor 2 by 1 degree

Develop a model in the form $K \Delta u = \Delta y$ and clearly identify $K$, $\Delta u$, and $\Delta y$.

2. (1 pt) While operating, $T_1$ is 3 degrees below the desired value and $T_3$ 4 degrees above the desired value. What problem would you solve to get the chamber back to the nominal operating temperature (what is your $\Delta y$ value?).
3. (4 pts.) To make changes in the main system temperatures, feedback control of the system is required. In the diagram below do the following. For combustion reactor 3, propose a feedback control system to control the reactor temperature. For the main furnace, propose a cascade control system to control main furnace temperature.
1. (5 pts.) For the following function of time:

\[ u(t) = \begin{cases} 
0 & t < 10 \\
\frac{1}{2}(t - 10) & 10 \leq t < 20 \\
5 & 20 \leq t < 30 \\
10 - 5 \exp \left( -\frac{1}{3}(t - 30) \right) & 30 \leq t < 50 \\
\sim 0 & 50 \leq t 
\end{cases} \]

Find \( u(s) \), the Laplace transform of \( u(t) \).

2. (4 pts) For the following differential equations:

\[ \begin{align*} 
3 \frac{dx_1}{dt}(t) &= -x_1(t) + 12u(t) \\
4 \frac{dx_2}{dt}(t) &= x_1(t) - 2x_2(t) \\
y(t) &= x_2(t)
\end{align*} \]

show that the transfer function relating \( u(s) \) to \( y(s) \) is:

\[ y(s) = \frac{1}{(s + \frac{1}{2})(s + \frac{1}{3})} u(s) \]

Note: You may assume that \( x_1(t = 0) = x_2(t = 0) = 0 \).

b). (1 pt) For the process in part a), what is the ultimate response of the dynamic system to a unit impulse in the input at time \( t = 0 \)?

**BONUS:** Use partial fraction expansion to show that your answer to part b) is correct.
1. *(25 pts. total)* Consider growth of a new strain of algae for biofuel production. Three growing ponds are used for production during the day.

Assume the following:
- Liquid enters pond 1 at flow rate $F_1$
- The flow out of pond 1 is split. $F_2$ goes to pond 2 and $F_3$ flows to pond 3
- The ponds are constant volume and $F_1 = F_2 + F_3$
- No algae enters pond 1
- The three ponds can be assumed to be well-mixed with volumes $V_1$, $V_2$, and $V_3$
- Pond 2 receives a minimal volume nutrient feed. The *volumetric growth rate* for this pond is $r(t) = k_1 T(t) + k_2 C_{A2}(t)$
- The *volumetric growth rate* in the other ponds depends only on deviation in the ambient temperature: $r(t) = k_1 T(t)$
- Physical properties and parameters do not change with time
- All concentrations are normally zero
- The total concentration of biomass leaving the system can be measured, $C_{AP}(t)$
- Treat the ambient temperature deviation as a process input, $u(t) = T(t)$

a. *(15 pts.)* Develop a dynamic model for this system.

b. *(10 pts.)* Put your linear model from part a in state space form and clearly identify $x(t)$, $A$, $B$, $C$, and $D$. 
2. *(20 pts. total)* For the pond system, the ambient temperature deviation has the following trajectory during the day. What is the Laplace transform of the following function of time, $u(s)$?

\[ u(t) = \begin{cases} 
0 & t \leq 6 \\
2(t-6) & 6 \leq t < 12 \\
12 & 12 \leq t < 17 \\
0 & 17 \leq t 
\end{cases} \]

3. *(20 pts. total)* From the following dynamic response data, determine an empirical **transfer function** for the response to changes in the ambient temperature.
4. (10 pts. total) You must control the system to maintain the product biomass. Draw a cascade feedback control scheme on the following diagram.

5. (25 pts. total) You are given the following model for product concentration $y(t)$ as a function of ambient temperature deviation $u(t)$.

   a. (5 pts.) For the following differential equation:

   $$\frac{d^2 y}{dt^2}(t) + 8 \frac{dy}{dt}(t) + 15y(t) = 4u(t)$$

   show that for the initial conditions $y(t = 0) = \frac{dy}{dt}(t = 0) = 0$, the following transfer function relationship holds:

   $$y(s) = \frac{4}{s^2 + 8s + 15} u(s)$$

   b. (5 pts.) Given that you implement a negative step change of magnitude 10 at time zero in the input $u(t)$, what is the initial value for $y$? Use the Initial Value Theorem to find $y(t = 0)$.

   c. (5 pts.) Given that you implement a negative step change of magnitude 10 at time zero in the input $u(t)$, what is the ultimate response? Use the Final Value Theorem to find $y(t = \infty)$.

   d. (10 pts.) Given that you implement a negative step change of magnitude 10 at time zero in the input $u(t)$, what is analytical value of the response? Find $y(t)$. 
1. (7 pts. total) As the economic downturn continues, you were forced to take a job at Clem’s Son Sewage Co. in the water processing plant. Water is pumped from two reservoirs to a large storage tank before it is distributed to the community. It is desired to maintain the level in the main storage tank, which is measured by an ultrasonic sensor.

a. (5 pts.) Draw a simple feedback control scheme on the figure above for F2. Simultaneously, propose a separate cascade control system to maintain the tank level.

b. (2 pts.) Propose a feedforward control scheme on the figure above, assuming variable demand.
2. (3 pts. total) For the following matrix:

\[
\begin{bmatrix}
  0 & -1 & 0 \\
  1 & 2 & 0 \\
  3 & 1 & 0 \\
\end{bmatrix}
\]

a. (1 pts.) What is the determinant? Show your work.

b. (2 pts.) What are the eigenvalues? Show your work.
1. (4 pts.) For the following function of time:

\[
u(t) = \begin{cases} 
0 & t < 20 \\
40 - 40 \exp\left(-\frac{1}{4}(t - 20)\right) & 20 \leq t < 50 \\
\sim 40 - 2(t - 50) & 50 \leq t < 70 \\
\sim 0 & 70 \leq t 
\end{cases}
\]

Note, you may assume \(\exp(-t)\) for \(t > 6\) \(\sim 0\)

Find \(u(s)\), the Laplace transform of \(u(t)\).

2. (5 pts) You are in the sheet metal painting business. The sheets of metal roll through painters then into heaters. After the paint is applied, the heaters are used to rapidly dry the paint. There are three heaters across the plane of the moving sheet metal and three corresponding temperature sensors in positions 1-3. You may independently adjust the amount of energy for each heater. From process data, you know the following steady state relations:

- Increasing heater 1 \((Q_1)\) by 5% results in \(T_1\) increase of 3°
- Increasing heater 1 \((Q_1)\) by 5% results in \(T_2\) increase of 1°
- Decreasing heater 2 \((Q_2)\) by 10% results in \(T_2\) decrease of 4°
• Decreasing heater 2 \((Q_2)\) by 10% results in \(T_1\) and \(T_3\) decreases of 2°
• Increasing heater 3 \((Q_3)\) by 5% results in \(T_3\) increase of 3°
• Increasing heater 3 \((Q_3)\) by 5% results in \(T_2\) increase of 1°

Develop a linear stead-state model in the form \(\Delta y = K \Delta u\) and clearly identify \(\Delta y\), \(\Delta u\), and \(K\).

b). (1 pt) For the process in part a), assuming the current temperatures are all 8° too high, what problem would you solve to find new desired heater values?

**BONUS:** Solve the problem in part b
1. (25 pts. total) Your job at GameCock Co. Inc. requires that you improve the operation of an extruder used to produce thermoset plastic. This process takes a feed of plastic pellets and mixes them to produce a final product. The amount of crosslinked polymer determines the final product quality.

Assume the following:
- The mass flow rate of pellets entering is constant, \( M \)
- The feed temperature is \( T_o(t) \)
- The percent of crosslinked polymer in the feed is \( W_o(t) \)
- The length of the extruder is \( L \) with cross-section \( A \)
- The screw extruder can be assumed to be well-mixed
- The volumetric crosslinking rate (mass based) is \( r(t) = kT(t) \)
- The heat of reaction for the crosslinking is negligible
- The extruder is poorly insulated and the heat loss rate to the environment is given as \( hA(T(t) - T_{atm}) \)
- The atmospheric temperature \( T_{atm} \) is constant
- The deviation feed percentage and deviation feed temperature can be treated as inputs
- The deviation temperature of the product is measured, \( y(t) = T(t) - T_{ss} \)
- Physical properties are constant

a. (15 pts.) Develop a dynamic model for this system.

b. (10 pts.) Put your linear model from part a in state space form and clearly identify \( x(t), u(t), A, B, C, \) and \( D \).
2. *(15 pts. total)* For the extruder, the feed temperature for a product run in deviation terms has the following trajectory. What is the Laplace transform of the following function of time, \( u(t) \)?

\[
u(t) = \begin{cases} 
0 & t \leq 0 \\
2e^{-3t} - 2 & 0 \leq t < 10 \\
\sim 3 - .2(t - 10) & 10 \leq t < 20 \\
\sim 1 & 20 \leq t < 25 \\
\sim 0 & 25 \leq t 
\end{cases}
\]

3. *(15 pts. total)* From the following dynamic response data, determine a **FOTD transfer function model** for the response to changes in the feed temperature. Show your work on the figure to receive full credit.
4. **(20 pts. total)** It is suggested that a steam jacket be added to the extruder to help maintain product temperature despite significant variations in the feedstock.

   a. **(10 pts.)** Propose a **cascade feedback control** scheme on the following diagram.

   ![Cascade Feedback Control Diagram]

   a. **(10 pts.)** Propose a **feedforward control** scheme on the following diagram.

   ![Feedforward Control Diagram]
5. **(25 pts. total)** You are given the following model for the product temperature \( y(t) \) as a function of the feed temperature deviation \( u(t) \).

a. **(7 pts.)** For the following set of differential and algebraic equations:

\[
\begin{align*}
\frac{dx_1}{dt}(t) &= -15x_1(t) - 56x_2(t) + u(t) \\
\frac{dx_2}{dt}(t) &= x_1(t) \\
y(t) &= x_1(t) + 4x_2(t)
\end{align*}
\]

show that for the initial conditions \( x_1(t = 0) = x_2(t = 0) = 0 \), the following transfer function relationship holds:

\[
y(s) = \frac{s + 4}{s^2 + 15s + 56} u(s)
\]

b. **(5 pts.)** Given that you implement a step increase of magnitude 6 at time zero in the input \( u(t) \), what is the initial value for \( y \)? Use the Initial Value Theorem to find \( y(t = 0) \).

c. **(5 pts.)** Given that you implement a step increase of magnitude 6 at time zero in the input \( u(t) \), what is the ultimate response? Use the Final Value Theorem to find \( y(t = \infty) \).

d. **(8 pts.)** Given that you implement a step increase of magnitude 6 at time zero in the input \( u(t) \), what is analytical value of the response? Find \( y(t) \).
1. (6 pts. total) Jessie goes online to find dates. People respond for dates from three areas: Aiken ($D_A$), Beaufort ($D_B$), and Charleston ($D_C$). Jessie collected a lot of data after changing dating profile information varying three factors: Income ($F_1$), Pictures ($F_2$), and Years of Education ($F_3$). Jessie realizes the following relationships:

- Increasing income ($F_1$) by 10% increases the number of monthly responses from Aiken ($D_A$) by 1.
- Increasing income ($F_1$) by 10% increases the number of monthly responses from Beaufort ($D_B$) by 3.
- Increasing income ($F_1$) by 10% decreases the number of monthly responses from Charleston ($D_C$) by 1.
- Decreasing pictures ($F_2$) by 4 decreases the number of monthly responses from Aiken ($D_A$) by 2.
- Decreasing pictures ($F_2$) by 1 decreases the number of monthly responses from Charleston ($D_C$) by 2.
- Increasing years of education ($F_3$) by 2 increases the number of monthly responses from all by 1.

a. (2 pts.) What are the adjustable variables? What are the measured variables?
b. (4 pts.) Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

2. (4 pts. total) For the following matrix:

$$
\begin{bmatrix}
0 & -1 & 0 \\
0 & 2 & 0 \\
0 & 1 & 3
\end{bmatrix}
$$

a. (2 pts.) What is the determinant? Show your work.
b. (2 pts.) What are the eigenvalues? Show your work.
1. (4 points total) You are working at Gamecock Drug Co. Your company just came up with a drug to cure cancer and male pattern baldness. This drug is produced inside a genetically modified organism. You are tasked with developing a dynamic model at the cellular level to aid in drug production. The drug is $B$ and it is made from substrate $A$. Develop a dynamic model for a single cell, assuming the following:

- The volume of the cell is $V$ (L) and is assumed constant
- The concentrations inside the cell are $C_A(t)$ and $C_B(t)$ ($\text{mol L}^{-1}$)
- The total diffusion rate of $A$ into the cell is $j_{in}(t) = D_1 (C_{Ao}(t) - C_A(t))(\text{mol hr}^{-1})$
- The total reaction rate in the cell is $r(t) = kC_A(t)(\text{mol hr}^{-1})$
- The total diffusion rate of $B$ out of the cell is $j_{out} = D_2 C_B(t)(\text{mol hr}^{-1})$
- You can measure a signal proportional to the concentration of $B$ inside the cell, $y(t) = kC_B(t)$
- The temperature is constant, the cell is assumed well-mixed, physical properties are constant
- **Bonus:** Assume concentrations are initially 0 and $C_{Ao}(t)$ is adjustable, develop a state space model.
2. **(6 points total)** To scale up production, you must use a jacketed CSTR with tight controls to maintain ideal process conditions. Clearly present your control structures on the diagram below.

a). **(2 points)** Assuming the feed temperature varies significantly, propose a feedforward control scheme to mitigate the resulting effects.

b). **(2 points)** Propose a feedback control scheme to maintain the reactor feed flow rate.

c). **(2 points)** Propose a cascade control scheme to maintain the reactor temperature.
1. **(25 pts. total)** Your new internship at GameCock Co. Inc. is in the power and recovery section of the plant. Your area supplies chilled water to the plant. One solar heating tank and one cooling tank provide a mixed product stream. The product temperature must be maintained very accurately. You must develop a first-principles dynamic model for the system. The schematic is below.

Assume the following:

- Two well-mixed tanks are used of volume $V_1$ and $V_2$
- The volumetric flow rate into each tank is constant: $F$
- You have constant physical properties ($\rho$, $C_p$)
- The total rate of solar energy into tank 1 is $q_1(t) \left( \frac{dT}{dt} \right) = k \left( T_1(t) - T_a(t) \right)^2$
- Chilled water is available to cool tank 2. The total rate is adjustable: $q_2(t) \left( \frac{dT}{dt} \right)$, $q_2(t) > 0$
- The total product flow rate is $2F$ at temperature $T_p(t)$
- The total feed flow rate is $2F$ at temperature $T_o(t)$
- Tanks operate at steady states $T_{1ss}$ and $T_{2ss}$
• The following are inputs: \( u_1(t) = T_o(t) - T_{oss} \quad u_2(t) = T_o(t) - T_{ass} \quad u_3(t) = q_2(t) - q_{2ss} \)

• The following are outputs: \( y_1(t) = T_1(t) - T_{1ss} \quad y_2(t) = T_p(t) - T_{pss} \)

a. (12 pts.) Develop a dynamic model for this system. You should also model the product temperature relationship to other variables.

b. (13 pts.) If necessary, linearize your model from part a and put it in state space form, remembering to clearly identify \( x(t), u(t), A, B, C, \) and \( D \).

2. (15 pts. total) For the tank system, as the day goes along, the ambient temperature deviation follows the given trajectory. What is the Laplace transform of the following function of time, \( u(s) \)?

\[
u(t) = \begin{cases} 
0 & \text{if } t \leq 8 \\
5(t - 8) & \text{if } 8 \leq t < 12 \\
20 & \text{if } 8 \leq t < 16 \\
30 & \text{if } 16 \leq t < 20 \\
30 \left(e^{-4(t - 20)}\right) & \text{if } 20 \leq t
\end{cases}
\]
3. (15 pts. total) From the following dynamic response data, determine a First-Order plus Time-Delay transfer function model for the response to changes in the feed temperature. Show your work on the figure to receive full credit.
4. **(20 pts. total)** You must develop control systems to maintain the water supply system at the specified temperature using the following schematics. Remember to show your direction of information flow.

a. **(10 pts.)** Propose a **feedback control** scheme on the following diagram to maintain the product temperature.

![Feedback Control Diagram](image)

b. **(10 pts.)** Propose a **feedforward control** scheme (separate from part a) on the following diagram to maintain the product temperature despite changes in the cooling water temperature.

![Feedforward Control Diagram](image)
5. (25 pts. total) You are given the following model for the product temperature $y(t)$ as a function of the feed temperature deviation $u(t)$.

a. (7 pts.) For the following differential equations:

$$\frac{d^2y}{dt^2}(t) + 4\frac{dy}{dt}(t) + 20y(t) = \frac{du}{dt}(t) + 5u(t)$$

show that for the initial conditions $y(t = 0) = \frac{dy}{dt}(t = 0) = u(t = 0) = 0$, the following transfer function relationship holds:

$$y(s) = \left(\frac{s + 5}{s^2 + 4s + 20}\right) u(s)$$

b. (8 pts.) What are the gain, poles, and zeros of this transfer function?

c. (5 pts.) Given that you implement a step increase of magnitude 8 at time zero in the input $u(t)$, what is the initial value for $y$? Use the Initial Value Theorem to find $y(t = 0)$. Show your work.

d. (5 pts.) Given that you implement a step increase of magnitude 8 at time zero in the input $u(t)$, what is the ultimate response? Use the Final Value Theorem to find $y(t = \infty)$. Show your work.
2. (6 pts. total) Your old roommate dropped out of USC and started a moderately successful agricultural business, 420GrowHaus. He calls you up one day because the plants in his greenhouse are getting totally fried by the new steam heating system designed by a grad from the upstate. He would like you to come in and consult on how to improve the control system. In the diagram below:

a. (2 pts.) Propose a feedforward control system to help regulate the temperature in zone 2, assuming the steam supply manifold pressure fluctuates.
b. (2 pts.) Propose a feedback control system to regulate the temperature in zone 1.
c. (2 pts.) Propose a cascade feedback control system to regulate the temperature in zone 3.
2. (4 pts. total) Assume that you can adjust valves $v_1$, $v_2$, and $v_3$. Measurements $T_1$, $T_2$, and $T_3$ are available, with the following information.

- Increasing valve 1 ($v_1$) by 5% increases $T_1$ by 4° and increases $T_2$ by 2°
- Decreasing valve 2 ($v_2$) by 10% decreases $T_1$ by 7°, decreases $T_2$ by 1°, and decreases $T_3$ by 2°
- Increasing valve 3 ($v_3$) by 3% increases $T_3$ by 4° and increases $T_2$ by 1°

Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

**BONUS:** Assuming zone 1 is 5° too high, zone 2 is 3° too low and zone 3 is 7° too high, what problem would you solve? How would you solve it?
1. **(2 points total)** For the following matrix, determine the eigenvalues. Show all work!

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
2 & 1 & 1 \\
\end{bmatrix}
\]

2. **(8 points total)** Your buddy went back to his native country and start the Canadian Cock Company (C³), a chicken processing facility. He patented the novel EXCREMENT (EXtrusion of Chicken Rapidly Ensuring Meat is Edible -N- Tasty) process for nugget production. However, the feed stream to the extruder is too hot. He thinks that by adding a holding tank in parallel to the existing one, he may be able to use the freezing cold weather to chill the product stream. He asks for your help. Assume the following:

- The volume of the inner tank is constant at \( V_1 \) (L)
- The volume of the outer tank is constant at \( V_2 \) (L)
- The temperature of the inner tank is \( T_1(t) \)
- The temperature of the outer tank is \( T_2(t) \)
- Feed/product flow enters and leaves the first tank at a constant flow rate \( F_1 \)
- A separate recycle flow goes between the two tanks at a constant flow rate \( F_2 \)
- The temperature entering varies, \( T_0(t) \)
- The ambient temperature is \( T_A(t) \)
- The first tank is insulated without any loss of energy to the environment
- The second tank loses energy to the environment at the rate \( h_A(T_2(t) - T_A(t)) \)
- The tanks are well-mixed. Assume constant physical properties (\( \rho, C_p \))
- Deviation temperature of the tanks are measured: \( y_1(t) = T_1(t) - T_{1ss} \) and \( y_2(t) = T_2(t) - T_{2ss} \)
- Inputs are deviation, \( u_1(t) = T_0(t) - T_{oss} \) and \( u_2(t) = T_A(t) - T_{Ass} \)

a). **(4 points)** Develop a dynamic linear model for the system.
b). **(4 points)** Put the model in deviation form and then develop the state space representation. Clearly show \( A, B, C, D, x(t), \) and \( y(t) \).

![Diagram of the system](image-url)
1. (25 pts. total) You just started your internship at the local subsidiary of GameCock Co., at PolyCock Inc. working the new polymerization facility. You are working on a simple CSTR. You must develop a first-principles dynamic model for the system. The schematic is below.

\[ u(t) = (C_{Ao}(t) - C_{Aoss}) \]

A $\rightarrow_{r} B \rightarrow_{r_s} W$

### Assume the following:
- The system is well-mixed with constant volume \( V \)
- The volumetric flow rate into and out of the reactor is constant: \( F \)
- You have constant physical properties
- Two species of interest are present, \( A \) and \( B \)
- The volumetric rate of conversion of \( A \) to \( B \) is given by \( r(t) = k C_B(t) (C_A(t))^2 \)
- There is a side reaction of product \( B \), volumetric rate \( r_s(t) = k_s C_B(t) \)
- There is no volume change due to reaction
- Feed is dilute (changing \( C_{Ao}(t) \) has no effect on \( F \))
- The system operate at steady state \( C_{Ass} \) and \( C_{Bss} \)
- No \( B \) is fed into the system
- \( A \) enters at concentration \( C_{Ao}(t) \)
- The following is the input: \( u(t) = (C_{Ao}(t) - C_{Aoss}) \)
- The following are outputs: \( y_1(t) = (C_A(t) - C_{Ass}) \quad y_2(t) = (C_B(t) - C_{Bss}) \)

\[ r(t) = k C_B(t) (C_A(t))^2 \]
\[ r_s(t) = k_s C_B(t) \]

a. (12 pts.) Develop a dynamic model for this system.

b. (13 pts.) If necessary, linearize your model from part a and put it in state space form, remembering to clearly identify \( x(t), u(t), A, B, C, \) and \( D \).
2. (15 pts. total) For the reaction system, the input concentration follows a specific profile for a given polymer recipe.
   a. (6 pts.) What is a simple function of time that represents \( u(t) \)?
   b. (9 pts.) What is the Laplace transform of the following function of time, \( u(s) \)?

\[
\begin{align*}
  u(t) &= \begin{cases} 
  0 & t \leq 10 \\
  -4 & 10 \leq t < 20 \\
  -4e^{-2(t-20)} & 20 \leq t 
  \end{cases}
\end{align*}
\]
3. (15 pts. total) From the following dynamic response data, determine a **First-Order plus Time-Delay transfer function model** for the response to changes in the feed temperature. Show your work on the figure to receive full credit.

![Graph showing dynamic response data with feed concentration on the x-axis and product concentration on the y-axis.]

4. (15 pts. total) You must develop control systems to maintain product concentration at the specified value using the following schematics. Remember to show your direction of information flow and label the type of controller to receive full credit.

a. (8 pts.) Propose a **cascade feedback control** scheme on the following diagram to maintain the product concentration.

![Diagram showing a cascade feedback control system with inputs and outputs labeled.]

b. (7 pts.) Propose a **feedforward control** scheme (separate from part a) on the above diagram to maintain reactor temperature despite changes in the feed water temperature.

![Diagram showing a feedforward control system with inputs and outputs labeled.]
5. **(30 pts. total)** You are given the following model for the product temperature $y(t)$ as a function of the feed temperature deviation $u(t)$.

a. **(6 pts.)** For the following differential equations:

\[
2 \frac{d^2 y(t)}{dt^2} + 18 \frac{dy(t)}{dt} + 36y(t) = 2 \frac{du(t)}{dt} - 4u(t)
\]

show that for the initial conditions $y(t = 0) = \frac{dy}{dt}(t = 0) = u(t = 0) = 0$, the following transfer function relationship holds:

\[
y(s) = \left( \frac{s - 2}{(s + 3)(s + 6)} \right) u(s)
\]

b. **(8 pts.)** What are the gain, poles, and zeros of this transfer function?

c. **(4 pts.)** Given that you implement an **impulse of magnitude 3** at time zero in the input $u(t)$, what is the initial value for $y$? Use the Initial Value Theorem to find $y(t = 0)$. Show your work.

d. **(4 pts.)** Given that you implement an **impulse of magnitude 3** at time zero in the input $u(t)$, what is the ultimate response? Use the Final Value Theorem to find $y(t = \infty)$. Show your work.

e. **(8 pts.)** Given that you implement an **impulse of magnitude 3** at time zero in the input $u(t)$, what is the analytical response? Show your work.
2. (5 pts. total) You just started your new job a local health and fitness establishment, *HealthCountry*. Your boss says that the hot water pre-heat system for the giant sauna lead to many issues with lack of steam in the steam rooms. He would like you to come in and consult on how to improve the control system. In the diagram below:

a. (2 pts.) Propose a feedforward control system to help regulate the temperature in the third mixing tank, assuming that the cold water supply temperature varies considerably.

b. (3 pts.) At the same time, propose a cascade feedback control system to regulate the product temperature.
2. **(5 pts. total)** Assume that you can adjust valves \( v_1 \), \( v_2 \), and \( v_3 \). Measurements \( T_1 \), \( T_2 \), and \( T_3 \) are available, with the following information.

- Increasing valve 1 \( (v_1) \) by 10\% increases \( T_1 \) by 7\(^\circ\), increases \( T_2 \) by 4\(^\circ\), and increases \( T_3 \) by 3\(^\circ\),
- Increasing valve 2 \( (v_2) \) by 5\% decreases \( T_1 \) by 1\(^\circ\), increases \( T_2 \) by 5\(^\circ\), and increases \( T_3 \) by 2\(^\circ\),
- Decreasing valve 3 \( (v_3) \) by 10\% increases \( T_3 \) by 8\(^\circ\)

a. **(4 pts.)** Develop a model in the form \( \Delta y = K \Delta u \) and clearly identify \( \Delta y \), \( \Delta u \), and \( K \).

b. **(1 pt.)** Assuming \( T_1 \) is 20\(^\circ\) too low, \( T_2 \) is 22\(^\circ\) too high and \( T_3 \) is 2\(^\circ\) too low, what problem would you solve?

**BONUS:** What change in the valves do you implement?
1. (2 points total) For the following matrix, determine the eigenvalues. Show all work!

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
1 & 3 & 1 \\
2 & 1 & 0 
\end{bmatrix}
\]

2. (8 points total) You are pre-heating a liquid using a furnace. Liquid enters at the bottom of the heating tank inside the furnace. A blower feeds air to the furnace where fuel oil is burned. You know the following about the system:

- The volume of liquid in the inner heating tank is constant: \( V_1 \) (L)
- The volume of the furnace combustion chamber is constant: \( V_2 \) (L)
- The temperature of the liquid being heated is \( T_1(t) \)
- The temperature of the furnace is \( T_2(t) \)
- Feed liquid enters and leaves the system at a constant flow rate \( F_1 \) (L/min)
- The volume of burnt oil is negligible.
- The blower feeds air into the furnace at a constant flow rate \( F_2 \) (L/min)
- The blower air is at the ambient value, \( T_A(t) \)
- The energy generated by combustion in the furnace is adjustable: \( Q(t) \) (J/min)
- The temperature of the entering liquid varies: \( T_o(t) \)
- The furnace is poorly insulated and loses energy to the environment at the rate \( Q_B(t) = h_2 A_2 (T_2(t) - T_A(t)) \) (J/min)
- Heat is transferred from the furnace to the liquid at a rate of \( Q_A(t) = h A (T_2(t) - T_1(t)) \) (J/min)
- The volumes are well-mixed.
- Assume constant physical properties for the liquid \((\rho, C_p)\)
- Assume constant physical properties for air in the furnace \((\rho_2, C_{p2})\)
- Deviation temperature of the product is measured: \( y_1(t) = T_1(t) - T_{1ss} \)
- Inputs are deviation, \( u_1(t) = T_o(t) - T_{oss} \) and \( u_2(t) = T_A(t) - T_{Ass} \) and \( u_3(t) = Q(t) - Q_{ss} \)

a). (4 points) Develop a dynamic linear model for the system.

b). (4 points) Put the model in deviation form and then develop the state space representation. Clearly show \( A, B, C, D, x(t), \) and \( y(t) \).
\[ Q_A(t) = h A (T_2(t) - T_1(t)) \]

\[ Q_B(t) = h_3 A_3 (T_2(t) - T_4(t)) \]

\[ y_1(t) = T_1(t) - T_{1ss} \]

\[ u_3(t) = Q(t) - Q_{ss} \]

\[ T_A(t) \]

\[ F_2 \]

\[ V_1 \]

\[ V_2 \]

\[ \rho C_p \]

\[ \rho C_{p2} \]

\[ F_1 \]

\[ T_e(t) \]
1. (25 pts. total) You have landed a great job at Lou’s Lowcountry Lobster Lab working the feeding facility growing plankton in a CSTR type reactor. Lou wants you to develop a first-principles dynamic model for the system. The schematic is below.

Assume the following:
- The system is well-mixed with constant volume \( V \)
- The volumetric flow rate into and out of the reactor is constant: \( 2F \)
- The volumetric flow rate of a side-stream bypass around the reactor is constant: \( F \)
- You can assume constant physical properties
- One species of plankton is present at concentration \( C_F(t) \) in the reactor
- The volumetric growth rate of plankton is given by \( r(t) = k T(t) / (C_F(t))^2 = k T(t) (C_F(t))^{-2} \)
- There is no volume change due to growth
- The system operate at steady state \( C_{Fss}, T_{ss}, \) and \( T_{Pss} \)
- No plankton is fed into the system
- The input temperature is manipulated: \( u_1(t) = (T_o(t) - T_{oss}) \)
- Additional steam heating is available to the reactor: \( u_2(t) = (Q(t) - Q_{ss}) \)
- The deviation plankton concentration in the reactor is measured: \( y_1(t) = (C_F(t) - C_{Fss}) \)
- The deviation product stream temperature is measured: \( y_2(t) = (T_P(t) - T_{Pss}) \)

a. (12 pts.) Develop a dynamic model for this system.
b. (13 pts.) If necessary, linearize your model from part a and put it in state space form, remembering to clearly identify \( x(t), u(t), A, B, C, \) and \( D \).
2. **(16 pts. total)** For the reaction system, the input concentration follows a specific profile for a given polymer recipe.
   a. **(8 pts.)** What is a simple function of time that represents \( u(t) \)?
   b. **(8 pts.)** What is the Laplace transform of the following function of time, \( u(s) \)?

\[
\begin{align*}
u(t) &= \begin{cases} 
0 & t \leq 4 \\
6 + 4e^{-3(t-4)} & 4 \leq t < 8 \\
6 + 4e^{-3(t-4)} - 2(t-8) & 8 \leq t < 10 \\
2 & 10 \leq t
\end{cases}
\end{align*}
\]

3. **(14 pts. total)** Assume that you can adjust valves \( v_1 \) and \( v_2 \). Measurements \( T_1 \) and \( T_2 \) are available, with the following information.

- Increasing valve 1 \( (v_1) \) by 20\% decreases \( T_1 \) by 12\° and decreases \( T_2 \) by 4\°
- Increasing valve 2 \( (v_2) \) by 5\% increases \( T_1 \) by 2\° and increases \( T_2 \) by 3\°

a. **(10 pts.)** Develop a model in the form \( \Delta y = K \Delta u \) and clearly identify \( \Delta y \), \( \Delta u \), and \( K \).

b. **(4 pt.)** Assuming \( T_1 \) is 10\° too high and \( T_2 \) is 9\° too low, what problem would you solve?
4. (15 pts. total) You must develop control systems to maintain product temperature at the specified value using the following schematic. Remember to show your direction of information flow and label the type of controller to receive full credit.

a. (8 pts.) Propose a feedforward control scheme on the diagram to maintain the reactor temperature despite changes in the steam supply pressure.

b. (7 pts.) Propose a cascade feedback control scheme on the diagram to maintain the product temperature.
5. **(30 pts. total)** You are given the following model for the product temperature $y(t)$ as a function of the feed temperature deviation $u(t)$.

a. **(6 pts.)** For the following differential equation:

$$8\frac{dy}{dt}(t) + 4y(t) = 2\frac{dD}{dt}(t) - 6D(t) + 12u(t)$$

*show* that for zero initial conditions $y(t = 0) = D(t = 0) = u(t = 0) = 0$, the following transfer function relationships hold:

$$y(s) = \frac{-\frac{3}{2} \left(-\frac{1}{3}s + 1\right)}{(2s + 1)} D(s) + \frac{3}{(2s + 1)} u(s)$$

b. **(6 pts.)** What are the gain, poles, and zeros of the transfer function relating $D(s)$ to $y(s)$?

c. **(5 pts.)** Given that you implement an impulse of magnitude 7 at time $t = 1$ (not at 0) in the disturbance $D(t)$, what is the initial value for $y$? Use the Initial Value Theorem to find $y(t = 0)$. You may assume $u(t) = 0$. Show your work.

d. **(5 pts.)** Given that you implement an impulse of magnitude 7 at time $t = 1$ (not at 0) in the disturbance $D(t)$, what is the ultimate response? Use the Final Value Theorem to find $y(t = \infty)$. You may assume $u(t) = 0$. Show your work.

e. **(8 pts.)** Given that you implement a step of magnitude 12 at at time $t = 1$ (not at 0) in the disturbance $D(t)$, what is the analytic response? You may assume $u(t) = 0$. Show your work.

**BONUS:** Sketch your response from part e.
1. **(7 pts. total)** Your just started at GameCockAir, building the new polymer composite *DazeLiner*. Your boss says that the polymer pre-heat system has some problems. He would like you to consult on how to improve the control system. Liquid polymer is fed to a heating tank where steam coils heat the polymer to a desired temperature. Assume that the polymer is well-mixed in the tank.

a. **(3 pts.)** Propose a feedforward control system to help regulate the level in the tank despite unknown changes in the outlet flow rate. At the same time, propose a simple feedback control method to regulate the temperature of the polymer in the tank.
b. (4 pts.) Propose a cascade control system to regulate the product temperature $T_p$. At the same time, propose a feedforward scheme to maintain tank level despite changes in the inlet polymer feed flow due to pressure changes.

![Diagram](image)

2. (3 pts. total) The following steady state multivariable model in the form $\Delta y = K \Delta u$ was validated for a similar polymer heating system. Identify the matrix $K$ and determine the eigenvalues of $K$ to the best of your ability.

\[
\begin{align*}
\Delta y_1 &= 2\Delta u_1 - \Delta u_2 \\
\Delta y_2 &= \Delta u_2 + 2\Delta u_3 \\
\Delta y_3 &= -\Delta u_1 - \Delta u_2 - 3\Delta u_3
\end{align*}
\]
1. **(8 points total)** You are in charge of the waste treatment reactor system. The EPA limits your emissions, so a process stream is passed through the reactor before being released into a nearby river. The reaction rate is very temperature dependent, so a complex control system is being developed to maintain the temperatures at desired levels. The reactor has a cooling jacket and steam coils. You are expected to develop a dynamic model for your reaction system. You know the following about the system:

- The volume of liquid in reactor is constant: $V \ (L)$
- The volume of cooling jacket is constant: $V_j \ (L)$
- The concentration of the entering stream contaminant is $C_{Ao}(t)$
- The temperature of the entering stream is $T_o(t)$
- The temperature of feed cooling water is $T_{jo}(t)$
- The process stream enters and leaves at a constant flow rate $F \ (L/min)$
- The cooling water enters and leaves at a constant flow rate $F_j \ (L/min)$
- Steam coils can be used to heat the reactor: $Q(t) \ (J/min)$
- Heat is transferred from the reactor to the cooling jacket at a rate of $Q_A(t) = hA(T(t) - T_j(t)) \ (J/min)$
- The reactor and jacket are assumed are well-mixed.
- Assume constant physical properties for the liquid ($\rho, C_p$)
- Assume constant physical properties for the cooling water ($\rho_j, C_j$)
- The volumetric reaction rate is $r(t) = k_1 + k_2T(t) \ (mol/L \ min)$
- Deviation temperature of the product is measured: $y_1(t) = T(t) - T_{ss}$
- Deviation temperature of the jacket is measured: $y_2(t) = T_j(t) - T_{jss}$
- Inputs are deviation, $u_1(t) = T_o(t) - T_{oss}$ and $u_2(t) = T_{jo}(t) - T_{joss}$ and $u_3(t) = Q(t) - Q_{ss}$ and $u_4(t) = C_{Ao}(t) - C_{Aoss}$

a). **(4 points)** Develop a dynamic linear model for the system.

b). **(4 points)** Put the model in deviation form and then develop the state space representation. Clearly show $A, B, C, D, x(t)$, and $y(t)$. 

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b). (2 points) Now, assume that you can adjust the steam flow rate $F_1$ and the cooling water flow rate $F_2$. Measurements for $T$ and $T_j$ are available, with the following information.

- Increasing the steam flow ($F_1$) by 10% increases $T$ by 8°, and increases $T_j$ by 4°,
- Decreasing the cooling water flow ($F_2$) by 4% increases $T$ by 2°, and increases $T_j$ by 5°

Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

**BONUS:** Assuming $T$ is 15° too low and $T_j$ is 12° too high, what problem would you solve?
1. (20 pts. total) Your uncle managed to get you hired on at Slick Sammy’s Solar Systems Sales and Service. Slick Sammy landed a sweet government contract to build and maintain a solar thermal power generation plant. The plant uses hundreds of heliostats (mirrors) to reflect sun onto a tower where molten salt is heated. The heated salt is stored in a large insulated tank, allowing for steam to be produced even at night. A congressional oversight committee is investigating the $250 million investment, so Slick Sammy needs you to do some engineering analysis to model the system.

Assume the following:

- The solar collection tank is well-mixed with constant volume $V_1$
- The salt storage tank is well-mixed with constant volume $V_2$
- A recirculation flow moves salt between the two tanks at a constant flow rate $F$
- The rate of energy flowing into the collection tank is $Q_A(t)$
- $Q_A$ depends on the ambient temperature, $T_A(t)$, as $Q_A(t) = kT_A(t)^4$
- $Q(t)$ is the rate which energy is removed from the storage tank to make steam
- You can assume constant physical properties
- The temperature in the collection tank is $T_1(t)$
- The temperature in the storage tank is $T_2(t)$
- The system operate at steady state $T_{1ss}$, $T_{2ss}$, and $T_{Ass}$
- The ambient temperature changes with time: $u_1(t) = (T_A(t) - T_{Ass})$
- Energy is removed to make steam: $u_2(t) = (Q(t) - Q_{ss})$
- The deviation temperature in the storage tank is measured: $y_1(t) = (T_2(t) - T_{2ss})$
- The difference between deviation temperatures is measured: $y_2(t) = (T_1(t) - T_{1ss}) - (T_2(t) - T_{2ss})$

a. (10 pts.) Develop a dynamic model for this system.
b. (10 pts.) If necessary, linearize your model from part a and put it in state space form, remembering to clearly identify $x(t)$, $u(t)$, $A$, $B$, $C$, and $D$.

2. (16 pts. total) For a given day, the ambient temperature follows the following function of time:
   a. (8 pts.) What is a simple function of time that represents $u(t)$?
   b. (8 pts.) What is the Laplace transform of the following function of time, $u(s)$?

   $u(t) = \begin{cases} 
   0 & t \leq 20,000 \\
   0.004(t - 20,000) & 20,000 \leq t < 25,000 \\
   20 & 25,000 \leq t < 35,000 \\
   15 & 35,000 \leq t < 45,000 \\
   15e^{-0.0005(t-45,000)} & 45,000 \leq t 
   \end{cases}$

3. (16 pts. total) Determine a first-order-plus-time-delay transfer function model from the following data. Show how you determined your numerical values on the graph below to receive full credit.
4. **(14 pts. total)** You must develop control systems for the steam boiler using the following schematic. Molten salt is used to heat water in a boiler, generating steam for use in power generation. Remember to show your direction of information flow and label the type of controller to receive full credit.

a. **(8 pts.)** Propose a **feedback control** scheme on the diagram below to control the height of the tank $h$. Simultaneously, propose a **feedforward control** scheme on the diagram below to maintain boiler temperature when there are significant changes in feed water supply temperature.

b. **(6 pts.)** Propose a **cascade feedback control** scheme on the diagram below to maintain the boiler temperature.
5. **(10 pts. total)** Assume that you can adjust valves \(v_1\) and \(v_2\). Measurements \(h\) and \(T\) are available, with the following information.

- Increasing valve 1 (\(v_1\)) by 3\% increases \(h\) by 2 cm and increases \(T\) by 4\°
- Increasing valve 2 (\(v_2\)) by 5\% increases \(h\) by 1 cm and decreases \(T\) by 6\°

Develop a model in the form \(\Delta y = K \Delta u\) and clearly identify \(\Delta y\), \(\Delta u\), and \(K\).

6. **(24 pts. total)** You are given the following model for the salt temperature deviation \(y(t)\) as a function of the ambient temperature deviation \(u(t)\).

a. **(4 pts.)** For the following differential equation:

\[
\frac{1}{8} \frac{d^2 y}{dt^2}(t) + \frac{3}{4} \frac{dy}{dt}(t) + y(t) = 3u(t) - \frac{du}{dt}(t)
\]

*show* that for zero initial conditions \(y(t = 0) = \frac{dy}{dt}(t = 0) = u(t = 0) = 0\), the following transfer function relationships hold:

\[
y(s) = \frac{-8(s - 3)}{(s+2)(s+4)} u(s)
\]

b. **(4 pts.)** What are the gain, poles, and zeros of the transfer function relating \(u(s)\) to \(y(s)\)?

c. **(4 pts.)** Given that the ambient temperature experiences a step of magnitude 10 at time \(t = 0\), what is the initial value for \(y\)? Use the Initial Value Theorem to find \(y(t = 0)\). Show your work.

d. **(4 pts.)** If the ambient temperature experiences a step of magnitude 10 at time \(t = 0\), what is the eventual steady state value? Use the Final Value Theorem to find \(y(t = \infty)\). Show your work.

e. **(8 pts.)** Given that the ambient temperature experiences a step of magnitude 10 at time \(t = 0\), what is the response \(y(t)\)? Show your work.

**BONUS:** Sketch your response from part e.
1. (6 pts. total) Congratulations! You were hired at BioPolyCo, a small biotech startup that produces specialty polymers using biological organisms. The core process is a bio reactor for growing cells. The bio reactor is effectively a continuously stirred tank reactor with a liquid jacket to help regulate reactor temperature. Liquid nutrients are fed to the reactor to increase the organism growth rate. Product X is collected from the liquid product stream where the concentration of Product X is measured. To maintain the growth rate, the headspace is filled with an elevated level of oxygen. Draw three separate control schemes on the reactor below.

a. (2 pts.) Propose a feedforward control system to maintain the jacket temperature, assuming that the hot water feed exhibits significant temperature variation.

b. (2 pts.) Simultaneously propose a feedback control scheme with the goal of maintaining the concentration of oxygen fed to the bio reactor.

c. (2 pts.) Include a cascade control system to regulate the product polymer concentration, assuming that the liquid feed rate influences the growth rate.
2. (4 pts. total) Assume that you can adjust the jacket temperature $T_j$, the headspace oxygen concentration $C_{O2}$, and the mixing rate $M$. Measurements are available for the reactor temperature $T$, the intermediate species concentration $C_M$, and the product concentration $C_X$.

- Increasing the jacket temperature $T_j$ by $4^\circ$ results in an increase in $T$ of $2^\circ$, a decrease of $6\%$ in $C_M$, and an increase of $3\%$ in $C_X$

- Increasing the headspace oxygen concentration $C_{O2}$ by $11\%$ results in an increase of $7\%$ in $C_M$ and an increase of $1\%$ in $C_X$

- Increasing the mixing rate $M$ by 20 RPM results in an increase in $T$ of $0.3^\circ$, a decrease of $2.4\%$ in $C_M$, and a decrease of $4.3\%$ in $C_X$

a. (3 pts.) Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

b. (1 pt.) What problem would you solve, assuming $T$ is $10^\circ$ too low, $C_M$ is $12\%$ too high, and $C_X$ is $13\%$ too low?
1. **(10 points total)** You just started at the Gamecock Nucluar working with one of the auxiliary reactors, reactor $S$. The reaction to destroy species $S$ takes place in a cylindrical CSTR reactor with a thick solid lead outer covering. The covering is wired to heat the reactor. The reaction rate is very temperature dependent, so a control system is being developed to maintain the temperatures at desired levels. You are expected to develop a dynamic model for your reaction system. You know the following about the system:

- The volume of liquid in reactor is constant is $V \ (L)$
- The volume of lead lining around the reactor is $V_j \ (L)$
- The concentration of the entering stream is $C_{So}(t)$
- The temperature of the entering stream is $T_{So}(t)$
- The process stream enters and leaves at a constant flow rate $F \ (L/min)$
- Adjustable electric heating is used to heat the jacket: $Q(t)\ (J/min)$
- The reactor is assumed are well-mixed at temperature $T_S(t)$ and concentration $C_S(t)$.
- Assume constant physical properties for the liquid ($\rho, C_p$)
- The jacket temperature is assumed to be evenly distributed at temperature $T(t)$
- Assume constant physical properties for the jacket ($\rho_j, C_{pj}$)
- The jacket and reactor are assumed well-insulated, no loss to the environment.
- Heat is transferred from the jacket to the reactor at a rate of $Q_H(t) = hA(T(t) - T_S(t)) \ (J/min)$
- The volumetric reaction rate for consumption of $S$ is $r(t) = k_1 + k_2 T_S(t) \ (mol/L \ min)$
- The heat of reaction is $-\Delta H \ (J/mol)$
- Deviation temperature of the product is measured: $y_1(t) = T_S(t) - T_{Sss}$
- Inputs are deviation, $u_1(t) = T_{So}(t) - T_{So ss}$ and $u_2(t) = C_{So}(t) - C_{So ss}$ and $u_3(t) = Q(t) - Q_{ss}$

a). **(5 points)** Develop a dynamic linear model for the system.
b). **(5 points)** Put the model in deviation form and then develop the state space representation. Clearly show $A$, $B$, $C$, $D$, $x(t)$, and $y(t)$.

**BONUS:** Determine an expression for $V_j$ in terms of $d_0$, $d_1$, and $d_2$. Simplify your answer.
\[ y(t) = T_S(t) - T_{ss} \]

\[ r(t) = k_1 + k_2 T_S(t) \]

\[ Q_H(t) = hA(T(t) - T_S(t)) \]

\[ V \quad T_S(t) \quad C_S(t) \]

\[ T(t) \]

\[ V_j \]

\[ d_0 \]

\[ d_1 \]

\[ d_2 \]
1. (12 pts. total) Congratulations! You aced your interview and landed a position at Big Bobby’s Beer, Booze, and Biofuels located deep in the hills of backwoods Tennessee. You are working in the heart of their facility, the glucose fermentation reactor system. Here, glucose is converted to ethanol, which is later separated out using a novel distillation process. Your boss, Jackie Daniella, wants you to work on improving control of the fermentation system. You must develop control systems for the fermentor system using the following schematic. Glucose is fed to a mixing tank with a bypass stream in order to control the temperature and concentration of the reactor feed stream. Both of these values influence the fermentation reaction. The mixer outlet stream is fed to the reactor, where agitation occurs to aid the fermentation reaction. Remember to show your direction of information flow and label the type of controller to receive full credit.

a. (8 pts.) Propose a **feedforward control** scheme on the diagram below to maintain the mixer tank temperature when there are periodic variations in steam supply pressure. Simultaneously, propose a **feedback control** scheme on the diagram below to control the product concentration.
b. **(4 pts.)** Propose a *cascade feedback control* scheme on the diagram below to maintain product concentration.

![Diagram](image)

2. **(12 pts. total)** Assume that you can adjust the feed concentration $C_0$ and the steam flow $F$. Measurements $C_1$ and $T$ are available, with the following information.
   - Increasing concentration ($C_0$) by 3% increases $C_1$ by 6% and increases $T$ by 0°
   - Increasing steam flow ($F$) by 7% decreases $C_1$ by 1% and increases $T$ by 4°

a. **(8 pts.)** Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

b. **(4 pts.)** Assuming $C_1$ is 8% too low and $T$ is 9° too high, what problem would you solve? How would you solve it?
3. (12 pts. total) For a given product, the inlet concentration follows the following function of time. Note, $e^{-3t} \approx 0$ for $t > 4$.
   a. (6 pts.) What is a simple function of time that represents $u(t)$?
   b. (6 pts.) What is the Laplace transform of the following function of time, $u(s)$?

\[
u(t) = \begin{cases} 
0 & t \leq 2 \\
4 - 4e^{-3(t-2)} & 2 \leq t < 6 \\
\approx 6 & 6 \leq t < 12 \\
\approx 6 - 3(t - 12) & 12 \leq t < 14 \\
\approx 0 & 14 \leq t
\end{cases}
\]

4. (16 pts. total) Determine a first-order-plus-time-delay transfer function model from the following data. Show how you determined your numerical values on the graph below to receive full credit.
5. **(24 pts. total)** You are given the following model for the product concentration deviation $y(t)$ as a function of the feed concentration deviation $u(t)$.

a. **(4 pts.)** For the following differential equation:

\[
\frac{1}{9} \frac{d^2 y}{dt^2}(t) + \frac{dy}{dt}(t) + 2y(t) = 4u(t) + \frac{du}{dt}(t)
\]

show that for zero initial conditions $y(t = 0) = \frac{dy}{dt}(t = 0) = u(t = 0) = 0$, the following transfer function relationships hold:

\[
y(s) = \frac{9 (s + 4)}{(s + 3)(s + 6)} u(s)
\]

b. **(4 pts.)** What are the gain, poles, and zeros of the transfer function relating $u(s)$ to $y(s)$?

c. **(4 pts.)** Given that the feed concentration experiences a step of magnitude -3 at time $t = 0$, what is the initial value for $y$? Use the Initial Value Theorem to find $y(t = 0)$. Show your work.

d. **(4 pts.)** If the feed concentration experiences a step of magnitude -3 at time $t = 0$, what is the eventual steady state value? Use the Final Value Theorem to find $y(t = \infty)$. Show your work.

e. **(8 pts.)** Given that the feed concentration experiences a step of magnitude -3 at time $t = 0$, what is the response $y(t)$? Show your work.
6. **(24 pts. total)** The model you were provided does not represent the real system very accurately. It is proposed that you should do some engineering analysis and develop a detailed dynamic model of the system.

![System Diagram]

Assume the following:
- The mixing tank is well-mixed with constant volume $V_1 \ (L)$
- The reactor is well-mixed with constant volume $V_2 \ (L)$
- A glucose flow of concentration $C_{G0}(t) \ \left(\frac{mol}{L}\right)$ enters at a constant flow rate $F \ \left(\frac{L}{min}\right)$
- The flow rate entering the mixer is $\gamma F \ \left(\frac{L}{min}\right)$
- The bypass flows past the mixer with flow rate of $(1 - \gamma)F \ \left(\frac{L}{min}\right)$.
- The concentration entering the reactor is $C_{GF}(t) \ \left(\frac{mol}{L}\right)$
- The concentration in the mixer is $C_{G1}(t) \ \left(\frac{mol}{L}\right)$
- The concentration in the reactor is $C_{G2}(t) \ \left(\frac{mol}{L}\right)$
- There is no accumulation at the mixing point.
- You can assume constant physical properties.
- Mixing tank and reactor are assumed constant temperature.
- The mixing rate in the reactor is $M(t) \ \ (rpm)$
- The volumetric reaction rate for conversion of glucose is $r(t) = k \ M(t) \ (C_{G2}(t))^{2.3} \ \left(\frac{mol}{L \ min}\right)$
- The system operates at steady states $C_{G1ss}, C_{GFss},$ and $C_{G2ss}$
- The feed concentration changes with time: $u_1(t) = (C_{G0}(t) - C_{G0ss})$
- The mixing rate may change: $u_2(t) = (M(t) - M_{ss})$
- The deviation concentration in the mixing tank is measured: $y_1(t) = (C_{G1}(t) - C_{G1ss})$
- The deviation concentration in the reactor is measured: $y_2(t) = (C_{G2}(t) - C_{G2ss})$

a. **(10 pts.)** Develop a dynamic model for this system.

b. **(14 pts.)** If necessary, linearize your model from part a and put it in state space form, remembering to clearly identify $x(t), u(t), A, B, C,$ and $D$.  

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1. (6 pts. total) Congratulations! You are a very special athlete. Your swimming ability is limited but you float very well. Your home country of Elbonia has put you in charge of the system of pools for the upcoming Elbonian Olympiad. The warm up pool, the diving well, and the swimming pool must be maintained at constant temperature, level, and chlorine level. The system uses gaseous chlorine from an external supply tank. For the schematic below, draw three separate control schemes.

a. (2 pts.) Propose a feedforward control system to reduce variation in the chlorine levels in the swimming pool, knowing that the chlorine supply pressure may drift with time.

b. (2 pts.) Develop a feedback control scheme to regulate the chlorine concentration in the diving well.

c. (2 pts.) Present a cascade control system to maintain the height in the warm up pool.
2. *(4 pts. total)* You suggest the Elbonian government invest in a new salt water system where electrolysis is used to generate two chlorine species. The test machine includes two different type of chlorinator cells plates: a perforated mesh plate (MP) and a solid plate (SP). Two different species are produced, hypochlorous acid (HClO) and sodium hypochlorite (NaClO). You know the following information:

- Increasing the duty cycle of the mesh plate $D_{MP}$ by 20% results in an increase of 13 ppm in $C_{HClO}$ and a **decrease** of 9 ppm in $C_{NaClO}$

- **Decreasing** the duty cycle of the solid plate $D_{SP}$ by 10% results in a **decrease** of 3 ppm in $C_{HClO}$ and an **increase** of 11 ppm in $C_{NaClO}$

a. *(3 pts.)* Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

b. *(1 pt.)* What problem would you solve, assuming $C_{HClO}$ is 44 ppm too low and $C_{NaClO}$ is 84 ppm too high? How would you solve the problem?
1. (10 points total) You just started working for GameCock Energy making LiPo batteries. They have developed a proprietary reactor to create the Poly Methyl Methacrylate (PMMA) used as the electrolyte in the cell. The reactor is heated by a large solid metal core. You are expected to develop a dynamic model for your reaction system. You know the following about the system:

- The volume of liquid in reactor is constant is \( V \) (\( L \))
- The volume of the solid reactor core is \( V_c \) (\( L \))
- The concentration of PMMA in the entering stream is \( C_{PPo}(t) \)
- The temperature of the entering stream is \( T_o(t) \)
- The process stream enters and leaves at a constant flow rate \( F \) (\( L/min \))
- Adjustable steam heating is used to heat the metal core: \( Q(t) \) (\( J/min \))
- The reactor is assumed well-mixed at temperature \( T(t) \) and concentration \( C_{PP}(t) \).

- Assume constant physical properties for the liquid in the reactor \((\rho, C_p)\)
- The core temperature is assumed to be evenly distributed at temperature \( T_c(t) \)
- Assume constant physical properties for the core \((\rho_c, C_{pc})\)
- Heat is transferred from the core to the reactor at a rate of \( Q_H(t) = h_c A_c (T_c(t) - T(t)) \) (\( J/min \))
- Heat is lost from the reactor to the surrounding environment at a rate of \( Q_C(t) = h A (T(t) - T_A) \) (\( J/min \))
- The ambient temperature \( T_A \) is constant.
- The volumetric reaction rate for production of PMMA is \( r(t) = kT(t) \) (\( mol/L \text{ min} \))
- The heat of reaction is \(-\Delta H \) (\( J/mol \))
- The difference between the two deviation temperatures is measured: \( y_1(t) = (T_c(t) - T_{css}) - (T(t) - T_{ss}) \)
- Deviation temperature of the product is measured: \( y_2(t) = T(t) - T_{ss} \)
- Inputs are deviation, \( u_1(t) = T_o(t) - T_{oss} \) and \( u_2(t) = C_{PPo}(t) - C_{PPass} \) and \( u_3(t) = Q(t) - Q_{ss} \)

a). (5 points) Develop a dynamic linear model for the system.
b). (5 points) Put the model in deviation form and then develop the state space representation. Clearly show \( A, B, C, D, x(t), \) and \( y(t) \).

**BONUS:** Determine an expression for \( V \) in terms of \( d_0, d_1, \) and \( d_2 \). Simplify your answer.
\[ y_1(t) = (T_c(t) - T_{es}) - (T(t) - T_{ss}) \]

\[ y_2(t) = T(t) - T_{ss} \]

\[ Q_C(t) = hA(T(t) - T_A) \]

\[ Q_H(t) = h_c A_c (T_c(t) - T(t)) \]

\[ V_c \quad T_c(t) \]

\[ r(t) = kT(t) \]

\[ d_1 \quad d_2 \quad d_0 \]
1. (6 pts. total) For the following function of time:

\[ u(t) = \begin{cases} 
0 & t < 200 \\
-30 \left(1 - e^{-0.03(t-200)}\right) & 200 \leq t < 400 \\
20 - 0.1(t - 400) + e^{-0.03(t-200)} & 400 \leq t < 800 \\
-20 + e^{-0.03(t-200)} & 800 \leq t < 900 \\
0 + e^{-0.03(t-200)} & 900 \leq t 
\end{cases} \]

A. (3 pts) Express \( u(t) \) as a sum of simple functions of time.
B. (3 pts) Find \( u(s) \), the Laplace transform of \( u(t) \).

2. (4 pts total) For the following process data:
A. (3 pts) Determine the gain, time constant, and time delay. Show on the graph how you determined these values to receive full credit.
B. (1 pt) What is the first-order-plus-time-delay transfer function for this system?
1. (12 pts. total) Hurrah! Harry H. Hamilton of Happy Harry’s Hot-n-Hairy Hurricanes has hired you to work in his large-scale cocktail mixer production facility where various fruit juices are combined then reacted in a proprietary process. Harry wants you to improve the control of the mixing and reaction system. You must develop control systems for the system using the following schematic. Two juice mixing / holding tanks feed the reactor where steam is used to adjust the temperature. Remember to show your direction of information flow and label the type of controller to receive full credit.

a. (4 pts.) Propose feedback control scheme on the diagram below to maintain the concentration in mixing tank number 2.

b. (4 pts.) Propose a feedforward control scheme on the diagram below to maintain the temperature in the reactor drastic changes in the steam supply pressure.

c. (4 pts.) Propose a cascade feedback control scheme on the diagram below to maintain concentration supplied to the reactor.
2. (12 pts. total) Assume that you can adjust the reactor feed concentration $C_B$ and the steam flow $F$. Measurements $C_P$ and $T$ are available, with the following information.

- Increasing concentration ($C_B$) by 12% increases $C_P$ by 7% and increases $T$ by $1^\circ$
- Decreasing steam flow ($F$) by 23% increases $C_P$ by 2% and decreases $T$ by $9^\circ$

a. (8 pts.) Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

b. (4 pts.) Assuming $C_P$ is 13% too high and $T$ is $17^\circ$ too low, what problem would you solve? How would you solve it?

3. (12 pts. total) For a given product, the inlet concentration follows the following function of time.

a. (6 pts.) What is a simple function of time that represents $u(t)$?

b. (6 pts.) What is the Laplace transform of the following function of time, $u(s)$?

\[
u(t) = \begin{cases} 
0 & t \leq 20 \\
250 & 20 \leq t < 60 \\
400 + 100e^{-0.2(t-60)} & 60 \leq t < 100 \\
\approx 400 - 8(t-100) & 100 \leq t < 150 \\
\approx 0 & 150 \leq t 
\end{cases}
\]
4. (16 pts. total) Determine a first-order-plus-time-delay transfer function model from the following data. Show how you determined your numerical values on the graph below to receive full credit.
5. **(24 pts. total)** You are given the following model for the product concentration deviation \( y(t) \) as a function of the feed concentration deviation \( u(t) \).

a. **(4 pts.)** For the following differential equations:

\[
\frac{dx_1}{dt}(t) = -9x_1(t) - 18x_2(t) + u(t) \\
\frac{dx_2}{dt}(t) = x_1(t) \\
y(t) = x_1(t) + 3x_2(t)
\]

use the Laplace transform to show that for the initial conditions \( x_1(t = 0) = x_2(t = 0) = 0 \), the following transfer function relationship holds between \( u \) and \( y \):

\[
y(s) = \frac{s + 3}{s^2 + 9s + 18} u(s)
\]

b. **(4 pts.)** What are the gain, poles, and zeros of the transfer function relating \( u(s) \) to \( y(s) \)?

c. **(4 pts.)** Given that the feed concentration experiences a impulse of magnitude 10 at time \( t = 6 \), what is the initial value for \( y \)? Use the Initial Value Theorem to find \( y(t = 0) \). Show your work.

d. **(4 pts.)** If the feed concentration experiences a impulse of magnitude 10 at time \( t = 6 \), what is the eventual steady state value? Use the Final Value Theorem to find \( y(t = \infty) \). Show your work.

e. **(8 pts.)** Given that the feed concentration experiences a impulse of magnitude 10 at time \( t = 6 \), what is the response \( y(t) \)? Show your work.
6. **(24 pts. total)** The model you were provided does not represent the real system very accurately. It is proposed that you should do some engineering analysis and develop a detailed dynamic model of the system. Assume the following:

- Mixing tanks are well-mixed with constant volumes $V_1$ (L) and $V_2$ (L)
- The reactor is well-mixed with constant volume $V$ (L)
- A flow of concentration $C_{B10}(t) \left( \frac{\text{mol}}{L} \right)$ enters mixing tank 1 at a constant flow rate $F_1 \left( \frac{L}{\text{min}} \right)$
- A flow of concentration $C_{B20}(t) \left( \frac{\text{mol}}{L} \right)$ enters mixing tank 2 at a constant flow rate $F_2 \left( \frac{L}{\text{min}} \right)$
- The flow rate entering and leaving the reactor is $F \left( \frac{L}{\text{min}} \right)$ where $F = F_1 + F_2$
- The concentration entering the reactor is $C_B(t) \left( \frac{\text{mol}}{L} \right)$
- The concentration in the reactor is $C_{B3}(t) \left( \frac{\text{mol}}{L} \right)$
- There is no accumulation at the mixing point.
- You can assume constant physical properties.
- Mixing tanks are assumed constant temperature.
- The reactor is at temperature $T(t)$ (C) with feed temperature $T_0(t)$
- Steam is used to heat the reactor at rate $Q(t) \left( \frac{\text{J}}{\text{min}} \right)$
- Assume constant density $\rho$ and heat capacity $C_p$
- The volumetric reaction rate for conversion of juice is $r(t) = k \cdot T(t)^{1.2} \cdot (C_{B3}(t))^{1.8} \left( \frac{\text{mol}}{L \cdot \text{min}} \right)$
- The heat of reaction is $-\Delta H$
- The system operates at steady states $C_{B1ss}$, $C_{B2ss}$, $C_{B3ss}$, and $T_{ss}$
- The mixer feed concentration changes with time: $u_1(t) = (C_{B10}(t) - C_{10ss})$
- The mixer feed concentration changes with time: $u_2(t) = (C_{B20}(t) - C_{20ss})$
- The reactor feed temperature changes with time: $u_3(t) = (T(t) - T_{ss})$
- The reactor steam input may change with time: $u_4(t) = (Q(t) - Q_{SS})$
- The deviation concentration in the mixing tank is measured: $y_1(t) = (C_{B3}(t) - C_{B3ss})$
- The deviation concentration in the reactor is measured: $y_2(t) = (T(t) - T_{ss})$

a. **(10 pts.)** Develop a dynamic model for this system.

b. **(14 pts.)** If necessary, linearize your model from part a and put it in state space form, remembering to clearly identify $x(t)$, $u(t)$, $A$, $B$, $C$, and $D$. 

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1. (6 pts. total) Slick Sammy’s Snowflake Shop in Southern Saskatoon suffered saturating showers and storms Saturday and Sunday, shuttering the store for several shifts, spawning severe sadness and sorrow. The blower system that makes snow had trouble getting back online. Your boss thinks a modern control system could help speed up the startup process. The main pumps supply the feed manifold supporting three separate areas. Three pre-coolers chill the supply water to near freezing before the water is sent to the single-speed blower system. For the schematic below, propose three separate control schemes that best corrects the problem presented.

a. (2 pts.) Propose a feedforward control system to limit changes in the temperature of flow to zone one given large transients in the cooling water feed temperature.

b. (2 pts.) Present a cascade control system to maintain the surge tank level in zone two.

c. (2 pts.) Develop a feedback system to regulate the total flow to zone three.

2. (4 pts. total) You suggest using two variable speed pumps for supply. You are given the following testing data after installation of the new system.
• Increasing speed of pump $P_1$ by 10 RPM results in an increase of 5 GPM in $F_1$, an increase of 7 GPM in $F_2$, and an increase of 11 GPM in $F_3$.

• Increasing speed of pump $P_2$ by 10 RPM results in a decrease of 1 GPM in $F_1$, an increase of 17 GPM in $F_2$, and a decrease of 3 GPM in $F_3$.

a. (3 pts.) Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

b. (1 pt.) What problem would try to solve, assuming $F_1$ is 8 GPM too high, $F_2$ is 17 GPM too low, and $F_3$ is correct? What difficulties might you encounter when solving this problem?
1. **(10 points total)** Congratulations! Your new process engineering job has you working in the environmental control section of the plant. Species X must be removed from a process stream before the stream goes to the environmental control section of your plant. A reactor is run in parallel with a large cooler/chiller. You are expected to develop a dynamic model for your reaction system. You know the following about the system:

- The volume of liquid in reactor is constant is $V_1$ (L)
- The volume of the chiller is constant $V_2$ (L)
- The concentration of entering species X is $C_{X_0}(t)$
- The temperature of the entering stream is $T_o(t)$
- The process stream enters and leaves at a constant flow rate $F_1$ (L/min)
- The flow to and from the chiller from the reactor is a constant value $F_2$ (L/min)
- Adjustable cooling is used to cool the chiller: $Q(t)\, (J/min)$
- Heat is transferred from the chiller of $hA(T_2(t) - T_A)\, (J/min)$
- The reactor is well-insulated
- The reactor is assumed well-mixed at temperature $T_1(t)$ and concentration $C_{X1}(t)$.
- The chiller is assumed well-mixed at temperature $T_2(t)$ and concentration $C_{X2}(t)$.
- Assume constant physical properties for the liquid ($\rho, C_p$)
- The ambient temperature $T_A$ is constant
- The total rate expression describing conversion of X is $k_1 + k_2 C_{X1}(t) + k_3 T_1(t) \ (mol/min)$
- No reaction takes place in the chiller
- The heat of reaction is $-\Delta H \ (J/mol)$
- The deviation of the total amount of X flowing out is measured: $y_1(t) = F_1 (C_{X1}(t) - C_{X1ss})$
- The temperature difference between the two vessels is measured: $y_2(t) = (T_1(t) - T_{1ss}) - (T_2(t) - T_{2ss})$
- Inputs are deviation, $u_1(t) = T_o(t) - T_{oss}$ and $u_2(t) = C_{X0}(t) - C_{Xoss}$ and $u_3(t) = Q(t) - Q_{ss}$

a). **(6 points)** Develop a dynamic linear model for the system.

b). **(4 points)** Put the model in deviation form and then develop the state space representation. Clearly show $A$, $B$, $C$, $D$, $x(t)$, and $y(t)$.

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\[ y_2(t) = (T_1(t) - T_{1ss}) - (T_2(t) - T_{2ss}) \]

\[ y_1(t) = F_1 (C_{X1}(t) - C_{X1ss}) \]

\[ y_1(t) = \frac{k_1 + k_2 C_{X1}(t) + k_3 T_1(t)}{V_1} \]

\[ T_o(t) \quad C_{Xo}(t) \]

\[ F_1 \]

\[ F_2 \]

\[ V_1 \]

\[ V_2 \]

\[ hA (T_2(t) - T_A) \]

\[ Q(t) \]

\[ 257 \]
15.51 Fall 2017 Quiz 3

ECHE 550, Fall 2017 Chemical Process Dynamics and Control – Quiz #3,
October 4, 2017
Simplify all answers for full credit. Show all work.

1. (4 pts. total) For the following function of time:
   \[ u(t) = \begin{cases} 
   0 & t < 6 \\
   -3(t - 6) & 6 \leq t < 12 \\
   -18e^{-(t-12)/2} & 12 \leq t 
   \end{cases} \]

   (4 pts) Find \( u(s) \), the Laplace transform of \( u(t) \).

2. (6 pts total) For the following information about an isothermal Continuous Stirred Tank Reactor (CSTR) system with two species, \( A \) and \( B \) where \( A \) is converted to \( B \):
   A. (4 pts) Develop a linear dynamic model for the system.
   B. (2 pts) Put your linear model in state space form.

   • The volume of liquid in reactor is constant at \( V \) (L)
   • The concentration of entering species \( A \) is \( C_{Ao}(t) \)
   • The concentration of entering species \( B \) is \( C_{Bo}(t) \)
   • The process stream enters and leaves at a constant flow rate \( F \) (L/min)
   • The reactor is assumed well-mixed at concentration \( C_A(t) \) and \( C_B(t) \)
   • The reactor is isothermal
   • Assume constant physical properties
   • Assume there is no volume change during the reaction
   • The volumetric rate for conversion of \( A \) to \( B \) is \( k(C_A(t)C_B(t))^{1.7} \) (mol/L min)
   • The inlet concentration of \( A \) can be manipulated: \( u_1(t) = C_{Ao}(t) - C_{AoSS} \)
   • The inlet concentration of \( B \) can be manipulated: \( u_2(t) = C_{Bo}(t) - C_{BoSS} \)
   • The deviation concentration of \( B \) is measured: \( y_1(t) = C_B(t) - C_{BSS} \)
   • The deviation sum concentration of \( A \) and \( B \) is measured: \( y_2(t) = (C_A(t) - C_{ASS}) + (C_B(t) - C_{BSS}) \)
1. (8 pts. total) Congratulations! You landed an interview with Hot Harry’s Heating and Cooling. You accidently put on your resume that you are an expert in linear algebra. They want you to answer the following questions:
   a. (4 pts.) Determine result of the following matrix multiplication:
   \[
   \begin{bmatrix}
   -1 & 1 \\
   0 & 1 \\
   -2 & 3 
   \end{bmatrix}
   \begin{bmatrix}
   1 & -1 \\
   0 & -2 
   \end{bmatrix}
   \]

   a. (4 pts.) Determine eigenvalues of the following matrix:
   \[
   \begin{bmatrix}
   0 & -1 & 0 \\
   -2 & 1 & 0 \\
   4 & 1 & 0 
   \end{bmatrix}
   \]

2. (12 pts. total) Congratulations! You got the job in the heating systems area. There is a furnace for heating a process stream. The furnace has a pre-heater system before the furnace.
   a. (4 pts.) Propose a feedforward control scheme on the diagram below to maintain the fuel oil supply flow when there are periodic variations in the oil supply pressure.

   b. (4 pts.) Simultaneously propose feedback control to control the furnace exit temperature $T_F$ assuming the oxidizer supply flow influences this measurement.

   c. (4 pts.) Simultaneously propose cascade control system to control the temperature of the process stream entering the furnace.
3. (12 pts. total) Assume that you can adjust the oil flow $F_o$ and the steam flow $F_S$. Measurements $T_S$ and $T_F$ are available, with the following information.
   
   - Increasing the oil flow ($F_o$) by 9% increases $T_S$ by 4° and increases $T_F$ by 19°
   - Decreasing the steam flow ($F_S$) by 6% decreases $T_S$ by 11°

a. (8 pts.) Develop a model in the form $\Delta y = K \Delta u$ and clearly identify $\Delta y$, $\Delta u$, and $K$.

b. (4 pts.) Assuming $T_F$ is 38° too low and $T_S$ is 3° is too high, what problem would you solve? What is the solution?
4. **(12 pts. total)** For the standard startup process, the steam flow rate follows the following function of time. Note, \( e^{-4t} \approx 0 \) for \( t > 6 \).

What is the Laplace transform of the following function of time, \( u(s) \)?

\[
u(t) = \begin{cases} 
0 & \text{if } t \leq 3 \\
30 + 20e^{-4(t-3)} & \text{if } 3 \leq t < 12 \\
\approx 30 - 4(t - 12) & \text{if } 12 \leq t < 17 \\
\approx 10 & \text{if } 17 \leq t
\end{cases}
\]

5. **(9 pts. total)** Determine a first-order-plus-time-delay transfer function model from the following data. Show how you determined your numerical values on the graph below to receive full credit.

6. **(12 pts. total)** For a first-order-plus-time-delay system, assume the input is a unit impulse at time \( t = 0 \).

   a) What is the analytical expression for \( y(t) \)?
b) Determine values for the transfer function from the following data, assuming the input is a unit impulse at time $t = 0$.

![Graph showing deviation temperature over time](image)

5. **(24 pts. total)** You are given the following model for the product concentration deviation $y(t)$ as a function of the feed concentration deviation $u(t)$.

a. **(4 pts.)** For the following differential equations:

\[
\frac{dx_1}{dt}(t) = -x_1(t) + 2u(t) \\
\frac{dx_2}{dt}(t) = 3x_1(t) - 4x_2(t) \\
y(t) = x_2(t)
\]

with zero initial conditions

\[
\frac{dx_1}{dt}(t = 0) = \frac{dx_2}{dt}(t = 0) = 0
\]

determine a transfer function relating $u$ to $y$.

For parts b-e, use the following transfer function:

\[
g(s) = \frac{2s - 24}{s^2 + 5s + 6}
\]

b. **(4 pts.)** What are the gain, poles, and zeros of the transfer function?

c. **(4 pts.)** Given that the input an impulse of magnitude 3 at time $t = 0$, what is the initial value for $y$? Use the Initial Value Theorem to find $y(t = 0)$. Show your work.

d. **(4 pts.)** If the input experiences a step of magnitude 3 at time $t = 0$, what is
the eventual steady state value? Use the Final Value Theorem to find \( y(t = \infty) \). Show your work.

e. (8 pts.) Given the input experiences a unit impulse at time \( t = 0 \), what is the response \( y(t) \)? Show your work.

6. (16 pts. total) The model you were provided does not represent the real system very accurately. It is proposed that you should do some engineering analysis and develop a detailed dynamic model of the system.

Assume the following:

- The pre-heating tank is well-mixed with constant volume \( V_1 \) (L)
- The pre-heating tank is \( T_1(t) \) (°C)
- The heating tank in the furnace is well-mixed with constant volume \( V_2 \) (L)
- The heating tank is \( T_2(t) \) (°C)
- The process flow in and out the system is \( F \left( \frac{L}{min} \right) \).
- The flow between tanks is \( F_1 \left( \frac{L}{min} \right) \).
- You can assume constant physical properties (\( \rho, C_p \)).
- The pre-heating tank is poorly insulated, losing heat at the rate \( hA(T_1(t) - T_A(t)) \left( \frac{J}{min} \right) \)
- Steam is used in the pre-heating tank, providing heat input of \( Q(t) \left( \frac{J}{min} \right) \)
- The heating tank is well-insulated.
- The furnace heats the heating tank at a rate \( k(T_F(t) - T_2(t))^4 \left( \frac{J}{min} \right) \)

a. (8 pts.) Develop a dynamic model for this system.

a. (8 pts.) Given the following nonlinear expression:

\[
ke^{-\frac{t}{\tau}}C_A(t)
\]

Linearize this expression around the steady state values \( T_{SS} \) and \( C_{ASS} \)
Chapter 16

Lecture Notes

The following pages include lecture notes in the “Spark Note” format. Traditional note taking involves a lecturer writing notes onto a board while students copy the notes rapidly. Modern lectures often involve students watching electronic presentations presented on a screen.

Traditional lecturing has the disadvantage that only a limited amount of material can be presented in a set amount of time. The material presented must be hand-drawn by the instructor. The material cannot include complex diagrams or figures. Complex derivations can be very time consuming. However, the main advantage lies in the fact that the student is actively engaged in note taking. When a student stops actively taking notes, their attention drops and retention is limited.

Modern electronic presentations often lull students to sleep, allowing them to become passive consumers of information. Even when presentation copies are provided before the lecture, students rarely annotate the notes in any substantial fashion. The notes included here allow for a trade off: the bulk of the information is provided. However, the student must attend lecture and fill in substantial portions of the lecture material. This forces the student to be actively engaged. It also allows for complex diagrams and derivations to be presented.
16.1 Information Flow and Control Concepts

Information Flow and Control Concepts
Fundamental Concepts

Edward P. Gatzke
Department of Chemical Engineering
University of South Carolina
Lecture 1

Basic Control Concepts
Space for Notes Below
Process Signals

Space for Notes Below

PFD Conventions
Feedforward Control

Goal: Regulate T

Feedback Control

Goal: Regulate T
Cascade Control

Goal: Regulate T

Assignments and Review

- Homework
  -
  -

- Dynamic process require signals
  - Inputs and Outputs
  - Disturbances and Setpoints

- Feedforward Control: Measured disturbance

- Feedback Control: Needs $y(t)$ and $y_{sp}(t)$ going in

- Cascade Control: One controller controls another
16.2 Linear Algebra

Fundamental Linear Algebra Concepts

Solving $Ax = b$

Edward P. Gatzke

Department of Chemical Engineering
University of South Carolina
Lecture 2

Basic Linear Algebra Concepts

- Matrix Matrix Multiplication
  - The vector
  - Matrix \times vector and matrix \times matrix multiplication
  - The identity matrix $I$
- Solving sets of linear equations in form $Ax = b$
  - Matrix Inverse
- Finding the determinant of a matrix
- Finding the eigenvalues of a matrix
- Multivariable Steady State problems: $\Delta y = K \Delta u$
The Vector

Matrix × Vector
Matrix \times Matrix

Matrix \times Matrix, Nonsquare

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
2 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
= 
\begin{bmatrix}
6 & 8 \\
8 & 10 \\
7 & 10 \\
3 & 4
\end{bmatrix}
\]
Identity Matrix

Space for Notes Below

Solving $Ax = b$

Space for Notes Below
Solving $Ax = b$ Example

\[
\begin{align*}
1x_1 + 0x_2 + 1x_3 &= 3 \\
0x_1 + 2x_2 + 6x_3 &= 2 \\
2x_1 + 0x_2 + 4x_3 &= 8
\end{align*}
\]

$\Rightarrow$

\[
\begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 2 & 6 & 2 \\
2 & 0 & 4 & 8
\end{bmatrix}
\]

$\Rightarrow Ax = b$

Multiply row 3 by 1/2 (Rule 1)

\[
\begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 2 & 6 & 2 \\
1 & 0 & 2 & 4
\end{bmatrix}
\]

Subtract row 1 from row 3 (Rule 3)

\[
\begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 2 & 6 & 2 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Multiply row 2 by 1/2 (Rule 1)

\[
\begin{align*}
1x_1 + 0x_2 + 1x_3 &= 3 \\
0x_1 + 1x_2 + 3x_3 &= 1 \\
0x_1 + 0x_2 + 1x_3 &= 1
\end{align*}
\]

So $x_3 = 1$, $x_2 = 1 - 3x_3 = -2$, and $x_1 = 3 - 1x_3 = 2$

\[
x = \begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}
\]

Finished? Must check!

\[
Ax = \begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 6 \\
2 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
2 \\
-2 \\
1
\end{bmatrix}
= \begin{bmatrix}
3 \\
2 \\
8
\end{bmatrix} = b
\]

- Method:
  - Start on column 1
  - Get diagonal to 1 and make values below diagonal 0
  - Go to next column
Why Solve $Ax = b$?

- Steady State Multivariable Control
  - Know inputs and outputs
  - Make changes in inputs: $\Delta u_1, \Delta u_2 \ldots \Delta u_n$
  - Examine outputs response $\Delta y_1, \Delta y_2 \ldots \Delta y_n$
  - Get "gain" relationships for all pairs $\Delta y_i = k_{ij} \Delta u_j$
  - Set up in matrix form $\Delta y = K \Delta u$

- If you know changes in inputs ($\Delta u$) easy to find $\Delta y$
- If you know target / reference change ($\Delta y$) harder to find $\Delta u$

---

Matrix Inverse $A^{-1}$

Space for Notes Below
Matrix Inverse $A^{-1}$

Space for Notes Below

---

Matrix Inverse Example

- **Method:**
  - Same as row reduction, get diag to 1, values below diag 0
  - Start on last column and get 0 above using multiple rules
  - Go to previous column

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 6 \\
2 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\leftrightarrow A|I
\]

- Multiply row 2 and row 3 by $1/2$

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 3 \\
1 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0.5 & 0 \\
0 & 0 & 0.5
\end{bmatrix}
\]

- Subtract row 1 from row 3

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0.5 & 0 \\
-1 & 0 & 0.5
\end{bmatrix}
\]
Matrix Inverse Example

- Subtract 3×row 3 from row 2

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
- \begin{bmatrix}
3 & 0.5 & -1.5 \\
-1 & 0 & 0.5 \\
\end{bmatrix}
\]

- Subtract row 3 from row 1

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
- \begin{bmatrix}
3 & 0.5 & -1.5 \\
-1 & 0 & 0.5 \\
\end{bmatrix}
\]

- Check!

\[
AA^{-1} = \begin{bmatrix}
2 & 0 & -0.5 \\
3 & 0.5 & -1.5 \\
-1 & 0 & 0.5 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 6 \\
2 & 0 & 4 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Assignments and Review
16.3  Linear Algebra

Fundamental Linear Algebra Concepts

Solving $Ax = b$

Edward P. Gatzke

Department of Chemical Engineering
University of South Carolina
Lecture 3

Basic Linear Algebra Concepts

- **Matrix Matrix Multiplication**
  - The vector
  - Matrix $\times$ vector and matrix $\times$ matrix multiplication
  - The identity matrix $I$

- **Solving sets of linear equations in form** $Ax = b$
  - Matrix Inverse for solution: $x = A^{-1}b$

- Finding the determinant of a matrix
- Finding the eigenvalues of a matrix
- Multivariable Steady State problems: $\Delta y = K \Delta u$
Question #1, Matrix Multiplication

Question: What is the correct answer?

\[
\begin{bmatrix}
-2 & -1 \\
2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
-1 & 2 \\
1 & 0 \\
\end{bmatrix}
\]

A \[
\begin{bmatrix}
2 & -1 \\
-3 & 1 \\
\end{bmatrix}
\]
B \[
\begin{bmatrix}
-3 & -1 \\
-3 & -1 \\
\end{bmatrix}
\]
C \[
\begin{bmatrix}
-3 & 1 \\
-3 & 1 \\
\end{bmatrix}
\]
D None of the above

Answer: ???

Determinant of A

Space for Notes Below
Determinant Example

For simple $3 \times 3$

$$
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{vmatrix}
$$

$$
1 \det \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \det \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \det \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}
$$

$$
1(45 - 48) - 2(36 - 42) + 3(32 - 35) = -3 + 12 - 9 = 0
$$

A $4 \times 4$ requires four $3 \times 3$ determinants (alternate sign)

Why find the Determinant?

Space for Notes Below
Determinant Example

Question #2, Determinants

Question: What is the correct answer?

\[
\text{det} \begin{bmatrix}
1 & -2 & 0 \\
1 & 2 & 6 \\
2 & 1 & 0
\end{bmatrix} =
\]

A Less than 0  
B Equal to 0  
C Greater than 0  
D None of the above

Answer: ???
Eigenvalues of $A$

Space for Notes Below

Eigenvalue Example

Space for Notes Below
Why Find Eigenvalues of \( A \)?

Space for Notes Below

---

Question #3, Eigenvalues

Question: What is the minimum eigenvalue of the following matrix?

\[
\begin{bmatrix}
2 & 3 \\
1 & 0 \\
\end{bmatrix}
\]

A 3  
B 0  
C -1  
D None of the above

Answer: ???
Question #4, Eigenvalues

- **Question:** What is the minimum eigenvalue of the following matrix?

\[
\begin{bmatrix}
1 & -2 & 3 \\
0 & 0 & 6 \\
0 & 0 & -3
\end{bmatrix}
\]

A 3  
B 0  
C -1  
D None of the above

- **Answer:** ???
Assignments and Review

- Linear Algebra Review
  - Solve $A \cdot x = b$
  - Row reduction for $x$ or $A^{-1}$
  - Determinants and Eigenvalues of $A$
    - Know how to compute them and why we use them!
  - Multivariable Steady State Control problems

\[ \Delta y = K \Delta u \]

- Homework

Question #5

- Question: How is this course format (iClicker, presentations) working?
  - B I don’t like it, but I will cope.
  - C Meh. No major complaints so far.
  - D I actually like it. Seriously.

- Answer: ???
16.4 Dynamic Modeling

Dynamic Process Modeling
Accumulation = In - Out + Created - Destroyed

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University of South Carolina
Lecture 4

Control and Linear Algebra Concepts

- Basic control formulations
  - Feedforward (measured $d$)
  - Feedback (measure $y$, have $y_{sp}$, adjust $u$)
  - Cascade (controller telling another controller $y_{sp}$)

- Need arrows / information flow. Label signals, controller

- Set up steadystate multivariable control problems in form
  - $\Delta y = K \Delta u$ (same as $Ax = b$)

- Determinants and their use
- Eigenvalues and their use
Control and Linear Algebra Concepts

- Basic control formulations
  - Feedforward (measured $d$)
  - Feedback (measure $y$, have $y_{sp}$, adjust $u$)
  - Cascade (controller telling another controller $y_{sp}$)

- Need arrows / information flow. Label signals, controller
- Set up steadystate multivariable control problems in form
  - $\Delta y = K \Delta u$ (same as $Ax = b$)
- Determinants and their use
- Eigenvalues and their use

Dynamic Modeling Concepts

- First Principles Models
  - Mass and Energy Balances
- Linear vs. Nonlinear
- Deviation Variables
- State Space Representation
Question #1, Determinants

**Question:** What is the correct answer?

\[
\begin{vmatrix}
0 & -1 & 2 \\
0 & 2 & 6 \\
2 & 1 & 0
\end{vmatrix} =
\]

A: Less than 0  
B: Equal to 0  
C: Greater than 0  
D: None of the above

**Answer:** ???

Question #2, Eigenvalues

**Question:** What is the maximum eigenvalue?

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 2 & 0 \\
2 & 1 & -1
\end{bmatrix} =
\]

A: 0  
B: 1  
C: 2  
D: -1  
E: None of the above

**Answer:** ???
Tank Example
# Tank Example

Space for Notes Below

<table>
<thead>
<tr>
<th>Ed Getzke (USC CHE)</th>
<th>L4 - Dynamic Modeling A</th>
<th>ECHE 550</th>
<th>8 / 20</th>
</tr>
</thead>
</table>

# Tank Example

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<table>
<thead>
<tr>
<th>Ed Getzke (USC CHE)</th>
<th>L4 - Dynamic Modeling A</th>
<th>ECHE 550</th>
<th>8 / 20</th>
</tr>
</thead>
</table>
Mixing Tank Example

\[ r(t) = kC_A(t) \]

\[ V; C_A(t) \]

Space for Notes Below
Mixing Tank Example

Space for Notes Below

Question #3

**Question:** It is OK to assume the accumulation rate is 0 when developing dynamic mass and energy balances.

- A True
- B False
- C What?

**Answer:** ???
Energy Balance Example

\[ r(t) = kC_A(t) \]

\( C_{Ao}(t) \)

\( T_o(t) \)

\( F \)

\( A \rightarrow B \)

\( V, C_A(t), T(t) \)

\( T(t) \)

\( F \)

\( C_A(t) \)
Energy Balance Example

Space for Notes Below

Question #4

- **Question:** What is NOT a typical assumption in a dynamic model?
  A. Constant physical Properties
  B. Well-mixed
  C. Constant volume or constant area
  D. Steady state

- **Answer:** ???
Linear vs. Nonlinear

Question #5

Question: What is the most confusing topic?

A  Determinants
B  Eigenvalues
C  Solving $Ax = b$ and getting $A^{-1}$
D  Setting up $\Delta y = K\Delta u$
E  Basic control, Feedback, Feedforward, Cascade

Answer: ???
16.5 Dynamic Modeling

Dynamic Process Modeling
Accumulation = In - Out + Created - Destroyed

Edward P. Gatzke
Department of Chemical Engineering
University of South Carolina
Lecture 5

Dynamic Modeling Concepts

- First Principles Models
  - Mass and Energy Balances
- Linear vs. Nonlinear
- Deviation Variables
- State Space Representation
- Analytical vs. Numerical Solutions
Question #1

Question: To regulate $P_1$, what type of control is being used below?

![Diagram of control system]

A  Feedforward Control  
B  Feedback Control  
C  Cascade Control  
D  Other  
E  I have no clue and deserve a bad grade.

Answer: ???

---

Question #2

Question: To regulate $P_1$, what type of control is being used below?

![Diagram of control system]

A  Feedforward Control  
B  Feedback Control  
C  Cascade Control  
D  Other  
E  I have no clue and deserve a bad grade.

Answer: ???
Question #3

**Question:** To regulate $P_1$, what type of control is being used below?

A. Feedforward Control  
B. Feedback Control  
C. Cascade Control  
D. Other  
E. I have no clue and deserve a bad grade.

**Answer:** ???

---

**Deviation Variables**

![Diagram showing deviation variables with T(t) and y(t) graphs.]

$T(t)$ at steady state is 90°  
y(t) at steady state is 0°

d$T$  
d$y$

Space for Notes Below
State Space

\[
\dot{x}(t) = \begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
    x_1(t) \\
    \vdots \\
    x_n(t)
\end{bmatrix} + \begin{bmatrix}
    b_{11} & \cdots & b_{1m} \\
    \vdots & \ddots & \vdots \\
    b_{m1} & \cdots & b_{mm}
\end{bmatrix} \begin{bmatrix}
    u_1(t) \\
    \vdots \\
    u_m(t)
\end{bmatrix}
\]

\[
y(t) = \begin{bmatrix}
    c_{11} & \cdots & c_{1n} \\
    \vdots & \ddots & \vdots \\
    c_{n1} & \cdots & c_{nn}
\end{bmatrix} \begin{bmatrix}
    x_1(t) \\
    \vdots \\
    x_n(t)
\end{bmatrix} + \begin{bmatrix}
    d_{11} & \cdots & d_{1m} \\
    \vdots & \ddots & \vdots \\
    d_{m1} & \cdots & d_{mm}
\end{bmatrix} \begin{bmatrix}
    u_1(t) \\
    \vdots \\
    u_m(t)
\end{bmatrix}
\]

Space for Notes Below
State Space

Space for Notes Below

Tanks Example

\[F_0(t)\]

\[F_1(t) = k_1 h_1(t)\]

\[F_2(t) = k_2 h_2(t)\]

\[y(t) = F_2(t) - F_{2ss}\]
Tanks Example

Assume:

- Constant Tank Areas $A_1$ and $A_2 \, (m^2)$
- Flow out is proportional to tank height
  \[ F_1(t) = k_1 h_1(t) \left( \frac{m^3}{s} \right) \] and \[ F_2(t) = k_2 h_2(t) \left( \frac{m^3}{s} \right) \]
- Constant physical properties ($\rho$)

Space for Notes Below

Tanks Example

Must get to deviation variables!

\[ A_1 \frac{dh_1}{dt}(t) = F_0(t) - k_1 h_1(t) \]

At Steady State:

\[ 0 = F_{0,ss} - k_1 h_{1,ss} \]

Subtract SS from dynamic balance, term by term:

\[ A_1 \frac{dh_1}{dt}(t) - 0 = (F_0(t) - F_{0,ss}) - k_1 (h_1(t) - h_{1,ss}) \]

Assume deviation variables $x_1(t) = h_1(t) - h_{1,ss}$ and $u(t) = F_0(t) - F_{0,ss}$

Realize derivative of deviation variable is same as derivative of normal variable:

\[ \frac{dh_1}{dt}(t) = \frac{dx_1}{dt}(t) \]

\[ A_1 \frac{dx_1}{dt}(t) = u(t) - k_1 x_1(t) \]
Tanks Example

Must do the same for the second tank

\[ A_2 \frac{dx_2}{dt}(t) = k_1 x_1(t) - k_2 x_2(t) \]

Write out measurement equations

\[ y(t) = k_2 (h_2(t) - h_{2,ss}) = k_2 x_2(t) \]

Get derivatives on LHS:

\[ \frac{dx_2}{dt}(t) = \frac{k_1}{A_2} x_1(t) - \frac{k_1}{A_2} x_2(t) \]

Rewrite dynamic & measurement equations with all states and inputs included:

\[ \frac{dx_1}{dt}(t) = -\frac{k_1}{A_1} x_1(t) + 0 x_2(t) + \frac{1}{A_1} u(t) \]

\[ \frac{dx_2}{dt}(t) = \frac{k_1}{A_2} x_1(t) - \frac{k_2}{A_2} x_2(t) + 0 u(t) \]

\[ y(t) = 0 x_1(t) + k_2 x_2(t) + 0 u(t) \]

Tanks Example

Identify \( A, B, C, \) and \( D \)

\[
A = \begin{bmatrix}
-\frac{k_1}{A_1} & 0 \\
\frac{k_1}{A_2} & -\frac{k_2}{A_2}
\end{bmatrix}
B = \begin{bmatrix}
\frac{1}{A_1} \\
0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & k_2
\end{bmatrix}
D = \begin{bmatrix}
0
\end{bmatrix}
\]

\( C \) may only be a single 1, the rest 0s.
**Question #4**

- **Question:** Given the following state space model:

  \[
  A = \begin{bmatrix}
  -\frac{k_1}{A_1} & 0 \\
  -\frac{k_2}{A_2} & \frac{1}{A_1} \\
  \end{bmatrix}
  \]

  \[
  B = \begin{bmatrix}
  0 \\
  0 \\
  \end{bmatrix}
  \]

  \[
  C = \begin{bmatrix}
  0 & k_2 \\
  \end{bmatrix}
  \]

  \[
  D = \begin{bmatrix}
  0 \\
  \end{bmatrix}
  \]

- **What are the eigenvalues of the A matrix.**

  A \(\frac{k_1}{A_1} \quad \frac{k_2}{A_2}\)

  B \(-\frac{k_1}{A_1} \quad -\frac{k_2}{A_2}\)

  C \(-\frac{k_1}{A_1} \quad -\frac{k_2}{A_2}\)

  D \(\frac{1}{A_1} \quad k_2\)

- **Answer:** ???

---

**State Space Procedure**

Space for Notes Below
Question #5

- **Question:** Given the following balances

\[ V \rho C_p \frac{dT}{dt}(t) = F \rho C_p T_o(t) - F \rho C_p T(t) + (-\Delta H) V k C_A(t) \]

\[ V \frac{dC_A}{dt}(t) = F C_{A0}(t) - F C_A(t) - V k C_A(t) \]

- What is a state variable?
  A  \( V \)
  B  \( T_o \)
  C  \( T \)
  D  \( C_A \)
  E  I have no clue and deserve a bad grade.

- **Answer:** ???

---

Question #6

- **Question:** Given the following balances

\[ V \rho C_p \frac{dT}{dt}(t) = F \rho C_p T_o(t) - F \rho C_p T(t) + (-\Delta H) V k C_A(t) \]

\[ V \frac{dC_A}{dt}(t) = F C_{A0}(t) - F C_A(t) - V k C_A(t) \]

- Are the balances in deviation form
  A  Yes. Obviously.
  B  No. Are you crazy?
  C  I have no clue and deserve a bad grade.

- **Answer:** ???
Numerical Solution of ODEs

Question #7

- Most confusing topic currently:
  A Basic control structures. Feedforward, Feedback, Cascade
  B Linear algebra, $\Delta y = K \Delta u$, Eigenvalues, Determinants
  C Dynamic modeling of chemical processes
  D Converting dynamic models to deviation form
  E Converting linear dynamic models to state space form

- Answer: ???
16.6 Laplace Transforms

Laplace Transforms
Analytical Solutions for ODEs

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Lecture 6

Question #1

Question: Have you looked at chapter 4?

A No.
B Yes and it was review.
C Yes and it was all new.
D I don’t have a book yet.
E This is not an answer. I will get you.

Answer: ???
Question #2

- **Question:** How do you determine the states of an ODE dynamic model?
  - A. Read the problem
  - B. States are manipulated variables
  - C. States change with time unpredictably
  - D. States change based on accumulation rate expressions
  - E. I hate dynamics and controls

- **Answer:** ???

---

**Laplace Transforms**

- The Laplace Transform and the Inverse Transform
  - Properties
- Laplace of Functions
  - Heaviside, Exponential, Derivatives, Time Shift
- Initial and Final Value Theorems
- Solving ODEs using Laplace domain
  - Partial Fraction Expansion
Laplace Transform and Inverse Transform

Use of the Laplace Transform

\[
\begin{align*}
\text{t-domain} & \quad L \quad \text{s-domain} \\
\text{Problem} & \quad \rightarrow \quad \text{Transformed Problem} \\
\text{Solution} & \quad \text{Transformed Solution} \\
\text{t-domain} & \quad \leftarrow \quad \text{s-domain}
\end{align*}
\]
Heaviside Function

Space for Notes Below

Laplace of Heaviside

Space for Notes Below
Laplace Properties

Space for Notes Below

Derivatives and Integrals

Space for Notes Below
Time Shift Operator

Space for Notes Below

Question #3

**Question:** What function is the inverse Laplace of $\frac{1}{s}$?

A The derivative operator  
B The time shift operator  
C The integral operator  
D The Heaviside function  
E I have no idea

**Answer:** ???
Exponential Function

Space for Notes Below
Functions So Far

\[ L\left\{ \frac{df(t)}{dt} \right\} = sf(s) - f(t)|_{t=0} \]

\[ L\left\{ \int_{0}^{t'} f(t)\,dt' \right\} = \frac{1}{s} f(s) \]

Question #4 & 5

- **Question 4:** Why are we learning about Laplace transforms?
  
  A. They can be used to solve linear ordinary differential equations
  B. They can be used to analyze the stability of a dynamic system
  C. They can be used to determine steady state values for systems
  D. Gatzke told me to
  E. I have no idea

- **Answer:** ???

- **Question 5:** What is the most confusing topic so far?
  
  A. Laplace transforms
  B. Determinants and Eigenvalues
  C. Solving \( Ax = b \) and getting \( A^{-1} \)
  D. Setting up \( \Delta y = K\Delta u \)
  E. Basic control, Feedback, Feedforward, Cascade
Laplace Transforms

Analytical Solutions for ODEs

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University of South Carolina
Lecture 7

Laplace Transforms

- The Laplace Transform and the Inverse Transform
  - Properties
- Laplace of Functions
  - Heaviside, Exponential, Derivatives, Time Shift
- Initial and Final Value Theorems
- Solving ODEs using Laplace domain
  - Partial Fraction Expansion
Review: Laplace Transform and Inverse Transform

\[ f(s) = \int_0^\infty e^{-st} f(t) \, dt = L\{f(t)\} \quad f(t) = \int_C e^{st} f(s) \, ds = L^{-1}\{f(s)\} \]

- Not all functions have a Laplace transform and only for \( t > 0 \)
- Unique transform pair
- Linear (functions add, can multiply by constant)

\[ L\{c_1f_1(t) + c_2f_2(t)\} = c_1L\{f_1(t)\} + c_2L\{f_2(t)\} \]

Heaviside Function and Exponential Function

\[ f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad f(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0 \end{cases} \]

\[ f(t) = \mathcal{H}(t) \quad f(t) = \mathcal{H}(t) \, e^{-at} \]

\[ L\{\mathcal{H}(t)\} = \frac{1}{s} \quad L\{f(t)\} = \frac{1}{s+a} \]
Laplace So Far

\[ \mathcal{H}(t) \]

\[ \frac{1}{s} \]

\[ 0 \]

\[ \mathcal{H}(t-\alpha) \]

\[ \frac{1}{s} e^{-\alpha s} \]

\[ 0 \]

\[ \mathcal{H}(t) e^{-at} \]

\[ \frac{1}{s+a} \]

\[ L \left\{ \frac{df}{dt}(t) \right\} = sf(s) - f(t)|_{t=0} \]

\[ L \left\{ \int_0^{t'} f(t') dt' \right\} = \frac{1}{s}f(s) \]

Question #1

**Question:** Does the Laplace transform make any sense to you right now?

A. Yes, I am comfortable so far.
B. Sorta but not solid yet.
C. No.
D. Math is hard, you are going too fast.
E. This is not an answer. I will get you.

**Answer:** ???
Ramp Function

Space for Notes Below

Dirac Delta Function

Space for Notes Below
## Relationships

\[
u(t) = \mathcal{H}(t) t
gives \quad u(t) = \mathcal{H}(t)
\]

\[
u(t) = \delta(t)
\]

\[
u(s) = \frac{1}{s^2}
gives \quad u(s) = \frac{1}{s}
\]

\[u(s) = 1\]

## Sin Function

Space for Notes Below
Question #2

- **Question:** Who is this guy?
  - A  George Washington (1732-1799)
  - B  Pierre-Simon Laplace (1749-1827)
  - C  Paul Dirac (1902-1984)
  - D  Oliver Heaviside (1850-1925)
  - E  I have no clue and don't really care.

- **Answer:** ???

---

Question #3

- **Question:** What is the Laplace transform of this function?

![Laplace Transform Diagram](image)

- A  \( \sigma \)
- B  \( \frac{s}{s} \)
- C  \( \frac{\sigma}{s^2} \)
- D  \( \frac{\sigma}{s} e^{-\alpha s} \)
- E  \( \frac{\sigma}{s} e^{-\alpha s} \)

- **Answer:** ???
Rectangular Pulse

Space for Notes Below

Rectangular Pulse

\[ f_1(t) = \mathcal{H}(t) \]
\[ f_2(t) = -\mathcal{H}(t - \alpha) \]
\[ f(t) = f_1(t) + f_2(t) \]

\[ f(s) = \frac{1}{s} - \frac{1}{s} e^{-\alpha s} \]
“Composite” Function Rules

Initial and Final Value Theorem
Initial and Final Value Theorem Example

\[ f(s) = \frac{7}{(2s + 1)} \cdot \frac{4}{s} \]

\[ f(t)|_{t=0} = \lim_{s \to \infty} s \cdot f(s) = \lim_{s \to \infty} s \cdot \frac{7}{(2s + 1)} \cdot \frac{4}{s} = \frac{28}{\infty + 1} = 0 \]

\[ f(t)|_{t=\infty} = \lim_{s \to 0} s \cdot f(s) = \lim_{s \to 0} s \cdot \frac{7}{(2s + 1)} \cdot \frac{4}{s} = \frac{28}{0 + 1} = 28 \]

REMEMBER to multiply \( f(s) \) by \( s \)

Note, you may need to use L'Hôpital's Rule

Or reformat to powers of \( \frac{1}{\infty}, \frac{1}{\infty^n} \)

Question #4

\[ f(s) = \frac{7(s + 1)}{(2s + 1)} \cdot \frac{4}{s} \]

- Question: What is the INITIAL value for \( f(t) \)?
  - A 7
  - B 28
  - C 7/2
  - D 0
  - E 14

- Answer: ???
Question #5

**Question: What is least clear?**

A. Linear algebra, $Ax = b$, determinants, eigenvalues  
B. Setting up $\Delta y = K\Delta u$ word problems  
C. Setting up dynamic models  
D. State space representation  
E. Laplace transforms

**Answer: ???**

---

Question #6

**Question: How is class going?**

A. Too much homework, homework is too hard  
B. Lecture notes are not clear, lecture is too fast  
C. Wednesday labs are bad, I hate computers  
D. I am starting to get it, I should be ok-  
E. I need to read the book, review notes, come to class, ask the occasional question, do the HW and I should be fine

**Answer: ???**
16.8 Laplace Transforms

Laplace Transforms
Analytical Solutions for ODEs

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Lecture 8

Roadmap
Laplace Transforms

- The Laplace Transform and the Inverse Transform
  - Properties
- Laplace of Functions
  - Heaviside, Exponential, Derivatives, Time Shift
- Initial and Final Value Theorems
- Solving ODEs using Laplace domain
  - Partial Fraction Expansion

Review: Laplace Transform and Inverse Transform

\[ f(s) = \int_0^\infty e^{-st} f(t) \, dt = L \{ f(t) \} \quad f(t) = \int_C e^{st} f(s) \, ds = L^{-1} \{ f(s) \} \]

- Not all functions have a Laplace transform and only for \( t > 0 \)
- Unique transform pair. Linear
  \[ L \{ c_1 f_1(t) + c_2 f_2(t) \} = c_1 L \{ f_1(t) \} + c_2 L \{ f_2(t) \} \]
Question #1

- Question: Have you read through chapter 3 yet?
  
  A  Yes, I did what you suggested.
  B  I sorta skimmed it.
  C  No.
  D  Reading about math is hard.
  E  What is a book? My book is on order.

- Answer: ???
Initial and Final Value Theorem

$$f(t)\big|_{t=0} = \lim_{s \to \infty} s f(s)$$

$$f(t)\big|_{t=\infty} = \lim_{s \to 0} s f(s)$$

**REMEMBER to multiply** $f(s)$ **by** $s$

Note, you may need to use L’Hôpital’s Rule

Or reformat to powers of $\frac{1}{\infty}$, $\frac{1}{\infty^n}$

---

**Question #2**

$$f(s) = \frac{7(s+1)}{(2s+1)} \cdot \frac{4}{s}$$

**Question:** What is the INITIAL value for $f(t)$?

A 7  
B 28  
C 7/2  
D 0  
E 14  

**Answer:** ???
Question #3

\[ y(s) = \frac{7(s + 1)}{(2s + 1)} \frac{4}{s} \]

- **Question:** What is the FINAL value for \( y(t) \)?
  
  A 7  
  B 28  
  C 7/2  
  D 0  
  E 14  

- **Answer:** ???

Ordinary Differential Equations

Space for Notes Below
Ordinary Differential Equations

Space for Notes Below
Question #4

- **Question: Who is this guy?**
  A George Washington (1732-1799)
  B Pierre-Simon Laplace (1749-1827)
  C Paul Dirac (1902-1984)
  D Oliver Heaviside (1850-1925)
  E I have no clue and don’t really care.

- **Answer: ???**

---

Question #5

- **Question: What is the Laplace transform of this function?**

![Graph of a unit step function](image)

A $1$
B $1 - e^{-\alpha s}$
C $\frac{1}{s} - \frac{1}{s^2} e^{-\alpha s}$
D $\frac{1}{s} - \frac{1}{s} e^{-\alpha s}$
E $\frac{1}{s + \alpha}$

- **Answer: ???**
IVT and FVT Analysis

\[ y(t) = \mathcal{H}(t) \left( AK \left( 1 - e^{-t/\tau} \right) \right) \]
\[ y(t = 0) = 1 \left( AK \left( 1 - e^0 \right) \right) = 0 \]
\[ y(t = \infty) = 1 \left( AK \left( 1 - e^{-\infty} \right) \right) = AK \]

Space for Notes Below

Review: Composite Function Rules

- Functions we consider at all points in time \(-\infty\) to \(\infty\)
- "Jump" instantly for \(\mathcal{H}(t)\) and \(e^{-at}\)
- Change in slope means ramp "turns on"
- May have 2 jumps at once, cancel out

Ramp of slope 2 at \(t = 6\), ramp of slope -2 at \(t = 17\), step of size -12 at \(t = 17\)

\[ u(t) = (2)\mathcal{H}(t-6)(t-6) + (-2)\mathcal{H}(t-12)(t-12) + (-12)\mathcal{H}(t-17) \]
\[ u(s) = (2) \frac{1}{s^2} e^{-6s} + (-2) \frac{1}{s^2} e^{-12s} + (-12) \frac{1}{s} e^{-17s} \]
Dynamic Polymerization Extruder

- Balance on total polymerized mass, polymerization rate is $VkT(t)$
- Mass flow $M$, weight $W$ is mass %, total mass: $V\rho = L\rho$
- Mass balance on crosslinked polymer and SS eqn:
  \[ L\rho \frac{dW}{dt}(t) = MW_o(t) - MW(t) + LAt(t) \]

\[ 0 = MW_{oss} - MW_{ss} - LAt_{ss} \]

- Energy balance on extruder and SS eqn:
  \[ (L\rho C_p) \frac{dT}{dt}(t) = (MC_p) T_o(t) - (MC_p) T(t) - (hA) (T(t) - T_{atm}) \]

\[ 0 = (MC_p) T_{oss} - (MC_p) T_{ss} - (hA) (T_{ss} - T_{atm}) \]

Review: Dynamic Modeling

- Deviation form, dynamic eqn - ss equation, matching terms:
  \[ L\rho \frac{dW}{dt}(t) = (MC_p) T_o(t) - (MC_p) T(t) - (hA) (T(t) - T_{atm}) \]

\[ (L\rho C_p) \frac{dT}{dt}(t) = (MC_p) T_o(t) - (MC_p) T(t) - (hA) (T(t) - T_{atm}) \]

- Collect terms and identify state / input variables, include $y(t)$
  \[ L\rho \frac{dx_1}{dt}(t) = Mu(t) - Mx_1(t) + (LAt) x_2(t) \]

\[ (L\rho C_p) \frac{dx_2}{dt}(t) = (MC_p) u(t) - (MC_p) x_2(t) - (hA) x_2(t) \]

\[ y(t) = T(t) - T_{ss} = x_2(t) \]

- Algebra to get in matrix / state space form to identify $A, B, C, \text{ and } D$:
  \[
  \frac{dx_1}{dt}(t) = \left(-\frac{M}{L\rho}\right) x_1(t) + \left(\frac{LA}{L\rho}\right) x_2(t) + 0 \quad \begin{array}{c} u(t) + \left(\frac{M}{L\rho}\right) u_2(t) \\
\end{array}
  \]

\[
  \frac{dx_2}{dt}(t) = 0 \quad x_1(t) + \frac{(MC_p + hA)}{L\rho C_p} x_2(t) + \frac{MC_p}{L\rho C_p} u_1(t) + 0 \quad u_2(t)
  \]

\[ y(t) = 0 \quad x_1(t) + 1 \quad x_2(t) + 0 \quad u_1(t) + 0 \quad u_2(t) \]
**First-Order+ Time Delay Step Response**

\[
\tau \frac{dy}{dt}(t) = -y(t) + Ku(t - \alpha) \quad y(t = 0) = 0
\]

\[
\tau sy(s) - 0 = -y(s) + Ke^{-\alpha s} u(s)
\]

\[
y(s) = \left( \frac{K}{\tau s + 1} e^{-\alpha s} \right) u(s)
\]

\[
u(t) = A\mathcal{H}(t) \rightarrow u(s) = A\left( \frac{1}{s} \right)
\]

\[
y(s) = \left( \frac{K}{\tau s + 1} \right) \left( \frac{A}{s} \right) e^{-\alpha s} = \left( KA \frac{1}{s} + (-KA) \frac{1}{s + \frac{1}{\tau}} \right) e^{-\alpha s} = y_1(s) + y_2(s)
\]

\[
y(t) = (KA)\mathcal{H}(t - \alpha) + (-KA)\mathcal{H}(t - \alpha) e^{-\left( \frac{t - \alpha}{\tau} \right)} = \mathcal{H}(t - \alpha) (KA) \left( 1 - e^{-\left( \frac{t - \alpha}{\tau} \right)} \right)
\]

---

**Question #6**

**Question:** What is least clear?

A. Linear algebra, \( Ax = b \), determinants, eigenvalues  
B. Setting up \( \Delta y = K\Delta u \) word problems  
C. Setting up dynamic model word problems  
D. State space representation of dynamic system  
E. Laplace transforms

**Answer:** ???
16.9 Linearization

Linearization
Welcome to the real world!

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Lecture 9

Question #1 & 2

• Question: Have you looked through the handout online?
  A No.
  B Yes and it was NOT helpful.
  C Yes and it was helpful.
  D I will before the quiz.
  E This is not an answer. I will get you.

• Answer: ???

• Question: Why do we learn about Laplace transforms?
  A To help get analytical solutions to linear ODEs
  B To allow us a simple method to get analytical ODE solutions
  C To allow us quick analysis using IVT and FVT
  D It is on the FE exam
  E All of the above

• Answer: ???
Question #3

- **Question:** Who is this guy?
  A. George Washington (1732-1799)
  B. Pierre-Simon Laplace (1749-1827)
  C. Brook Taylor (1685-1731)
  D. Oliver Heaviside (1850-1925)
  E. I have no clue and don’t really care.

- **Answer:** ???

---

**First-Order+ Time Delay Step Response**

\[
\tau \frac{dy}{dt}(t) = -y(t) + K u(t - \alpha) \quad y(t = 0) = 0
\]
\[
\tau s y(s) - 0 = -y(s) + K e^{-\alpha s} u(s)
\]
\[
y(s) = \left( \frac{K}{\tau s + 1} \right) e^{-\alpha s} u(s)
\]
\[
u(t) = A H(t) \quad \Rightarrow \quad u(s) = A \left( \frac{1}{s} \right)
\]
\[
y(s) = \left( \left( \frac{K}{\tau s + 1} \right) \left( \frac{A}{s} \right) \right) e^{-\alpha s} = \left( KA \frac{1}{s} + (-KA) \frac{1}{s + \frac{1}{\tau}} \right) e^{-\alpha s} = y_1(s) + y_2(s)
\]
\[
y(t) = (KA) H(t - \alpha) + (-KA) H(t - \alpha) e^{-\left( \frac{t-\alpha}{\tau} \right)} = H(t - \alpha) (KA) \left( 1 - e^{-\left( \frac{t-\alpha}{\tau} \right)} \right)
\]
First-Order + Time Delay Step Response

$u(t)$, $\Delta u$

$y(t)$, $\alpha$, $\tau$, $63\% \Delta y$, $\Delta y = K \Delta u$

$t = 0$, $t = \alpha + \tau$

Linearization

Nonlinear function, $f(x(t))$

Approximate value for $f(x(t))$

$f(x(t)) \approx f(x(t)|_{x=x_{SS}}) + \left( \frac{\partial f}{\partial x}(x(t)) \right)_{x=x_{SS}} (x(t) - x_{SS}) + \ldots H.O.T.$

Current value, $x(t)$

$x_{SS}$, $(x(t) - x_{SS})$
Linearization

Space for Notes Below

Linearization Example

Space for Notes Below
2. **(30 pts. total)** Develop a dynamic model of your office. Assume the following:

- The offices have no air moving in or out, but the air in the offices is well-mixed (fans)
- The volume of air in each office is \( V_1 \) and \( V_2 \) respectively
- The heat capacity and density of the air in each office is \( C_p \) and \( \rho \)
- The rate of energy entering the each office from the outside is equal:
  \[ Q_{in}(t) = c_1(T_0(t))^4 \]
- The rate of energy transferred from office 2 to office 1 across the thin wall is:
  \[ Q_{w}(t) = kA(T_2(t) - T_1(t)) \]
- Physical properties and parameters do not change with time
- Deviation values are:
  \[ x_1(t) = T_1(t) - T_{1,ss} \]
  \[ x_2(t) = T_2(t) - T_{2,ss} - y(t) \]
  \[ u_1(t) = T_1(t) - T_{1,ss} \]
  \[ u_2(t) = Q(t) - Q_{ch} \] (a chiller in Office 1)

a. **(10pts.)** Develop a dynamic model for this system.
b. **(10pts.)** Develop a linear dynamic model for this system in deviation form.
Linearization Example

No mass is moving, two rooms, well-mixed. 2 Energy balances:

\[ V_1 \rho \, C_p \frac{dT_1}{dt}(t) = Q_{in}(t) + Q_a(t) - Q(t) \]
\[ V_2 \rho \, C_p \frac{dT_2}{dt}(t) = Q_{in}(t) - Q_a(t) \]
\[ V_1 \rho \, C_p \frac{dT_1}{dt}(t) = c_1 \left( T_o(t) \right)^4 + hA\left( T_2(t) - T_1(t) \right) - Q(t) \]
\[ V_2 \rho \, C_p \frac{dT_2}{dt}(t) = c_1 \left( T_o(t) \right)^4 - hA\left( T_2(t) - T_1(t) \right) \]

There is a nonlinear term, so linearize at \( T_{OSS} \)
\[ c_1 \left( T_o(t) \right)^4 \approx c_1 \left( T_{OSS} \right)^4 + \left( 4c_1 \left( T_{OSS} \right)^3 \right) \left( T_o(t) - T_{OSS} \right) \]

Linearization Example

Put linearization back in balance

\[ V_1 \rho \, C_p \frac{dT_1}{dt}(t) \approx \left[ c_1 \left( T_{OSS} \right)^4 + \left( 4c_1 \left( T_{OSS} \right)^3 \right) \left( T_o(t) - T_{OSS} \right) \right] + hA\left( T_2(t) - T_1(t) \right) - Q(t) \]
\[ V_2 \rho \, C_p \frac{dT_2}{dt}(t) \approx \left[ c_1 \left( T_{OSS} \right)^4 + \left( 4c_1 \left( T_{OSS} \right)^3 \right) \left( T_o(t) - T_{OSS} \right) \right] - hA\left( T_2(t) - T_1(t) \right) \]

Balance 1, SS, and deviation form:

\[ V_1 \rho \, C_p \frac{dT_1}{dt}(t) \approx c_1 \left( T_{OSS} \right)^4 + \left( 4c_1 \left( T_{OSS} \right)^3 \right) \left( T_o(t) - T_{OSS} \right) + hA\left( T_2(t) - T_1(t) \right) - Q(t) \]
\[ 0 \approx c_1 \left( T_{OSS} \right)^4 + \left( 4c_1 \left( T_{OSS} \right)^3 \right) \left( T_o(t) - T_{OSS} \right) + hA\left( T_{2ss} - T_{1ss} \right) - Q_{ss} \]
\[ V_1 \rho \, C_p \frac{dT_1}{dt}(t) \approx 0 + \left( 4c_1 \left( T_{OSS} \right)^3 \right) \left( T_o(t) - T_{OSS} \right) + hA\left( T_2(t) - T_{2ss} \right) - hA\left( T_1(t) - T_{1ss} \right) \]
\[ - \left( Q(t) - Q_{ss} \right) \]
\[ V_1 \rho \, C_p \frac{dx_1}{dt}(t) = \left( 4c_1 \left( T_{OSS} \right)^3 \right) \, u_1(t) + hA\, x_2(t) - hA\, x_1(t) - u_2(t) \]
Linearization Example

Equations in SS form:

\[
\begin{align*}
\frac{dx_1}{dt}(t) &= - \frac{hA}{V_1 \rho C_p} x_1(t) + \frac{hA}{V_1 \rho C_p} x_2(t) + \frac{(4c_1(T_{ss})^3)}{V_1 \rho C_p} u_1(t) - w_2(t) \\
\frac{dx_2}{dt}(t) &= \frac{hA}{V_2 \rho C_p} x_1(t) - \frac{hA}{V_2 \rho C_p} x_2(t) + \frac{(4c_1(T_{ss})^3)}{V_2 \rho C_p} u_1(t) + 0 w_2(t) \\
y(t) &= 0 \cdot x_1(t) + 1 \cdot x_2(t) + 0 u_1(t) + 0 w_2(t)
\end{align*}
\]

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} = 
\begin{bmatrix}
\frac{dx_1}{dt}(t) \\
\frac{dx_2}{dt}(t)
\end{bmatrix} = 
\begin{bmatrix}
- \frac{hA}{V_1 \rho C_p} + \frac{hA}{V_1 \rho C_p} \\
\frac{hA}{V_2 \rho C_p} - \frac{hA}{V_2 \rho C_p}
\end{bmatrix} 
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + 
\begin{bmatrix}
\frac{(4c_1(T_{ss})^3)}{V_1 \rho C_p} \\
\frac{(4c_1(T_{ss})^3)}{V_2 \rho C_p}
\end{bmatrix} 
\begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix} - 1 
\begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix}
\]

\[
y = 
\begin{bmatrix}
0 & 1
\end{bmatrix} 
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + 
\begin{bmatrix}
0 & 0
\end{bmatrix} 
\begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix}
\]
16.10 Partial Fraction Expansion

Partial Fraction Expansion
Algebra is Fun!

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Lecture 10

Laplace Transforms

- The Laplace Transform and the Inverse Transform
  - Properties
- Laplace of Functions
  - Heaviside, Exponential, Derivatives, Time Shift
  - Composite functions (linear combinations)
- Initial and Final Value Theorems
- Solving ODEs using Laplace domain
  - Partial Fraction Expansion
Linearization Example

\[ A \frac{dh}{dt}(t) = F_i(t) - ch(t)^{2/3} \]
\[ A \frac{dh}{dt}(t) \approx F_i(t) - \left[ ch_{ss}^{2/3} + \frac{2}{3} ch_{ss}^{-1/3} (h(t) - h_{ss}) \right] \]
\[ 0 = F_{so} - \left[ ch_{ss}^{2/3} + 0 \right] \]
\[ A \frac{dh}{dt}(t) \approx (F_i(t) - F_{so}) - \frac{2}{3} ch_{ss}^{-1/3} (h(t) - h_{ss}) \]
\[ A \frac{dy}{dt}(t) = u(t) - \frac{2}{3} ch_{ss}^{-1/3} y(t) \]
\[ A \Sigma y(s) = u(s) - \frac{2}{3} ch_{ss}^{-1/3} y(s) \]

\[ y(s) = \frac{1}{A_s + 4 ch^{-1/3}} u(s) - \frac{\frac{2}{3} ch_{ss}^{1/3}}{A_s + \frac{2}{3} ch_{ss}^{1/3}} u(s) = \frac{K}{\tau s + 1} u(s) \]

Given process values, the system gain is \( K = 2.4 \) and time constant \( \tau = 0.6 \). Given a change in the flow of 0.2 \((\Delta u)\) the resulting change in the measurement is \( \Delta y = 0.48 \). Since the initial height is 0.512, the resulting height is 0.992 and it does not overflow. However, solving the nonlinear equation at SS with that flow:

\[ 0 = 0.52 - (0.5) (h_{ss})^{2/3} \]

Gives a \( h_{ss} = 1.06 \) and it does overflow.

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Linearization Example

\[ \frac{dx_1}{dt}(t) = a_1 \xi_1(t) - a_2 \xi_1(t) \xi_2(t) + b_1(t) \approx a_1 \xi_1(t) - a_1 [ \xi_1(t) - \xi_{1ss}) + \xi_2(t) - \xi_{2ss}] + b_1(t) \]
\[ \frac{dx_2}{dt}(t) = a_2 \xi_2(t) + a_2 \xi_1(t) \xi_2(t) = a_0 \xi_2(t) + a_2 [ \xi_2(t) - \xi_{2ss}] + b_2(t) \]

Subtract steady state values and collect deviation variable terms:

\[ \frac{dx_1}{dt}(t) = a_1 \xi_1(t) - a_1 [ \xi_1(t) - \xi_{1ss}) + \xi_2(t) - \xi_{2ss}] + b_1(t) \]
\[ \frac{dx_2}{dt}(t) = a_2 [ \xi_2(t) - \xi_{2ss}] + b_2(t) \]

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Review: Linearization

Nonlinear function of 1 variable:

\[
f(x(t)) \approx f(x(t))|_{x=x_{SS}} + \left. \frac{\partial f}{\partial x}(x(t)) \right|_{x=x_{SS}} (x(t) - x_{SS}) + \ldots
\]

Nonlinear function of 2 variables:

\[
f(x(t), y(t)) \approx f(x(t), y(t))|_{x=x_{SS}, \ y=y_{SS}} + \left. \frac{\partial f}{\partial x}(x(t), y(t)) \right|_{x=x_{SS}, \ y=y_{SS}} (x(t) - x_{SS}) + \left. \frac{\partial f}{\partial y}(x(t), y(t)) \right|_{x=x_{SS}, \ y=y_{SS}} (y(t) - y_{SS}) + \ldots
\]
Question #1

- Question: Given the following model and input forcing function, what is the value of $y(t = 0)$? (Hint, IVT)

$$y(s) = \frac{7}{2s + 1} u(s) \quad u(s) = 1$$


- Answer: ???

---

Question #2

- Question: What is the minimum eigenvalue of the following matrix?

$$\begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$$

[A] 3  [B] 0  [C] -2  [D] None of these  [E] What?

- Answer: ???
Partial Fraction Expansion

Space for Notes Below

Partial Fraction Expansion Example

\[ y(s) = \left( \frac{6}{2s+1} \right) \left( \frac{3}{s} \right) = \frac{18}{(2s+1)(s)} = \frac{9}{(s+\frac{1}{2})(s+0)} \]

\[ \frac{9}{(s+\frac{1}{2})(s+0)} = \frac{A_1}{(s+\frac{1}{2})} + \frac{A_2}{(s+0)} \]

\[ \frac{9(s+\frac{1}{2})}{(s+\frac{1}{2})(s+0)} = A_1(s+\frac{1}{2}) + A_2(s+0) \]

\[ A_1 = \left. \frac{9}{s+0} \right|_{s=-\frac{1}{2}} = -18 \]

\[ A_2 = \left. \frac{9(s+0)}{(s+\frac{1}{2})} \right|_{s=0} = 18 \]
Partial Fraction Expansion Example

\[
y(s) = \frac{-18}{s + \frac{1}{2}} + \frac{18}{s + 0}
\]

\[
y(s) = \frac{-18(s + 0)}{(s + \frac{1}{2})(s + 0)} + \frac{18(s + \frac{1}{2})}{(s + 0)(s + \frac{1}{2})} = \frac{9}{(s + \frac{1}{2})(s + 0)}
\]

\[
y(t) = (-18)H(t)e^{-\frac{1}{2}t} + (18)H(t)
\]

Partial Fraction Expansion Example

Find the analytical unit impulse response of the following differential equation. 0 ICs:

\[
\frac{d^2 y}{dt^2}(t) + 5 \frac{dy}{dt}(t) + 6y(t) = 4u(t)
\]

\[
s^2y(s) + 5sy(s) + 6y(s) = 4u(s)
\]

\[
y(s)\left(s^2 + 5s + 6\right) = 4u(s)
\]

\[
y(s) = \frac{4}{s^2 + 5s + 6} u(s) = g(s) u(s)
\]

Unit impulse input:

\[
u(t) = \delta(t) \quad \text{so} \quad u(s) = 1
\]

\[
y(s) = \left(\frac{4}{s^2 + 5s + 6}\right)(1) = \frac{4}{(s + 2)(s + 3)} = \frac{A_1}{s + 2} + \frac{A_2}{s + 3}
\]
Partial Fraction Expansion Example

\[ A_1 = \left. \frac{4(s + 2)}{(s + 2)(s + 3)} \right|_{s = -2} = \frac{4}{-2 + 3} = -4 \]

\[ A_2 = \left. \frac{4(s + 3)}{(s + 2)(s + 3)} \right|_{s = -3} = \frac{4}{-3 + 2} = -4 \]

\[ y(s) = \left( \frac{4}{s^2 + 5s + 6} \right) (1) = \frac{4}{(s + 2)} + \frac{-4}{(s + 3)} \]

\[ y(t) = (4) \mathcal{H}(t) e^{-2t} + (4) \mathcal{H}(t) e^{-3t} \]

Check IVT and FVT

\[ y(t = 0) = 0 \quad y(t = \infty) = 0 \]

\[ sy(s) \bigg|_{s = \infty} = \frac{\infty}{(\infty)(\infty)} = 0 \quad sy(s) \bigg|_{s = 0} = \frac{0}{(2)(3)} = 0 \]
16.11 Low Order Systems

Low-Order Systems
Simple Systems

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Lecture 11

Accumulation Terms

mass: \( \frac{d}{dt} \rho V(t) \rightarrow \frac{d}{dt} (\rho A) \frac{dh}{dt}(t) \)

moles (PV = nRT): \( \frac{dn}{dt}(t) \rightarrow \frac{V}{RT} \frac{dP}{dt}(t) \)

moles, concentration: \( \frac{d}{dt} (VC_A(t)) \rightarrow (V) \frac{dC_A}{dt}(t) \)

energy: \( \frac{d}{dt} (\rho VC_p T(t)) \rightarrow (\rho VC_p) \frac{dT}{dt}(t) \)
Partial Fraction Expansion

\[ y(s) = g(s)u(s) = \left( \frac{N(s)e^{-\alpha s}}{D(s)} \right) u(s) \]

**Zeroes of** \( g(s) \): **Roots of** \( N(s) \) **from** \( g(s) \)

**Poles of** \( g(s) \): **Roots of** \( D(s) \) **from** \( g(s) \)

\[ y(s) = \frac{\hat{N}(s)}{(s-p_1)(s-p_1)...(s-p_n)} = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_2)} + ... + \frac{A_n}{(s-p_n)} \]

For each root of denominator, multiply by \((s-p_i)\) and evaluate at \(s = p_i\)

\[ \frac{\hat{N}(s)}{\hat{D}(s)}(s-p_1) \bigg|_{s=p_1} = A_1 + \frac{A_2(s-p_1)}{(s-p_2)} \bigg|_{s=p_1} + ... + \frac{A_n(s-p_1)}{(s-p_n)} \bigg|_{s=p_1} = A_1 \]

---

Partial Fraction Expansion Example

\[ y(s) = \left( \frac{6}{2s+1} \right) \left( \frac{3}{s} \right) - \frac{18}{(2s+1)(s+4)} - \frac{9}{s+\frac{1}{2}}(s+0) \]

\[ \frac{9}{s+\frac{1}{2}}(s+0) = A_1 + \frac{A_2}{s+\frac{1}{2}}(s+0) \]

\[ A_1 = \frac{9}{s=\frac{1}{2}} = -18 \]

\[ \frac{9}{s+\frac{1}{2}}(s+0) = A_1 \left( s + \frac{1}{2} \right) + A_2 \left( s + \frac{1}{2} \right) \]

\[ A_2 = \frac{9}{s=0} = -18 \]

\[ y(s) = \frac{-18}{s + \frac{1}{2}}(s+0) + \frac{18}{(s+\frac{1}{2})(s+0)} \]

\[ y(t) = (-18) \mathcal{H}(t)e^{-\frac{t}{2}} + 18 \mathcal{H}(t) \]

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Question #1

- **What is the gain of this system response?**


- **Answer: ???**

Question #2

- **What is the time delay of this system response?**


- **Answer: ???**
Question #3

What is the time constant of this system response?


Answer: ???

First-Order System
First-Order System Examples

Liquid level in a single tank:

\[ \begin{aligned}
\frac{dA}{dt}(t) &= F_m(t) - k h(t) \\
A(s)h(s) &= F_m(s) - k h(s) \\
A(s)h(s) + k h(s) &= F_m(s) \\
h(s) (A + k) &= F_m(s) \\
\end{aligned} \]

\[ h(s) = \left( \frac{1}{As + k} \right) F_m(s) = \left( \frac{\frac{1}{s}}{\frac{1}{s} + 1} \right) F_m(s) \]

Simple CSTR:

\[ \frac{dC_A}{dt}(t) = F C_{A0} (t) - \frac{F C_A(t)}{V + kV} \]

\[ V s C_A(s) = F C_{A0}(s) - (F + kV) C_A(s) \]

\[ \frac{V s C_A(s) + (F + kV) C_A(s)}{V s (V + (F + kV))} = F C_{A0}(s) \]

\[ C_A(s) = \left( \frac{F}{V + (F + kV)} \right) C_{A0}(s) \]

\[ C_A(s) = \left( \frac{\frac{F}{V + (F + kV)}}{\frac{1}{s} + 1} \right) C_{A0}(s) \]

Hot water mixing tank:

\[ \begin{aligned}
\rho V C_p \frac{dT}{dt}(t) &= F_p C_p (T_o(t) - F_p C_p T(t)) \\
\rho V C_p s T(s) &= F_p C_p T_o(s) - F_p C_p T(s) \\
T(s) &= \left( \frac{\frac{F_p C_p}{\rho V C_p s + F_p C_p}}{\frac{1}{s} + 1} \right) T_o(s) \end{aligned} \]

First-Order Impulse Response

Space for Notes Below
First-Order Step Response

\[ y(s) = \left( \frac{K}{\tau s + 1} \right) u(s) \]

\[ u(t) = A\mathcal{H}(t) \quad \rightarrow \quad u(s) = A\left( \frac{1}{s} \right) \]

\[ y(s) = \left( \frac{K}{\tau s + 1} \right) \left( \frac{A}{s} \right) = (KA)\left( \frac{1}{s} \right) + (-KA)\left( \frac{1}{s + \frac{1}{\tau}} \right) \]

\[ y(t) = \mathcal{H}(t)(KA) \left( 1 - e^{-\left( \frac{t}{\tau} \right)} \right) \]

First-Order Pulse Response

\[ y(s) = \left( \frac{K}{\tau s + 1} \right) u(s) \]

\[ u(t) = A\mathcal{H}(t) - A\mathcal{H}(t - \alpha) \quad \rightarrow \quad u(s) = A\left( \frac{1}{s} \right) - A\left( \frac{1}{s} \right) e^{-\alpha s} \]

\[ y(s) = \left( \frac{KA}{(\tau s + 1)s} \right) + \left( \frac{-KA}{(\tau s + 1)s} \right) e^{-\alpha s} \]

\[ y(t) = \mathcal{H}(t)(KA) \left( 1 - e^{-\frac{t}{\tau}} \right) + \mathcal{H}(t - \alpha)(-KA) \left( 1 - e^{-\left( \frac{t - \alpha}{\tau} \right)} \right) \]
First-Order Ramp Response

\[ y(s) = \left( \frac{K}{\tau s + 1} \right) u(s) \]

\[ u(t) = \sigma \ t \ \mathcal{H}(t) \ \rightarrow \ u(s) = \sigma \left( \frac{1}{s^2} \right) \]

\[ y(s) = ??? \ Repeated \ Roots \]

\[ y(t) = \mathcal{H}(t)(AK\tau) \left( e^{-\left( \frac{t}{\tau} \right)} + \left( \frac{t}{\tau} \right) - 1 \right) \]

\[ y(t) \]

\[ m = \sigma K (t - \tau) \]

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First-Order+ Time Delay Step Response

\[ y(s) = \left( \frac{K}{\tau s + 1} e^{-\alpha s} \right) u(s) \]

\[ u(t) = A\mathcal{H}(t) \ \rightarrow \ u(s) = A \left( \frac{1}{s} \right) \]

\[ y(s) = \left( \frac{K}{\tau s + 1} \right) \left( \frac{A}{s} \right) e^{-\alpha s} = \]

\[ y(t) = \mathcal{H}(t - \alpha)(KA) \left( 1 - e^{-\left( \frac{t - \alpha}{\tau} \right)} \right) \]

\[ u(t) \]

\[ a(t) \]

\[ t = 0 \quad t = \alpha \]

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Why 63%?

\[ y(t) = \mathcal{H}(t - \alpha)(KA) \left(1 - e^{-\left(\frac{t-\alpha}{\tau}\right)}\right) \]

Assuming step at \( t = 0 \), set \( t = \alpha + \tau \)

\[ y(t = \alpha + \tau) = \mathcal{H}(\alpha + \tau - \alpha)(KA) \left(1 - e^{-\left(\frac{\alpha + \tau - \alpha}{\tau}\right)}\right) \]

\[ y(t = \alpha + \tau) = \mathcal{H}(\tau)(KA) \left(1 - e^{-\left(\frac{\tau}{\tau}\right)}\right) = (1)(KA) \left(1 - e^{-1}\right) = 0.632 \text{ KA } \approx 66\% \text{ KA} \]

---

First-Order+ Time Delay Step Response

- Both \( u \) and \( y \) must start and end at steady state
- Determine which is \( u \) and which is \( y \)
- Determine Step Time (change in input \( u \), not always 0)
- Determine \( \Delta y \) and \( \Delta u \)
- Determine Steady State gain using \( \Delta y = K \Delta u \)
- Determine time delay \( \alpha \) relative to step time
- Determine time constant \( \tau \) using 63% rule
- Present FOTD model:

\[ g(s) = \frac{K}{\tau s + 1} e^{-\alpha s} = \frac{y(s)}{u(s)} \]

\[ \tau \frac{dy}{dt}(t) + y(t) = Ku(t - \alpha) \]
Pure Gain System

Space for Notes Below

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Pure Capacity System

Space for Notes Below

---
Pure Capacity System Responses

\[ u(t) \]
\[ y(t) \]
\[ f = 0 \]
\[ y(s) = g(s)u(s) \]
\[ u(s) = \frac{1}{s} \]
\[ y(s) = \left( \frac{1}{s^2} \right) u(s) \]

Lead-Lag System

Space for Notes Below
16.12 High Order Systems

Higher-Order Systems
More than one state!

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Lecture 12

Low- and High-Order Systems

- Transfer function representation
  - $y(s) = g(s)u(s)$ or maybe $y(s) = g(s)u(s) + g_d(s)d(s)$
- Gain of $g(s)$: Just evaluate $g(s)$ with $s = 0$
  - Like unit step input ($u(s) = \frac{1}{s}$) then FVT
- Poles: roots of Denominator of $g(s)$
  - Stability, oscillations
- Zeros: roots of Numerator of $g(s)$
  - Inverse response
Question #1 & 2

- **Question:** What will help you most in this class?

[D] More examples and labs  [E] I am dropping this class

- **Answer:** ???

- **Question:** What is the gain of this system?

$$g(s) = \frac{2s + 3}{s^2 + 2s + 9}$$


- **Answer:** ???

---

**Pure Capacity System Responses**

Unit impulse response:
$$u(t) = \delta(t)$$
$$y(s) = \left( \frac{K}{s} \right) 1$$
$$y(t) = K \mathcal{H}(t)$$

Unit step response,
$$u(t) = \mathcal{H}(t)$$
$$y(s) = \left( \frac{K}{s} \right) \left( \frac{1}{s} \right) = K \left( \frac{1}{s^2} \right)$$
$$y(t) = K t \mathcal{H}(t)$$
Lead-Lag System

\[
\begin{align*}
\tau \frac{dy}{dt}(t) + y(t) &= K \left( \xi \frac{du}{dt}(t) + u(t) \right) \\
\tau s y(s) + y(s) &= K (\xi su(s) + u(s)) \\
y(s) &= \left( \frac{K (\xi s + 1)}{(\tau s + 1)} \right) u(s) \\
y(s) &= K \left( \frac{\xi + \frac{1 - \xi}{\tau}}{(\tau s + 1)} \right) u(s)
\end{align*}
\]

Example: CSTR with bypass

Low- and High-Order Systems

- Transfer function representation
  \[ y(s) = g(s) u(s) \] or maybe \[ y(s) = g(s) u(s) + g_d(s) d(s) \]

- Low-Order
  - First order \[ g(s) = \frac{K}{\tau s + 1} \]
  - Pure gain \[ g(s) = K \] and pure capacity \[ g(s) = \frac{1}{s} \]
  - Lead-Lag \[ g(s) = \frac{K (\xi s + 1)}{(\tau s + 1)} = A_0 + \frac{A_1}{(\tau s + 1)} \]

- High-Order
  - Second Order \[ g(s) = \frac{K}{\tau^2 s^2 + 2\tau\xi s + 1} \]
  - Time-delay \[ g(s) = e^{-\alpha s} \]
  - FOTD \[ g(s) = \frac{K}{\tau s + 1} e^{-\alpha s} \]
  - General high order \[ g(s) = \frac{K}{(\tau s + 1)^N} \]
**Easy: Two Tanks in Series**

- Two tanks in series

\[
F_w(t) \quad h_1(t) \quad A_1 \frac{dh_1}{dt}(t) = F_{in}(t) - k_1 h_1(t) \\
F_i(t) - k_i h_i(t) \quad \quad h_2(t) \quad \quad A_2 \frac{dh_2}{dt}(t) = k_1 h_1(t) - k_2 h_2(t) \\
F_w(t) = k_2 h_2(t)
\]

- Second has no effect on first, resulting \( g(s) = g_1(s) g_2(s) \)

\[
g(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} = \frac{K_1}{(\tau_1 s + 1)} \frac{K_2}{(\tau_2 s + 1)}
\]

- Step response, \( u(s) = \frac{1}{s} \)

\[
y(t) = K \left( 1 - \left( \frac{\tau_1}{\tau_1 - \tau_2} \right) e^{-t/\tau_1} - \left( \frac{\tau_2}{\tau_2 - \tau_1} \right) e^{-t/\tau_2} \right)
\]

**Easy: Two Interacting Tanks**

- Two interacting tanks, flow between is \( k(h_1(t) - h_2(t)) \)

\[
F_{in}(t) \quad h_1(t) \quad A_1 \frac{dh_1}{dt}(t) = F_{in}(t) - k(h_1(t) - h_2(t)) \\
F(t) - k(h_1(t) - h_2(t)) \quad \quad h_2(t) \quad \quad A_2 \frac{dh_2}{dt}(t) = k(h_1(t) - h_2(t)) - k_2 h_2(t) \\
F_2(t) = k_2 h_2(t)
\]

- Second now has effect on first, resulting \( g(s) \)

\[
g(s) = \frac{K}{\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2 + K \tau_2) s + 1}
\]
General Second Order

Second-Order Under-Damped Examples

- $\zeta > 1$ is over damped
- $\zeta = 1$ is critically damped
- $\zeta < 1$ is underdamped
  - Imaginary poles: oscillations
- U-tube monometer
  - Low friction, liquid “bounces”
- Spring + shock absorber
  - Bad shocks, can oscillate
General Second-Order Step Response

Space for Notes Below

General $N^{th}$ Order Step Response

Space for Notes Below
Question #3

Question: What best describes this system?

\[ g(s) = \frac{3}{s^2 + 1s + 9} \]

A First-order
B Lead-lag
C Pure gain
D Pure capacity
E Second-order

Answer: ???

Question #4

Question: What best describes this system?

\[ g(s) = \frac{3}{s^2 + 1s + 9} \]

A Under-damped
B Critically-damped
C Over-damped
D Unstable
E ????

Answer: ???
Question #5

- Question: What is still most confusing?
  A Dynamic Modeling
  B Laplace Transforms
  C Linear Algebra, $\Delta y = K \Delta u$
  D Step response data for $K, \tau, \alpha$
  E Other
- Answer: ???

Homework

A. Determine the analytical unit step response $y(t)$ for the following transfer functions and plot $y(t)$
B. Determine the analytical unit impulse response $y(t)$ for the following transfer functions and plot $y(t)$

$$g(s) = \frac{y(s)}{u(s)} = \frac{5}{(3s+1)(2s+1)}$$
$$g(s) = \frac{y(s)}{u(s)} = \frac{3(4s+1)}{(2s+1)}$$
$$g(s) = \frac{y(s)}{u(s)} = \frac{1}{(5s+1)(8s+1)(s+1)}$$
$$g(s) = \frac{y(s)}{u(s)} = \frac{5}{(3s+1)(2s+1)}$$
$$g(s) = \frac{y(s)}{u(s)} = \frac{3(4s+1)}{(2s+1)}$$
$$g(s) = \frac{y(s)}{u(s)} = \frac{1}{(5s+1)(8s+1)(s+1)}$$

For part A, $u(t) = H(t)$ so $u(s) = \frac{1}{s}$. Do PFE, get $y(s) = y_1(s) + y_2(t) + ... + y_n(s)$. Then $L^{-1}$ to get $y(t)$. Check with IVT and FVT. For part B, $u(t) = \delta(t)$ so $u(s) = 1$. Do PFE, get $y(t)$
16.13 High Order Systems

Higher-Order Systems
More than one state!

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Lecture 13

Low- and High-Order Systems

- Transfer function representation
  \[ y(s) = g(s)u(s) \text{ or maybe } y(s) = g(s)u(s) + g_d(s)d(s) \]
- **Low-Order**
  - First order \[ g(s) = \frac{K}{\tau s + 1} \]
  - Pure gain \[ g(s) = K \] and pure capacity \[ g(s) = \frac{1}{s} \]
  - Lead-Lag \[ g(s) = \frac{K(\zeta s + 1)}{(\tau s + 1)} = A_0 + \frac{A_1}{(\tau s + 1)} \]
- **High-Order**
  - Second Order \[ g(s) = \frac{K}{\tau^2 s^2 + 2\tau\zeta s + 1} \]
  - Time-delay \[ g(s) = e^{-\alpha s} \]
  - FOTD \[ g(s) = \frac{K}{\tau s + 1} e^{-\alpha s} \]
  - General high order \[ g(s) = \frac{K}{(\tau s + 1)^N} \]
**Batch Reactor Schematic**

*Inputs: deviation $C_{Ao}(t)$, $T_{o}(t)$*

\[ j(t) = D (C_{Ao}(t) - C_A(t)) \]

\[ y_1(t) = C_A(t) - C_{As} \]
\[ y_2(t) = T(t) - T_{ss} \]

\[ Q(t) = hA (T(t) - T_{o}(t)) \]

\[ A \rightarrow B \]

\[ r(t) = k (C_A(t))^2 \]

\[ V, \rho, C_p, -\Delta H \]


---

**BOP: Big Overall Problem**

- **Notes:**
  - No flow in and out
  - No balance on $B$ needed, but you can do it

- Develop a linear dynamic model
  - Two inputs (deviation $C_{Ao}(t)$ and $T_{o}(t)$)
  - Two outputs (deviation $C_A(t)$ and $T(t)$)
  - Linearize if needed and introduce deviation variables

- Put the model in state space form and identify $A$, $B$, $C$, $D$

- Now, assume the following for $y$, $u$, and $d$
  - $y(t) = T(t) - T_{ss}$, $u(t) = C_{Ao}(t) - C_{As}$ and $d(t) = T_{o}(t) - T_{oss}$
  - Develop a model in the form $y(s) = g(s) u(s) + g_d(s) d(s)$
  - Find $y(t)$ for a unit step in $u$ while $d(t) = 0$
Mass Balance

Mass balance on $A$

$$V \frac{dC_A}{dt}(t) = D(C_{Ao}(t) - C_A(t)) - Vk(C_A(t))^2$$

Need to linearize $(C_A(t))^2$

$$(C_A(t))^2 \approx (C_{Ass})^2 + (2C_{Ass})(C_A(t) - C_{Ass}) + ...$$

$$V \frac{dC_A}{dt}(t) \approx D(C_{Ao}(t) - C_A(t)) - Vk[(C_{Ass})^2 + (2C_{Ass})(C_A(t) - C_{Ass})]$$

$$V \frac{dC_A}{dt}(t) \approx DC_{Ao}(t) - DC_A(t) - Vk(C_{Ass})^2 - Vk(2C_{Ass})(C_A(t) - C_{Ass})$$

$$0 \approx DC_{Ass} - DC_{Ass} - Vk(C_{Ass})^2 - Vk(2C_{Ass})(0)$$

$$V \frac{dC_A}{dt}(t) \approx D(C_{Ass}(t) - C_{Ass}) - D(C_A(t) - C_{Ass}) - 0 - Vk(2C_{Ass})(C_A(t) - C_{Ass})$$

$$V \frac{dx_1}{dt}(t) \approx Du_1(t) - Dx_1(t) - Vk(2C_{Ass})x_1(t)$$

Energy Balance

$$(\rho C_p) \frac{dT}{dt}(t) = -hA(T(t) - T_{o}(t)) + (-\Delta H) Vk(C_A(t))^2$$

Need to include linearized $(C_A(t))^2$

$$(\rho C_p) \frac{dT}{dt}(t) \approx -hA(T(t) - T_{o}(t)) + (-\Delta H) Vk[(C_{Ass})^2 + (2C_{Ass})(C_A(t) - C_{Ass})]$$

$$(\rho C_p) \frac{dT}{dt}(t) \approx -hAT(t) + hAT_{o}(t) + (-\Delta H) Vk(C_{Ass})^2 + (-\Delta H) Vk(2C_{Ass})(C_A(t) - C_{Ass})$$

$$0 \approx -hAT_{Ass} + hAT_{Ass} + (-\Delta H) Vk(C_{Ass})^2 + \Delta H$$

$$(\rho C_p) \frac{dT}{dt}(t) \approx -hA(T(t) - T_{Ass}) + hA(T_{o}(t) - T_{Ass}) + 0 + (-\Delta H) Vk(2C_{Ass})(C_A(t) - C_{Ass})$$

$$V \frac{dx_2}{dt}(t) \approx -hAx_2(t) + hAx_2(t) + (-\Delta H) Vk(2C_{Ass})x_1(t)$$
State Space

\[
V \frac{dx_1}{dt}(t) \simeq D u_1(t) - D x_1(t) - V k(2 C_{Ass}) x_1(t)
\]

\[
V p C_p \frac{dx_2}{dt}(t) \simeq -hA x_2(t) + hA u_2(t) + (-\Delta H) V k(2 C_{Ass}) x_1(t)
\]

\[
V \frac{dx_1}{dt}(t) \simeq (-V k(2C_{Ass}) - D) x_1(t) + 0x_2(t) + D u_1(t) + 0 u_2(t)
\]

\[
V p C_p \frac{dx_2}{dt}(t) \simeq ((-\Delta H) V k(2C_{Ass})) x_1(t) - hAx_2(t) + 0u_1(t) + hAu_2(t)
\]

\[
\begin{bmatrix}
\frac{dx_1}{dt}(t) \\
\frac{dx_2}{dt}(t)
\end{bmatrix} \simeq \begin{bmatrix}
\frac{-V k(2C_{Ass}) - D}{V} x_1(t) + 0x_2(t) + \frac{D}{V} u_1(t) + 0 u_2(t) \\
\frac{-hA}{V p C_p} x_2(t) + 0u_1(t) + \frac{hA}{V p C_p} u_2(t)
\end{bmatrix}
\]

State Space

\[
y_1(t) = C_A(t) - C_{Ass} = x_1(t)
\]

\[
y_2(t) = T(t) - T_{ss} = x_2(t)
\]

\[
y_1(t) = x_1(t) + 0x_2(t) + 0u_1(t) + 0 u_2(t)
\]

\[
y_2(t) = x_1(t) + 0x_2(t) + 1u_1(t) + 0 u_2(t)
\]

\[
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} \simeq \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix}
\]

Remember: States found from variables in accumulation terms. In this problem, \(x_1(t) = C_A(t) - C_{Ass}\) and \(x_2(t) = T(t) - T_{ss}\). Inputs and outputs were specified in the problem statement.
Transfer Function

\[ V \frac{dx_1}{dt}(t) = (-V_k(2C_{Ass}) - D)x_1(t) + 0x_2(t) + Du_1(t) + 0u_2(t) \]

\[ VpC_p \frac{dx_2}{dt}(t) = ((-\Delta H) V_k(2C_{Ass})) x_1(t) - hA x_2(t) + 0u_1(t) + hA u_2(t) \]

Now use \( u(s) = u_1(s) \) and \( d(s) = u_2(s) \) from the problem statement.

\[ (V) s x_1(s) = (-V_k(2C_{Ass}) - D)x_1(s) + Du(s) \]

\[ (VpC_p) s x_2(s) = ((-\Delta H) V_k(2C_{Ass})) x_1(s) + (-hA) x_2(s) + (hA) d(s) \]

\[ x_1(s) = \frac{D}{Vs + (D + V_k(2C_{Ass}))} u(s) \quad \text{(from above)} \]

\[ (VpC_p) s x_2(s) + (hA) x_2(s) \simeq ((-\Delta H) V_k(2C_{Ass})) x_1(s) + (hA) d(s) \quad \text{(from above)} \]

Use equation for \( x_1(s) \) in second equation involving \( x_2(s) \):

\[ ((VpC_p) s + (hA)) x_2(s) = \frac{D((-\Delta H) V_k(2C_{Ass}))}{Vs + (D + V_k(2C_{Ass}))} u(s) + (hA) d(s) \]

\[ y(s) = x_2(s) = \frac{D((-\Delta H) V_k(2C_{Ass}))}{((VpC_p) s + (hA)) (Vs + (D + V_k(2C_{Ass})))} u(s) + \frac{(hA)}{((VpC_p) s + (hA))} d(s) \]

**Transfer Function Input Step Response**

\[ y(s) = \frac{D((-\Delta H) V_k(2C_{Ass}))}{((VpC_p) s + (hA)) (Vs + (D + V_k(2C_{Ass})))} \frac{u(s)}{s} + \frac{(hA)}{((VpC_p) s + (hA))} \frac{d(s)}{s} \]

Know \( u(t) \) and \( d(t) \) from problem statement:

\[ u(s) = \frac{1}{s} \quad d(s) = 0 \]

\[ y(s) = \frac{D((-\Delta H) V_k(2C_{Ass}))}{(hA)(Vs + (D + V_k(2C_{Ass}))} \frac{1}{s} + 0 \]

General form with gain and time constants:

\[ y(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \frac{1}{s} \]

\[ = \frac{\left(\frac{K}{\tau_1}\right)}{s + \frac{1}{\tau_1}} \frac{1}{s} \]

\[ = \frac{A_0}{s + \frac{1}{\tau_1}} + \frac{A_1}{s + \frac{1}{\tau_2}} + \frac{A_2}{s} \]
Transfer Function Input Step Response

This is in form of 6.11 in book. Step response of two non-interacting systems

\[ y(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \left( \frac{1}{s} \right) = g(s) \left( \frac{1}{s} \right) \]

Unit step response will be:

\[ y(t) - K \left( 1 - \left( \frac{\tau_1}{\tau_1 - \tau_2} \right) e^{-t/\tau_1} - \left( \frac{\tau_2}{\tau_2 - \tau_1} \right) e^{-t/\tau_2} \right) \]

Derivation using PFE:

\[ \frac{\left( \frac{K}{\tau_1 \tau_2} \right)}{(s + \frac{1}{\tau_1})(s + \frac{1}{\tau_2})} \left( \frac{1}{s} \right) = \frac{A_0}{(s + \frac{1}{\tau_1})} + \frac{A_1}{(s + \frac{1}{\tau_2})} + \frac{A_2}{s} \]

Transfer Function Step Response

\[ \left. \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) \right|_{s = -1/\tau_1} = A_0 = \left. \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) \right|_{s = -1/\tau_1} = \frac{-K\tau_1}{(-\tau_1 + \tau_2)} = -K \left( \frac{\tau_1}{\tau_1 - \tau_2} \right) \]

\[ \left. \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) \right|_{s = -1/\tau_2} = A_1 = \left. \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) \right|_{s = -1/\tau_2} = \frac{-K\tau_2}{(-\tau_2 + \tau_1)} = -K \left( \frac{\tau_2}{\tau_2 - \tau_1} \right) \]

\[ \left. \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) \right|_{s = 0} = A_2 = \left. \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) \right|_{s = 0} = -K \]

\[ y(s) = \left( \frac{K}{\tau_1 \tau_2} \right) \left( \frac{1}{s} \right) = \frac{-K}{(s + \frac{1}{\tau_1})} + \frac{-K}{(s + \frac{1}{\tau_2})} + \frac{K}{s} \]

\[ y(t) = \left( \frac{-K\tau_1}{\tau_1 - \tau_2} \right) \mathcal{H}(t)e^{-\left( \frac{1}{\tau_1} \right)t} + \left( \frac{-K\tau_2}{\tau_2 - \tau_1} \right) \mathcal{H}(t)e^{-\left( \frac{1}{\tau_2} \right)t} + K \mathcal{H}(t) \]

\[ y(t) = K \left( 1 - \left( \frac{\tau_1}{\tau_1 - \tau_2} \right) e^{-t/\tau_1} - \left( \frac{\tau_2}{\tau_2 - \tau_1} \right) e^{-t/\tau_2} \right) \mathcal{H}(t) \]
16.14 Time Delay

Poles, Zeros, and Time-Delay Systems
Stability, Inverse Response, Stuff in Pipes

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Question #1

Question: What is NOT equivalent to the following?

\[(s + 2)(s + 4) \frac{1}{s}\]

A. \[(s + 2)\left(1 + \frac{4}{s}\right)\]
B. \[\left(1 + \frac{2}{s}\right) (s + 4)\]
C. \[(s^2 + 6s + 8) \frac{1}{s}\]
D. \[\left(s + 6 + \frac{8}{s}\right)\]
E. \[\left(1 + \frac{2}{s}\right) \left(1 + \frac{4}{s}\right)\]

Answer: ???
Question #2

● Question: If your input experience a step of magnitude 10, what is \( u(s) \)?

A \( u(s) = 1 \)
B \( u(s) = 10 \)
C \( u(s) = \frac{1}{s} e^{-1s} \)
D \( u(s) = 10 \frac{1}{s} \)
E I really don’t get this at all.

● Answer: ???

General Second Order

● For general two-state system

\[
\tau^2 \frac{d^2 y}{dt^2}(t) + 2\zeta \tau \frac{dy}{dt}(t) + y(t) = K u(t)
\]

● Take Laplace:

\[
\tau^2 s^2 y(s) + 2\zeta \tau s y(s) + y(s) = K u(s)
\]

● Algebra:

\[
y(s) = \left( \frac{K}{\tau^2 s^2 + 2\zeta \tau s + 1} \right) u(s)
\]

● Dampening coefficient \( \zeta \)

- \( \zeta > 1 \) is over damped.
- \( \zeta = 1 \) is critically damped.
- \( \zeta < 1 \) is underdamped

- Imaginary poles: oscillations (underdamped)

- Two poles (roots of \( D(s) \)) and no zeros (roots of \( N(s) \))
General Second-Order Step Response

Second order unit step response, $K = 1$

$$y(t)$$

$$y(s) = \left( \frac{K}{\tau^2 s^2 + 2\tau\zeta s + 1} \right) u(s)$$

Poles: Stability

$$s = a \pm b\sqrt{-1}$$
Zeros: Inverse Response

Space for Notes Below

Inverse Response

Slow and fast together in parallel, opposite gains
Zeros: Inverse Response

\[ g(s) = \frac{-1}{s+1} + \frac{8}{10s+1} \]

\[ g(s) = \frac{-1(10s+1) + 8(s+1)}{(s+1)(10s+1)} \]

\[ g(s) = \frac{-10s - 1 + 8s + 8}{(s+1)(10s+1)} \]

\[ g(s) = \frac{-2s + 7}{(s+1)(10s+1)} \]

Zero? \( s = \frac{7}{2} \)

---

Question #3

**Question:** Does this system have inverse response to a input step?

\[ g(s) = \frac{2s + 10}{s^2 + 4s + 4} \]

A Yes  
B No  
C Maybe  
D What?  
E This is not an answer

**Answer:** ???

Question #4

- **Question:** Does this system have inverse response to a input step?

\[ g(s) = \frac{s^2 - 2s + 2}{s^2 + 4s + 4} \]

- A Yes
- B No
- C Maybe
- D What?
- E This is not an answer

- **Answer:** ???
Time-Delay System

\[ FC_{A_1}(t) \]
\[ F = 10 \left( \frac{m^3}{min} \right) \]
\[ V_1 = A_1 L_1 = 10 \left( m^3 \right) \]
\[ \alpha_1 = 1 \ (min) \]

Sensor Delay
\[ V \frac{dC_A}{dt}(t) = FC_{A_1}(t) - FC_A(t) \]
\[ \alpha_3 = 5 \ (min) \]
\[ V_2 = A_2 L_2 = 20 \left( m^3 \right) \]
\[ \alpha_2 = 2 \ (min) \]

\[ C_{A_2}(t) \]

\[ C_{A_2}(s) \]

\[ K \]
\[ \tau s + 1 \]
\[ e^{-\alpha_1 s} \]
\[ e^{-\alpha_2 s} \]
\[ C_{A_p}(s) \]

\[ C_{A_2}(s) \]

\[ \frac{K}{\tau s + 1} \]
\[ e^{-\alpha_3 s} \]
\[ C_{A_p}(s) \]

\[ C_{A_p}(s) = \left( \frac{K}{\tau s + 1} e^{-\left(\alpha_1 + \alpha_2 + \alpha_3\right)s} \right) C_{A_2}(s) \]
Padè Approximation

Space for Notes Below

Review

- Pole in Right Half Plane means unstable
- Zero in Right Half Plane means inverse step response
- Time delays happen
  - Caused usually by flow in pipes
  - Can be approximated using Padè Approximation
- Systems in series multiply, in parallel add
- Next: More on stability
16.15 Stability

Stability
Poles and Eigenvalues are Important

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Lecture 15

Question #1

Who is this guy?

A Laplace
B Padé
C Bode
D Dirac
E I don’t have a clue.

Answer: ???
**Time Delay & Padé Approximation**

**High-order approximation:**

\[ y(s) = \left( \frac{1}{(\pi s + 1)^N} \right) u(s) \]

**Padé Approximation:**

**First-Order**

\[ e^{-\alpha s} \approx \frac{1 - \frac{\alpha}{2} s}{1 + \frac{\alpha}{2} s} \]

**Second-Order**

\[ e^{-\alpha s} \approx \frac{1 - \frac{\alpha}{2} s + \frac{\alpha^2}{4} s^2}{1 + \frac{\alpha}{2} s + \frac{\alpha^2}{4} s^2} \]

---

**Inverse Response**

Two parallel effects:

\[ g(s) = g_1(s) + g_2(s) \]

Example:

\[ g(s) = \frac{-1}{s+1} + \frac{8}{10s+1} = \frac{-2s+7}{(s+1)(10s+1)} \]

Slow and fast together in parallel, opposite gains

Zero in right half plane \((Re > 0)\) means inverse response to step
**Question #2**

- **Laplace Transforms?**
  - A Love them! I am a Laplace master!
  - B I can use them to solve ODEs mostly.
  - C Bleh. Just give me a D on that part.
  - D I don’t get them, but I am improving.
  - E Hate them, won’t understand them, can’t do them.

- **Answer:** ???

**Question #3**

- **Question:** For this system, what value is not one the following: gain, pole, or a zero?

\[ g(s) = \frac{-4s + 24}{s^2 - 5s + 6} \]

- A 2
- B 3
- C 4
- D 5
- E 6

- **Answer:** ???
**Stability**

Space for Notes Below

---

**CSTR System**

\[ F \frac{dC_A}{dt}(t) = FC_{Ao}(t) - FC_A(t) - VkT(t) \]

\[ V \rho C_p \frac{dT}{dt}(t) = F \rho C_p T_o(t) - F \rho C_p T(t) + (\Delta H) V kT(t) \]
CSTR System

\[
\begin{align*}
\frac{dC_A}{dt}(t) &= -\frac{F}{V} C_A(t) - \frac{k}{V} \frac{C_A(t)}{T(t)} + \frac{F}{V} C_{A_o}(t) + 0 T_o(t) \\
\frac{dT}{dt}(t) &= -\frac{F}{V} C_A(t) + \frac{\Delta H}{V \rho C_p} T(t) + \frac{F}{V} C_{A_o}(t) + \frac{F}{V} T_o(t) \\
y(t) &= C_A(t) - C_{A_{ss}}
\end{align*}
\]

\[
\begin{align*}
\frac{dx_1}{dt}(t) &= -\frac{F}{V} x_1(t) - \frac{k}{V} x_2(t) + \frac{F}{V} u_1(t) + 0 u_2(t) \\
\frac{dx_2}{dt}(t) &= 0 x_1(t) + \left(-\frac{F}{V} + \frac{\Delta H}{V \rho C_p} \right) x_2(t) + 0 u_1(t) + \frac{F}{V} u_2(t) \\
y(t) &= 1 x_1(t) + 0 x_2(t) + 0 u_1(t) + 0 u_2(t)
\end{align*}
\]

\[
A = \begin{bmatrix}
-\frac{F}{V} & \frac{k}{V} \\
0 & \left(-\frac{F}{V} + \frac{\Delta H}{V \rho C_p} \right)
\end{bmatrix}
\]

Now, find transfer function in form: 

\[
y(s) = g(s) u(s) + g_d(s) d(s)
\]

with 

\[
y = C_A - C_{A_{ss}}, \quad u = T_o - T_{o_{ss}}, \quad d = C_{A_o} - C_{A_{o_{ss}}}
\]
Poles: Stability

Peale at:
\[ s = a \pm b \sqrt{-1} \]

Example Cases (left to right)

Left \( \times \) - Two distinct poles with \( \text{Re} < 0, \text{Im} = 0 \) (asympt. stable, not oscillatory)

Left \( \bullet \) - Two distinct poles with \( \text{Re} < 0, \text{Im} \neq 0 \) (asympt. stable, oscillatory)

Middle \( \times \) - Two distinct poles with \( \text{Re} = 0, \text{Im} \neq 0 \) (BIBO stable, oscillatory)

Middle \( \circ \) - One pole at origin (\( a = 0 \)), stability depends on input

Right \( \bullet \) - Two distinct poles with \( \text{Re} > 0, \text{Im} = 0 \) (unstable, oscillatory)

Right \( \times \) - One distinct pole with \( \text{Re} > 0, \text{Im} = 0 \) (unstable, not oscillatory)

Stability: Special Case 1

Space for Notes Below
Question #5

Question: What best describes the given system:

\[ g(s) = \frac{2s + 10}{s^2 + 4s + 4} \]

A Unstable, oscillatory, no inverse-response
B Unstable, non-oscillatory, with inverse-response
C Oscillatory with inverse response.
D Stable, non-oscillatory, no inverse-response
E Stable, non-oscillatory, with inverse-response

Answer: ???
Question #6

- Question: For this system, what value is not one the following: gain, pole, or a zero?
  
  \[
g(s) = \frac{s^2 - 2s + 1}{s^2 - 4s + 4}
  \]

  A 1  
  B 2  
  C 4  
  D 1/4

- Answer: ???

Summary

- Pole in RHP (\(Re > 0\)) is unstable
- Pole on RHP LHP boundary (\(Re = 0\)) is BIIO stable
- Pole at origin depends on \(u(t)\)
- Imaginary poles oscillate
- Zero in RHP is inverse response
- More \(s\) in numerator than denominator: improper
- More \(s\) in denominator than numerator: strictly proper
- Same order of \(s\) in numerator and denominator: semi-proper
16.16 Frequency Response

Frequency Response
Sinusoidal Response of Linear Systems

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Lecture 16

Pole Zero Summary

- Gain for most transfer functions: \( k = g(s)|_{s=0} \)
- Pole in RHP (\( Re > 0 \)) is unstable
- Pole on RHP/LHP boundary (\( Re = 0 \)) is BIBO stable (oscillates)
- Pole at origin (\( s = 0 \)) depends on \( u(t) \)

- Imaginary poles oscillate (can be stable or unstable)
  **Zero in RHP has inverse step response**

- Higher power of \( s \) in denominator vs. num.: **strictly proper**
- Same order of \( s \) in numerator and denominator: **semi-proper**
- Higher power of \( s \) in numerator than denominator: **improper**

\[
\text{Strictly Proper: } \frac{K}{\tau s + 1} \quad \text{Semi-proper: } \frac{K(\xi s + 1)}{(\tau s + 1)} \quad \text{Improper: } \frac{s^2 + s + 1}{(2s + 1)}
\]
**Poles: Stability**

Pole at:

\[ s = a \pm b\sqrt{-1} \]

Example Cases (left to right)

- **Left \times -** Two distinct poles with \( Re < 0, Im = 0 \) (asympt. stable, not oscillatory)
- **Left \circ -** Two distinct poles with \( Re < 0, Im \neq 0 \) (asympt. stable, oscillatory)
- **Middle \times -** Two distinct poles with \( Re = 0, Im = 0 \) (BIBO stable, oscillatory)
- **Middle \circ -** One pole at origin \((s = 0)\), stability depends on input
- **Right \times -** Two distinct poles with \( Re > 0, Im = 0 \) (unstable, oscillatory)
- **Right \circ -** One distinct pole with \( Re > 0, Im = 0 \) (unstable, not oscillatory)

---

**Frequency Response**

Sinusoidal input \rightarrow Sinusoidal output (eventually)

\[
\begin{align*}
  u(t) &= A \sin(\omega t) \\
  y(t) &= A_o \sin(\omega t + \phi) + \sum_{i=1}^{N} (A_i e^{-a_i t})^0
\end{align*}
\]

Output is different amplitude and phase shifted (eventually, if stable)
Frequency Response Example

\[ g(s) = \frac{100}{10s + 1} \quad \text{and} \quad g(s)|_{s=j\omega} = Re(\omega) + Im(\omega) j \]

<table>
<thead>
<tr>
<th>\omega</th>
<th>Re(\omega)</th>
<th>Im(\omega)</th>
<th>AR(\omega)</th>
<th>\phi(\omega)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>99.9</td>
<td>-9.99</td>
<td>100</td>
<td>-0.57°</td>
</tr>
<tr>
<td>0.01</td>
<td>99.5</td>
<td>-9.99</td>
<td>99.5</td>
<td>-5.71°</td>
</tr>
<tr>
<td>0.1</td>
<td>50</td>
<td>-50.0</td>
<td>70.7</td>
<td>-45.0°</td>
</tr>
<tr>
<td>1</td>
<td>.99</td>
<td>-9.90</td>
<td>9.95</td>
<td>-84.3°</td>
</tr>
<tr>
<td>10</td>
<td>.01</td>
<td>-0.99</td>
<td>1.00</td>
<td>-89.4°</td>
</tr>
</tbody>
</table>

\[ AR = \sqrt{Re^2 + Im^2} \]

Question #1

- Who is this guy?
  - A Laplace
  - B Padé
  - C Bode
  - D Dirac
  - E I don’t have a clue.

- Answer: ???
First-Order Amplitude Ratio and Phase Angle

\[ g(s) = \frac{K}{\tau s + 1} \]
\[ g(j\omega) = \frac{K}{\tau j\omega + 1} \]
\[ g(j\omega) = \frac{K}{1 + \tau j\omega} \]
\[ g(j\omega) = \frac{K(1 - \tau j\omega)}{(1 + \tau j\omega)(1 - \tau j\omega)} \]
\[ g(j\omega) = \frac{K - \tau j\omega}{1 - \tau^2 \omega^2} \]
\[ g(j\omega) = \frac{K - K \tau j\omega}{1 + \tau^2 \omega^2} \]
\[ g(j\omega) = \frac{K}{1 + \tau^2 \omega^2} + \frac{-K \tau j\omega}{1 + \tau^2 \omega^2} \]

Amplitude Ratio

\[ A_R(\omega) = |g(j\omega)| = \sqrt{\left( \frac{K}{1 + \tau^2 \omega^2} \right)^2 + \left( \frac{K \tau \omega}{1 + \tau^2 \omega^2} \right)^2} \]
\[ A_R(\omega) = |g(j\omega)| = \sqrt{\left( \frac{K^2 + (K \tau \omega)^2}{(1 + \tau^2 \omega^2)^2} \right) + \left( \frac{K}{1 + \tau^2 \omega^2} \right)^2} \]
\[ A_R(\omega) = |g(j\omega)| = \sqrt{\left( \frac{K^2 + K^2 \tau^2 \omega^2}{1 + \tau^2 \omega^2} \right)^2} = \sqrt{\left( \frac{K^2 + (1 + \tau^2 \omega^2)^2}{1 + \tau^2 \omega^2} \right)^2} \]
\[ A_R(\omega) = |g(j\omega)| = K \sqrt{\frac{1}{1 + \tau^2 \omega^2}} \]

For phase angle as a function of frequency \( \omega \)

\[ \phi(\omega) = \angle g(j\omega) = \arctan \left( \frac{b}{a} \right) = \arctan \left( \frac{-K \tau \omega}{K \tau \omega} \right) = \arctan \left( \frac{-K \tau \omega}{1 + \tau^2 \omega^2} \right) \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan (-\tau \omega) \]

First-Order Frequency Response
Time-Delay Amplitude Ratio and Phase Angle

Using the Euler Identity:

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]

\[ g(j\omega) = e^{-j\alpha\omega} = \cos(-\alpha\omega) + j\sin(-\alpha\omega) = \cos(-\alpha\omega) + \sin(-\alpha\omega)j \]

\[ AR(\omega) = |g(j\omega)| = \sqrt{\cos(-\alpha\omega)^2 + \sin(-\alpha\omega)^2} \]

\[ AR(\omega) = |g(j\omega)| = \sqrt{1} - 1 \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan \left( \frac{b}{a} \right) = \arctan \left( \frac{\sin(-\alpha\omega)}{\cos(-\alpha\omega)} \right) \]

\[ \phi(\omega) = \angle g(j\omega) = \arctan (\tan(-\alpha\omega)) \]

\[ \phi(\omega) = \angle g(j\omega) = -\alpha\omega \]
Lead-Lag Amplitude Ratio and Phase Angle

\[ g(s) = \frac{K(\xi s + 1)}{(\tau s + 1)} \]

\[ AR(\omega) = \frac{K\sqrt{1 + \omega^2\xi^2}}{\sqrt{1 + \omega^2\tau^2}} \]

\[ \phi(\omega) = \tan^{-1}(\omega\xi) - \tan^{-1}(\omega\tau) \]

Systems in Series

Space for Notes Below
Amplitude Ratio Rules

Question #2

What is the system Gain?

A 0.01  B 0.1  C 1.0  D 10  E 100

Answer: ???
Phase Angle Rules

Space for Notes Below

Question #3

What is the system time constant?

A 0.01  B 0.1  C 1.0  D 10  E 100

Answer: ???
16.17 Frequency Response

Frequency Response
Sinusoidal Response of Linear Systems

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Lecture 17

Frequency Response
Sinusoidal input → Sinusoidal output (eventually)

\[ u(t) = A \sin(\omega t) \quad y(t) = A_0 \sin(\omega t + \phi) + \sum_{i=1}^{N} \left( A_i e^{-\omega_i t} \right) \]

Output is different amplitude and phase shifted

\[ g(s) |_{s=j\omega} = Re(\omega) + Im(\omega) j \quad AR(\omega) = \sqrt{(Re)^2 + (Im)^2} \quad \phi(\omega) = \tan^{-1} \left( \frac{Im}{Re} \right) \]

\[ AR(\omega) = \frac{A_0(\omega)}{A(\omega)} \]
Amplitude Ratio Rules

- Amplitude Ratios for systems in series multiply
- The gain of the system is usually the low frequency AR value
- The high-frequency slope is the relative degree
  - \# zeros - \# poles
- The AR "bends" down at a pole. Near frequency \( \frac{1}{\tau} \)
- The AR "bends" up at a zero. Near frequency \( \frac{1}{\xi} \)
- The AR for an under-damped 2nd order system may “bump.”
- A time delay is not seen in the AR: \( AR(\omega) = 1 \)
- Pure-capacity system (pole at \( s = 0 \)) low frequency AR:
  - Slope of \(-1 \frac{\text{decade}}{\text{decade}}\) for a single integrator

Phase Angle Rules

- Phase angles for systems in series add
- Pole: \( 0^\circ \) to \(-90^\circ \) at high frequency
- LHP zero: \( 0^\circ \) to \( 90^\circ \) at high frequency (\( \xi > 0 \))
- RHP zero: \( 0^\circ \) to \(-90^\circ \) at high frequency (\( \xi < 0 \))
- Pure capacity: \( \phi \) starts at \(-90^\circ \) and does not change
- Pole or Zero: the frequency at \( \phi \approx 45^\circ \) is same as AR “bend”
  - At \( \omega = \frac{1}{\tau} \) or \( \frac{1}{\xi} \)
- \( \phi \) goes to \(-\infty \) at high frequency for a time delay
First-Order Amplitude Ratio and Phase Angle

\[ g(s) = \frac{K}{\tau s + 1} \]

\[ AR(\omega) = \frac{K}{\sqrt{1 + \omega^2 \tau^2}} \]

\[ \phi(\omega) = \text{atan}(-\omega \tau) = -\text{atan}(\omega \tau) \]

Time-Delay Amplitude Ratio and Phase Angle

\[ g(s) = e^{-\alpha s} \]

\[ AR(\omega) = 1 \]

\[ \phi(\omega) = -\alpha \omega \]
**Lead-Lag Amplitude Ratio and Phase Angle**

\[ g(s) = \frac{K(\xi s + 1)}{(\tau s + 1)} \]

\[ AR(\omega) = K \frac{\sqrt{1 + \omega^2\xi^2}}{\sqrt{1 + \omega^2\tau^2}} \]

\[ \phi(\omega) = \text{atan}(\omega\xi) - \text{atan}(\omega\tau) \]

**Systems in Series**

\[ g(s) = g_1(s) g_2(s) g_3(s) \]

\[ AR(\omega) = AR_1(\omega) \times AR_2(\omega) \times AR_3(\omega) \]

\[ \phi(\omega) = \phi_1(\omega) + \phi_2(\omega) + \phi_3(\omega) \]
Question #1

Question: What is $\tau$ for this system?

[A] 1  [B] 2  [C] 3  [D] 0.5  [E] 1/3

Answer: ???

Question #2

Question: What is the gain $K$ for this system?

[A] 0.006  [B] 0.03  [C] 0.04  [D] 0.05  [E] .4

Answer: ???
16.18 Closed-Loop Stability

Closed-Loop Systems and Stability

Finally! Feedback Control and Control Systems Can Be Dangerous!

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Lecture 18

Open-Loop Systems

\[
\begin{align*}
y(s) &= g(s) u(s) + g_d(s) d(s) \\
y_1(s) &= g_d(s) d(s) \\
y_2(s) &= g(s) u(s) \\
y(s) &= y_1(s) + y_2(s)
\end{align*}
\]
Closed-Loop Transfer Function

Space for Notes Below

Closed-Loop Block Diagrams

- You (usually) can’t change $g(s)$ and $g_d(s)$
- You can adjust $g_c(s)$ parameters
  \[ u(s) = g_c(s) e(s) = g_c(s) (y_{sp}(s) - y(s)) \]
- Controller types:
  - Proportional (P)
  - Proportional Integral (PI)
  - Proportional Integral Derivative (PID)
Closed-Loop Transfer Function with Disturbance

- Again: system model relates $u$ to $y$

\[ y(s) = g(s) u(s) + g_d(s) d(s) \]

Do not assume $d = 0$ this time:

\[ y(s) = g(s) g_c(s) e(s) + g_d(s) d(s) \]

\[ y(s) = g(s) g_c(s) (y_{sp}(s) - y(s)) + g_d(s) d(s) \]

\[ y(s) = g g_c y_{sp} - g g_c y + g_d(s) d(s) \]

\[ y(s) = \left( \frac{g(s) g_c(s)}{1 + g(s) g_c(s)} \right) y_{sp}(s) + \left( \frac{g_d(s)}{1 + g(s) g_c(s)} \right) d(s) \]

Controller Types

- For Proportional (P) control:

\[ g_c = K_c \]

- For Proportional + Integral (PI) control:

\[ g_c = K_c \left( 1 + \frac{1}{\tau_I} \frac{1}{s} \right) \]

- For Proportional + Integral + Derivative (PID) control:

\[ g_c = K_c \left( 1 + \frac{1}{\tau_I} \frac{1}{s} + \tau_D s \right) \]
CLTF PI Control Offset for FOTD

Space for Notes Below

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CLTF PI Control Offset for FOTD

Space for Notes Below
Question #1

Question: What is still most confusing?
A Dynamic Mass and Energy Balances
B Laplace transforms (finding $u(s)$ or $y(t)$, IVT, FVT)
C Linear Algebra
D Frequency response
E Feedback Control

Answer: ???

Question #2

Question: Who is this guy?

A Hendrik Alfred Bode, 1905-1982
B Paul Dirac, 1902-1984
C Harry Nyquist, 1889-1976
D Henri Padé, 1863-1953
E Grover Cleveland, 1837-1908

Answer: ???
Question #3

Question: What type of controller is the following?

\[ g_c(s) = K_c \left( 1 + \frac{1}{\tau I s} \right) \]

A. Proportional (P)
B. Proportional + Integral (PI)
C. Proportional + Integral + Derivative (PID)
D. Proportional + Derivative (PD)
E. Integral + Derivative (ID)

Answer: ???

CLTF P Control Offset for FOTD

Space for Notes Below
CLTF P Control Offset for FOTD

- Disturbance response?
  - Step in \( d \) of size \( A \), let \( y_{sp} = 0 \), find SS offset
  - P controller, \( g_c(s) = K_c \)

\[
s y(s) \bigg|_{s=0} = \frac{K_d e^{-\alpha d s}}{1 + \frac{K_d e^{-\alpha s}}{\tau s + 1} K_c} A = \frac{K_d e^{-\alpha d s}}{1 + \frac{K_d e^{-\alpha s}}{\tau s + 1} K_c} A
\]

- Normal change in \( y \), no control: \( y(t = \infty) = K_d A \)
- Disturbance offset (desired - actual)

\[
0 = \frac{K_d}{1 + K K_c} A
\]

CLTF P Control Stability for FO

Space for Notes Below
Closed-Loop Stability

Consider FOTD with P control: 
\[ g(s) = \frac{1}{s + 1} e^{2s}, \quad g_c(s) = 2 \]

- \( g_c \) is gain in series with FO in series with time delay

\[ g_c \rightarrow AR(\omega) = \frac{KK_c}{\sqrt{1 + \omega^2 \tau^2}} = \frac{4 \times 2}{\sqrt{1 + 2500 \times \omega^2}} \]

\[ \phi(\omega) = \text{atan}(-\tau \omega) - \alpha \omega - (\text{atan}(-50 \omega) - 2\omega)(180/\pi) \]
Question #4

- Question: For the following Bode plot of \( g_c g \), is the system closed-loop stable?

\[
\begin{align*}
\text{[A]} & \quad \text{Yes} \\
\text{[B]} & \quad \text{No} \\
\text{[C]} & \quad ???
\end{align*}
\]

- Answer: ???

General CLTF

- Negative Feedback!
- Systems in series multiply!
- Systems in parallel add!
- Reconcile inner loops first!
- Loop rule:

\[
\Pi All \ g_i \ between \ in \ & \ out \\
1 + \Pi All \ g_j \ in \ loop
\]
Question #5

Question: What is the most confusing topic?

A Laplace transforms, IVT, FVT, PFE
B Dynamic modeling, transfer functions, state space
C Bode plots, frequency response
D Closed-loop block diagrams
E Closed-loop stability

Answer: ???
16.19 Tuning and Feedforward

Tuning and Feed-Forward

Finally to Feed-Forward!

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Lecture 19

General CLTF

- Negative Feedback!
- Systems in series multiply!
- Systems in parallel add!
- Reconcile inner loops first!
- Loop rule:

\[ \prod \frac{All \ g_i \ between \ in \ & \ out}{1 + \prod \ All \ g_j \ in \ loop} \]
Closed-Loop Stability

- Consider only $gg_c$ when $\phi(\omega) = -180^\circ$

For $gg_c$ if $AR(\omega) > 1$ when $\phi(\omega) = -180^\circ$ then UNSTABLE
Question #2

- **Question:** Is the following closed-loop system stable? (Plot of $gg_c$)

![Plot of $gg_c$]

[A] Yes  [B] No  [C] ????

- **Answer:** ???

---

Open-Loop And Closed-Loop

- Must know models $g(s)$ and $g_d(s)$
- Must know signals $d(s)$ and $u(s)$

**Open-Loop Model**

$$y(s) = g(s) u(s) + g_d(s) d(s)$$

- For PI control, $g_c = K_c \left(1 + \frac{1}{\tau I \frac{1}{s}}\right)$ and FOTD $g(s) = \frac{K}{\tau s + 1} e^{-\alpha s}$

**Closed-Loop Transfer Function (CLTF):**

$$y(s) = \frac{g g_c}{1 + g g_c} y_{sp}(s) + \frac{g d}{1 + g g_c} d(s)$$
Closed-Loop Response

Closed-Loop Setpoint Tracking Response

Feed-forward Control

Space for Notes Below
**FF Control Example**

- Develop a feed forward controller based on the models:

\[ g = \frac{6}{3s+1} \quad g_d = \frac{2}{7s+1} \]

\[ g_{ff} = -\left(\frac{2}{7s+1}\right) \left(\frac{3s+1}{6}\right) = -\frac{1}{9} (3s+1) \]

- Offset for unit step in \( d \) without and with the ff controller?
  - Without:

\[ y(t = \infty) = sy(s)|_{s=0} = \left(\frac{2}{7s+1}\right) \left(\frac{1}{s}\right) s = 2 \]

  - With:

\[ sy(s)|_{s=0} = \left(\frac{2}{7s+1}\right) + \left(\frac{6}{3s+1}\right) \left(-\frac{1}{9} (3s+1)\right) \left(\frac{1}{s}\right) s = 0 \]

---

**FF Control Issues**

Space for Notes Below
FF Control Example 2

- Develop a feed forward controller based on the models:

\[
g_d = \frac{3e^{1s}}{4s+1} \quad g = \frac{(-14s+7)e^{3s}}{(3s+1)(5s+1)} = \frac{7(-2s+1)e^{3s}}{(3s+1)(5s+1)}
\]

\[
g_{ff} = -\left(\frac{3e^{1s}}{4s+1}\right) \frac{(3s+1)(5s+1)e^{3s}}{7(-2s+1)} = \left(\frac{-\frac{3}{7}(3s+1)(5s+1)e^{2s}}{(4s+1)(-2s+1)}\right)
\]

- Drop time prediction and RHP pole \((-2s+1)\), add filter

\[
g_{ff} = \left(\frac{-\frac{3}{7}(3s+1)(5s+1)}{(4s+1)(\lambda s+1)}\right)
\]

FF Control Example 2

- Now, assume new models for \(g\) and \(g_d\):

\[
\hat{g}_d = \frac{4e^{-2s}}{5s+1} \quad \hat{g} = \frac{8(-3s+1)e^{-4s}}{(4s+1)(6s+1)}
\]

- Find SS offset using your feed-forward controller and the new models for step in \(d(s)\)

\[
y(t = \infty) = sy(s)|_{s=0} = (g_d + g_{ff})d(s) = \left(\frac{4e^{-2s}}{5s+1} + \frac{8(-3s+1)e^{-4s}}{(4s+1)(6s+1)}\right)\left(\frac{-\frac{3}{7}(3s+1)(5s+1)}{(4s+1)(\lambda s+1)}\right)\left(\frac{1}{s}\right) = 4 + 8\left(-\frac{3}{7}\right) = 0.571
\]

- Still better than without, gain of \(g_d = 4\)
CLTF + FF Control

- Both feedback and feed-forward

\[
y(s) = \frac{g g_c}{1 + g g_c} y_{sp}(s) + \frac{g d}{1 + g g_c} d(s) + \frac{g_{ff} g}{1 + g g_c} d(s)
\]

\[
y(s) = \frac{g g_c}{1 + g g_c} y_{sp}(s) + \frac{g d}{1 + g g_c} d(s) + \frac{g_{ff} g}{1 + g g_c} d(s)
\]

Question #3

- Question: What is wrong with the following feed-forward controller?

\[
g_{ff} = \left( -\frac{1}{8} \frac{(6s + 1)^2 (8s + 1) e^{2s}}{(8s + 1) (-s + 1)} \right)
\]

A RHP Pole (open-loop unstable)
B Time prediction
C Improper
D None of the above
E A, B, and C

- Answer: ???
Question #4

- Question: What would be the gain of the properly-formatted version of the following feed-forward controller?

\[ g_{ff} = \left( \frac{-3(8s + 4)e^{2s}}{(-s + 2)} \right) \]

A  -12  
B  -3  
C  0  
D  -6  
E  6

- Answer: ???

Question #5

- Question: What is the following block diagram?

[Diagram]

A  Open-loop  
B  Closed-loop  
C  Semi-proper loop  
D  Infinite loop  
E  Both A and B

- Answer: ???
Closed-Loop Tuning and IMC

Closing the Loop

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Lecture 20

Closed-Loop Transfer Function (CLTF)

- For PID control, \( g_c = K_c \left( 1 + \frac{1}{\tau_I s} + \tau_D s \right) \)
- Three adjustable parameters: \( K_c \quad \tau_I \quad \tau_D \)
- CLTF (Setpoint Response)

\[
y(s) = \frac{g g_c}{1 + g g_c} y_{sp}(s) + \frac{g d}{1 + g g_c} d(s)
\]
Question #1

- **Question: Who is this guy?**

  ![Image of a professor](image)

  A  John Ziegler  
  B  Nathaniel Nichols  
  C  Cohen  
  D  Coon  
  E  Rudolph Kalman  

  **Answer:** ???

---

Question #2

- **Question: What type of controller is the following?**

  $$g_c(s) = K_c \left( 1 + \frac{1}{\tau_I s} + \tau_D s \right)$$

  A  Proportional (P)  
  B  Proportional + Integral (PI)  
  C  Proportional + Integral + Derivative (PID)  
  D  Proportional + Derivative (PD)  
  E  Integral + Derivative (ID)

  **Answer:** ???
Closed-Loop Response

Overshoot: \( \frac{a_1}{A} \)

Decay ratio: \( \frac{a_2}{a_1} \)

Rise time: \( t_r \)

Settling time: \( t_s \)

Steady-state offset: \( \frac{y_{sp}(\infty) - y(\infty)}{A} \)

Closed-Loop Metrics

- Overshoot percentage: \( \frac{a_1}{A} \)
- Decay ratio: \( \frac{a_2}{a_1} \)
- Percent Offset: \( \frac{y_{sp}(\infty) - y(\infty)}{A} \)
- Rise time
- Settling time (to around ±5%)

Others:
- Sum Square Error (SSE)
- Sum Absolute Error (SAE)
Tuning Procedures

- Develop model
  - First-principles
  - Empirical
  - Ultimate frequency

- Determine controller parameters (Chapter 15)
  - Ziegler Nichols PI: $K_c = \frac{0.6}{\tau} \left( \frac{\omega_n}{\omega_p} \right)$ and $\tau_I = 3.33\omega_p$
  - Or my simple PI: $K_c = \frac{1}{\tau}$ and $\tau_I = \tau$

- Evaluate and adjust parameters

- Too aggressive? Reduce $K_c$ and increase $\tau_I$

Closed-Loop Stability

- $AR(\omega)$ usually in form $AR(\omega) = K_c AR_1(\omega) AR_2(\omega) \cdots AR_N(\omega)$
- Use P control and increase $K_C$ to verge of stability

- Analytically: examine AR for $gg_c$ when
  - $\phi(\omega_c) = -180^\circ$ and $AR(\omega_c) = 1$
  - Ultimate Gain: $K_{cu}$ and Ultimate Period $P_u$
  - Rules for “backing” off in Chapter 15 (Ziegler Nichols)
    - P: $K_c = 0.5K_{cu}$
    - PI: $K_c = 0.45K_{cu}$ and $\tau_I = P_u/1.2$
Internal Model Control

\[ d(s) \rightarrow g_d(s) = 1 \]

\[ u(s) \rightarrow g(s) \rightarrow y(s) \]

\[ \tilde{g}(s) \rightarrow \tilde{d}(s) \]

Model
Question #3

- **Question:** What is wrong with the following IMC controller?

  \[
g_{IMC} = \left( \frac{-\frac{1}{3} (3s + 1)(4s + 1) e^{-4s}}{(7s + 1)(-2s - 1)} \right)
  \]

  A RHP Pole (open-loop unstable)
  B Time prediction
  C Improper
  D None of the above
  E A, B, and C

- **Answer:** ???

---

Question #4

- **Question:** What is the most confusing topic?

  A Laplace transforms, lvt, fvt, ffe
  B Dynamic modeling, transfer functions, state space
  C Bode plots, frequency response
  D Closed-loop block diagrams and stability
  E Feedforward or IMC

- **Answer:** ???
16.21 Combined Concepts

Putting It Together
PFD to P&ID with PID, FF, CC, and maybe IMC

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Lecture 21

Simple System

- Identify inputs, outputs, disturbances
- Draw open-loop block diagram
- For the following feedback system, feedforward, cascade, feedforward with feedback
  - Add controller and signal routing to the PFD
  - Draw block diagrams
  - Determine the closed-loop transfer function
Open-Loop

Feedback
Feedforward

![Feedforward Diagram]

Cascade

![Cascade Diagram]
Feedforward with Feedback

Open-Loop Control

We want \( y(s) = y_{sp}(s) \) so

\[
g_{ocl}(s) g(s) = 1
\]

\[
g_{ocl}(s) = \frac{1}{g(s)}
\]
Internal Model Control

\[
d(s) \rightarrow g_d(s) = 1 \\
g(s) \\
\tilde{g}(s) \\
\text{Model} \\
\tilde{d}(s) \\
\]

\[
d(s) \rightarrow g_d(s) = 1 \\
e(s) = \frac{1}{\tilde{g}(s)} \\
g(s) \\
\tilde{g}(s) \\
\tilde{d}(s) \\
\]

\[
e(s) = \frac{c(s)}{1 - r(s)\tilde{g}(s)} \\
g(s) \\
g_d(s)d(s) \\
\]
Question

- Question: Given the following process model, determine a realizable \( c(s) \) and \( g_c(s) \) for an IMC system.

\[
\hat{g} = \left( \frac{(-3s + 4)e^{-4s}}{(8s + 2)(2s + 1)} \right)
\]

Closed-Loop Stability

- Consider only \( gg_c \) when \( \phi(\omega) = -180^\circ \)

For \( gg_c \) if \( AR(\omega) > 1 \) when \( \phi(\omega) = -180^\circ \) then UNSTABLE
Closed-Loop Stability

- Consider FOTD with P control: \( g(s) = \frac{4}{50s+1} e^{-2s} \)  \( g_c(s) = 2 \)
- \( gg_c \) is gain in series with First-Order in series with time delay
  \[
  gg_c \rightarrow AR(\omega) = AR_1 AR_2 AR_3 = \frac{KK_c}{\sqrt{1+\omega^2\tau^2}} = \frac{4 \times 2}{\sqrt{1 + 2500 \times \omega^2}}
  \]
  \( \rightarrow \phi(\omega) = \phi_1 + \phi_2 + \phi_3 = atan(-\tau \omega) - \alpha \omega = (atan(-5\omega) - 2\omega) (180/\pi) \)

Amplitude Ratio Rules

- Amplitude Ratios for systems in series multiply
- The gain of the system is usually the low frequency AR value
- The high-frequency slope is the relative degree
  - \# zeros - \# poles
- The AR “bends” down at a pole. Near frequency \( \frac{1}{\tau} \)
- The AR “bends” up at a zero. Near frequency \( \frac{1}{\xi} \)
- The AR for a under-damped 2nd order system may “bump.”
- A time delay is not seen in the AR: \( AR(\omega) = 1 \)
- Pure-capacity system (pole at \( s = 0 \)) low frequency AR:
  - Slope of \(-1 \text{ decade}^{-1}\) for a single integrator
Phase Angle Rules

- Phase angles for systems in series add
- Pole: $0^\circ$ to $-90^\circ$ at high frequency
- LHP zero: $0^\circ$ to $90^\circ$ at high frequency ($\xi > 0$)
- RHP zero: $0^\circ$ to $-90^\circ$ at high frequency ($\xi < 0$)
- Pure capacity: $\phi$ starts at $-90^\circ$ and does not change
- Pole or Zero: the frequency at $\phi \approx 45^\circ$ is same as AR “bend”
  - $\omega = \frac{1}{\tau}$ or $\frac{1}{\xi}$
- $\phi$ goes to $-\infty$ at high frequency for a time delay

Bode Plot

- If this is the open-loop Bode plot for a system, what is a possible transfer function model $g(s)$. Justify.
- If this is the bode plot for $gg_c$ for a closed-loop system, is the system stable. Justify.
16.22 Multivariable Systems

Multivariable Systems
Now this stuff gets complicated!

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Lecture 22

ECHE 550 Chemical Process Dynamics and Control

Multivariable State Space

\[
\begin{bmatrix}
\frac{dx_1}{dt}(t) \\
\vdots \\
\frac{dx_n}{dt}(t) \\
y_1(t) \\
\vdots \\
y_o(t)
\end{bmatrix} = \frac{dx}{dt}(t) = Ax(t) + Bu(t)
\]

\[
\begin{bmatrix}
y_1(t) \\
\vdots \\
y_o(t)
\end{bmatrix} = y(t) = Cx(t) + Du(t)
\]

\[
\frac{dx}{dt} = \begin{bmatrix}
\frac{dx_1}{dt}(t) \\
\vdots \\
\frac{dx_n}{dt}(t)
\end{bmatrix} = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
\vdots \\
x_n(t)
\end{bmatrix} + \begin{bmatrix}
b_{11} & \cdots & b_{1m} \\
\vdots & \ddots & \vdots \\
b_{m1} & \cdots & b_{mm}
\end{bmatrix} \begin{bmatrix}
u_1(t) \\
\vdots \\
u_m(t)
\end{bmatrix}
\]

\[
y(t) = \begin{bmatrix}
y_1(t) \\
\vdots \\
y_o(t)
\end{bmatrix} = \begin{bmatrix}
c_{11} & \cdots & c_{1n} \\
\vdots & \ddots & \vdots \\
c_{o1} & \cdots & c_{om}
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
\vdots \\
x_n(t)
\end{bmatrix} + \begin{bmatrix}
d_{11} & \cdots & d_{1m} \\
\vdots & \ddots & \vdots \\
d_{m1} & \cdots & d_{mm}
\end{bmatrix} \begin{bmatrix}
u_1(t) \\
\vdots \\
u_m(t)
\end{bmatrix}
\]

\[m \text{ inputs, } n \text{ states, } o \text{ outputs}\]
Multivariable Open-Loop

Space for Notes Below

Multivariable Systems

Space for Notes Below
Question #1

Question: What is not a pole of the following system?

\[ G(s) = \begin{bmatrix} \frac{3}{2s+1} & \frac{3}{4s+1} \\ \frac{3}{s+1} & \frac{3}{5s+1} \end{bmatrix} \]

A -2.0  
B -1.0  
C -0.20  
D -0.25  
E -.50

Answer: ???

Multivariable \( G(s) \) from State Space

\[ \frac{dx}{dt}(t) = Ax(t) + Bu(t) \]
\[ sx(s) = Ax(s) + Bu(s) \]
\[ sIx(s) = Ax(s) + Bu(s) \]

\[ sIx(s) - Ax(s) = Bu(s) \]
\[ (sI - A)x(s) = Bu(s) \]
\[ (sI - A)^{-1} (sI - A)x(s) = (sI - A)^{-1} Bu(s) \]
\[ Lx(s) = x(s) = (sI - A)^{-1} Bu(s) \]

\[ y(t) = Cx(t) + Du(t) \]
\[ y(s) = Cx(s) + Du(s) \]
\[ y(s) = C (sI - A)^{-1} Bu(s) + Du(s) \]
\[ y(s) = \left( C (sI - A)^{-1} B + D \right) u(s) \]
Multivariable Closed-Loop

- 2x2 MIMO: How do you close the loop? Option 1

![Diagram showing 2x2 MIMO system with option 1.]

Option 1  \( y_1 \leftrightarrow u_1 \ y_2 \leftrightarrow u_2 \)

Multivariable Closed-Loop

- 2x2 MIMO: How do you close the loop? Option 2

![Diagram showing 2x2 MIMO system with option 2.]

Option 2  \( y_1 \leftrightarrow u_2 \ y_2 \leftrightarrow u_1 \)
Relative Gain Array (RGA)

Space for Notes Below
RGA Example 1

\[ K = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} \quad \zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}} = \frac{2(4)}{-1(-3)} = \frac{8}{3} \]

Determine the RGA matrix, \( \Lambda \)

\[ \Lambda = \begin{bmatrix} \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \\ \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\frac{3}{8}} & \frac{\frac{8}{3}}{1+\frac{3}{8}} \\ \frac{1}{1+\frac{3}{8}} & \frac{\frac{8}{3}}{1+\frac{3}{8}} \end{bmatrix} = y_1 \begin{bmatrix} \frac{3}{8} \\ \frac{8}{11} \end{bmatrix} \]

This means that you should pair \( u_1 \leftrightarrow y_2 \), \( u_2 \leftrightarrow y_1 \) since the (1,2) element (row 1, column 2) and (2,1) elements are \( \frac{8}{11} \), closer to 1 than the (1, 1) and (2, 2) elements. Option 2 from 2x2 previously.

RGA Example 2

\[ K = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} \quad \zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}} = \frac{2(4)}{-1(-3)} = \frac{8}{3} \]

Determine the RGA matrix, \( \Lambda \)

\[ \Lambda = \begin{bmatrix} \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \\ \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\frac{3}{8}} & \frac{\frac{8}{3}}{1+\frac{3}{8}} \\ \frac{1}{1+\frac{3}{8}} & \frac{\frac{8}{3}}{1+\frac{3}{8}} \end{bmatrix} = \begin{bmatrix} -\frac{3}{8} & \frac{8}{3} \\ -\frac{3}{8} & \frac{8}{3} \end{bmatrix} \]

Here, the (1,1) and (2,2) elements are negative. Avoid the \( u_1 \leftrightarrow y_1 \), \( u_2 \leftrightarrow y_2 \) pairing in this case, so you should use the \( u_1 \leftrightarrow y_2 \), \( u_2 \leftrightarrow y_1 \) pairing. Option 2 from 2x2 shown previously.
RGA Example 3

\[ K = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ -3 & 1 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0.25 & -2.25 & 3 \\ 0 & 3.5 & -2.5 \\ 0.75 & -0.25 & 0.5 \end{bmatrix} \]

- In row 2, the only good option appears to be pair \( y_2 \) with \( u_2 \).
- There will be interaction on this loop, as the value of 3.5 predicts
- Now, there are two different ways to consider the problem
  - If you consider column 1 first, you would pair \( y_3 \) with \( u_1 \) as a value of 0.75 is better than 0.25, then end up with \( y_1 \) paired with \( u_3 \) for a value of 3.
  - The alternative that would also be valid is pair \( y_1 \) with \( u_1 \) for a value of 0.25 and \( y_3 \) with \( u_3 \) for a value of 0.5. Either option is valid, neither is especially good.

RGA Example 4

\[ K = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ -3 & -3 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1 & 2 & 0 \\ 3.2 & -2.8 & 0.6 \\ -1.2 & 1.8 & 0.4 \end{bmatrix} \]

- First, consider column 1. Row elements (1,1) and (3,1) are both negative, implying that you should pair \( y_2 \) with \( u_1 \).
- Now, examine row 1. \( u_1 \) is already paired with \( y_2 \), so \( y_1 \) should be paired with \( u_2 \) since the (1,3) element is 0.
- This leave \( y_3 \) to be paired with \( u_3 \) for a value of 0.4.
- Every pairing will have interaction. This could be foreseen to some extent. Examine the "direction" of columns 1 and 2. Increasing either \( u_1 \) or \( u_2 \) will force the output measurements in almost the same direction.
Question #2

- **Question:** Given the following RGA, what goes in the 3, 3 position?

\[ K = \begin{bmatrix} 2.2 & 3.4 & 2.3 \\ 5 & 7 & 6 \\ 1 & 3 & 2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2.2 & 3.4 & ?? \\ -0.125 & -3.675 & 4.8 \\ -1.075 & 1.275 & ?? \end{bmatrix} \]

[A] -1  [B] -4.6  [C] 0.8  [D] -2  [E] 0.5

- **Answer:** ???

---

Question #3

- **Question:** What pairing is unacceptable?

\[ \Lambda = \begin{bmatrix} 3 & 2.5 & -4.5 \\ -1 & -2 & 4 \\ -1 & 0.5 & 1.5 \end{bmatrix} \]

[A] \( y_1 \leftrightarrow u_1 \)  [B] \( y_3 \leftrightarrow u_2 \)  [C] \( y_1 \leftrightarrow u_2 \)  [D] \( y_2 \leftrightarrow u_3 \)

- **Answer:** ???
Multivariable Decoupling
Now this stuff gets really complicated!

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Lecture 23
ECHE 550 Chemical Process Dynamics and Control

Review: CLTF PI Control

- For PI control, \( g_c = K_c \left(1 + \frac{1}{\tau_l} \frac{1}{s}\right) \) and FOPTD \( g(s) = \frac{K}{\tau s + 1} e^{-\alpha s} \)

\[
\begin{align*}
g_c(s) & = K_c \left(1 + \frac{1}{\tau_l s}\right) \\
g_d(s) & = \frac{K_d}{\tau_d s + 1} e^{-\alpha_d s} \\
g(s) & = \frac{K}{\tau s + 1} e^{-\alpha s}
\end{align*}
\]

- CLTF

\[
y(s) = \frac{g g_c}{1 + g g_c} y_{sp}(s) + \frac{g d}{1 + g g_c} d(s)
\]
**Review: Feed-forward Control**

![Feed-forward Control Diagram]

- Must have models \( g(s) \) and \( g_d(s) \)
- Must know disturbance value \( d(s) \)
- Response?

\[
y(s) = g_d d + g_{ff} d = \left(g_d + g_{ff}\right) d
\]

- Want \( \left(g_d + g_{ff}\right) = 0 \) to minimize effect of \( d \) so \( g_{ff} = \frac{-g_d}{g} \)

**Review: Internal Model Control**

![Internal Model Control Diagram]

- Find \( c(s) \) from model \( \tilde{g}(s) \)
- Problems: RHP poles, time prediction, improper

---

**Notes:**

- Ed Getzke (USC CHE)
- L23- Decoupling
- ECH 550 3 / 16

---

**Page:** 442
Question #1

Question: What is a zero of the following system?

\[ G(s) = \begin{bmatrix} \frac{1}{3s+1} & \frac{1}{4s+1} \\ \frac{4s+1}{s+1} & \frac{3s+1}{2s+1} \end{bmatrix} \]

A  -2.0  
B  -1.0  
C  0  
D  1  
E  2

Answer: ???

Question #2

Question: What issues may arise when developing an IMC controller?

A RHP Pole (open-loop unstable)  
B Time prediction  
C Improper formulation  
D None of the above  
E A, B, and C

Answer: ???
Multivariable Open-Loop

- Multiple-Input-Multiple-Output (MIMO)

\[ y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s) \]
\[ y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s) \]

State-Space Open-Loop

- State space to \( G(s) \)??

\[
\frac{dx}{dt}(t) = Ax(t) + Bu(t) \\
mx(s) = Ax(s) + Bu(s) = sIx(s)
\]

\[ sIx(s) - Ax(s) = Bu(s) \]
\[ x(s)(sI - A) = Bu(s) \]

\[ x(s) = (sI - A)^{-1}Bu(s) \]
\[ y(s) = Cx(s) + Du(s) \]
\[ y(s) = \left(C(sI - A)^{-1}B + D\right)u(s) \]
**Multivariable Closed-Loop**

- 2x2 MIMO: How do you close the loop? 2 Options

**Option 1**

- \( y_1 \rightarrow u_1, y_2 \rightarrow u_2 \)

**Option 2**

- \( y_1 \rightarrow u_2, y_2 \rightarrow u_1 \)

---

**Relative Gain Array (RGA)**

- Get steady-state gain matrix, \( K = G(s = 0) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \)

- Find \( \zeta \) and RGA matrix \( \Lambda \), **Rows and columns sum to 1**

\[
\zeta = \frac{K_{12} K_{21}}{K_{11} K_{22}}, \quad \Lambda = \begin{bmatrix} \frac{1}{1-\zeta} & \frac{-\zeta}{1-\zeta} \\ \frac{-\zeta}{1-\zeta} & \frac{1}{1-\zeta} \end{bmatrix}
\]

- If the \( \lambda_{ij} \) element is \( \leq 0 \), avoid pairing output \( i \) with input \( j \).
  - Very bad interaction between \( i \) and \( j \)
- If the \( \lambda_{ij} = 1 \) pair output \( i \) with input \( j \), no interaction
Question #3

- Question: Given the following RGA, what goes in the 3, 3 position?

\[ \Lambda = \begin{bmatrix} 5 & -3 & ?? \\ ?? & ?? & -2 \\ ?? & ?? & ?? \end{bmatrix} \]


- Answer: ???

Question #4

- Question: What pairing is unacceptable?

\[ \Lambda = \begin{bmatrix} 0.3 & 0 & 0.7 \\ -1 & 3 & -1 \\ 1.7 & -2 & 1.3 \end{bmatrix} \]

[A] \( y_1 \leftrightarrow u_1 \)  [B] \( y_3 \leftrightarrow u_3 \)  [C] \( y_2 \leftrightarrow u_2 \)  [D] \( y_3 \leftrightarrow u_2 \)

- Answer: ???
Multivariable Decoupling

- If you have models, treat other controller input as measured disturbance...

![Diagram of multivariable decoupling](image)

**Space for Notes Below**
State Space for Control?

Space for Notes Below

Question #5

- **Question:** What is most confusing still?
  
  - A Negative feedback (PID controller tuning, complex block diagrams, CLTF response characterization, CL stability)
  - B Frequency response (Bode Plots, $A_R(\omega)$ and $\phi(\omega)$)
  - C Feedforward (calculating $g_{ff} = -g_d/g$)
  - D IMC (Internal Model Control)
  - E Multivariable and Decoupling Control

- **Answer:** ???
Advanced Control Review

Feedforward, IMC, Closed-Loop, Multivariable, RGA Pairing

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Lecture 24

ECHE 550 Chemical Process Dynamics and Control

Closed-Loop Transfer Function PI Control

- For PI control, $g_c = K_c \left(1 + \frac{1}{\tau_l \frac{1}{s}}\right)$ and FOTD $g(s) = \frac{K}{\tau s + 1} e^{-\alpha s}$

\[ y(s) = \frac{g g_c}{1 + g g_c} y_{sp}(s) + \frac{g d}{1 + g g_c} d(s) \]

CLTF
Closed-Loop Response

Closed-Loop Setpoint Tracking Response

$y_p(t)$

$y(t)$

Overshoot Percentage: $\frac{a_2}{a_1}$

Offset Percentage: $\frac{O}{A}$

Decay Ratio: $\frac{a_2}{a_1}$

Rise Time: $t_r$

Settling Time: $t_s$

Review: Feed-forward Control

- Must have models $g(s)$ and $g_d(s)$
- Must know disturbance value $d(s)$
- Response?

$$y(s) = g_d d + g g_f d = \left( g_d + g g_f \right) d$$

Want $\left( g_d + g g_f \right) = 0$ to minimize effect of $d$ so $g_f = -\frac{g_d}{g}$
**FF Control Example**

- Develop a feed forward controller based on the models:
  
  \[ g_d = \frac{3e^{-1s}}{4s+1} \quad g = \frac{(-14s+7)e^{-3s}}{(3s+1)(5s+1)} = \frac{7(-2s+1)e^{-3s}}{(3s+1)(5s+1)} \]

  \[ g_{ff} = -\left(\frac{3e^{-1s}}{4s+1}\right)\left(\frac{(3s+1)(5s+1)e^{3s}}{7(-2s+1)}\right) = \left(-\frac{3}{7}(3s+1)(5s+1)e^{2s}\right) \]

- Drop time prediction and RHP pole \((-2s+1)\)
  
  \[ g_{ff} = \left(-\frac{3}{7}(3s+1)(5s+1)\right)(4s+1) \]

- Add filter to make semi-proper
  
  \[ g_{ff} = \left(-\frac{3}{7}(3s+1)(5s+1)\right)(4s+1)(\lambda s+1) \]

**FF Control Example**

- Now, assume new models for \(g\) and \(g_d\):
  
  \[ \hat{g}_d = \frac{4e^{-2s}}{5s+1} \quad \hat{g} = \frac{8(-3s+1)e^{-4s}}{(4s+1)(6s+1)} \]

- Find SS offset using your feed-forward controller and the new models for step of size \(A\) in \(d(s)\)
  
  \[ y(t = \infty) = sy(s)|_{s=0} = (g_d + g_{ff}) d(s) s = \left(\frac{4e^{-2s}}{5s+1}\right)\left(\frac{8(-3s+1)e^{-4s}}{(4s+1)(6s+1)}\right) \left(\frac{-\frac{3}{7}(3s+1)(5s+1)}{(4s+1)(\lambda s+1)}\right) \left(\frac{A}{s}\right)s \]

  \[ = \left(4 + 8\left(-\frac{3}{7}\right)\right) A = 0.571 A \]

- Still better than without, gain of \(g_d = 4A\)
CLTF + FF Control

- Both feedback and feed-forward

\[ y(s) = \frac{g_d g_c}{1 + g g_c} y_{sp}(s) + \frac{g_d}{1 + g g_c} d(s) + \frac{g_{ff} g}{1 + g g_c} d(s) \]

- CLTF?

Internal Model Control

- Find \( c(s) \) from model \( \tilde{y}(s) \)
- Problems: RHP poles, time prediction, improper
Multivariable Open-Loop

- Multiple-Input-Multiple-Output (MIMO)

\[
y_1(s) = g_{11}(s)u_1(s) + g_{12}(s)u_2(s)
\]
\[
y_2(s) = g_{21}(s)u_1(s) + g_{22}(s)u_2(s)
\]

Multivariable Systems

- Steady State Gain Matrix

\[
K = G(s)\big|_{s=0}
\]

- Poles of \(G(s)\) are poles of \(g_{ij}(s)\)
  - Repeated poles only if repeated in a \(g_{ij}(s)\)

- Zeros of \(G(s)\) from solving

\[
det(G(s)) = 0
\]
Multivariable Closed-Loop

- 2x2 MIMO: How do you close the loop? 2 Options

Relative Gain Array (RGA)

- Get steady-state gain matrix, $K = G(s = 0) = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$
- Find $\zeta$ and RGA matrix $\Lambda$, **Rows and columns sum to 1**
  \[
  \zeta = \frac{K_{12}K_{21}}{K_{11}K_{22}} \quad \Lambda = \begin{bmatrix}
  1 & -\zeta \\
  1 & 1 - \zeta
  \end{bmatrix}
  \]
- If the $\lambda_{ij}$ element is $\leq 0$, avoid pairing output $i$ with input $j$.
  - Very bad interaction between $i$ and $j$.
- If the $\lambda_{ij} = 1$ pair output $i$ with input $j$, no interaction.
Multivariable Decoupling

- If you have models, treat other controller input as measured disturbance…

![Block Diagram]

**Multivariable Decoupling**

- Top loop $y_1 \leftrightarrow u_1$ (Assumes “Option 1” pairing)
  - $u_2$ is disturbance, $g_d = g_{12}$
  
  $$g_{I1} = -\frac{g_{12}}{g_{11}}$$

- Bottom loop $y_2 \leftrightarrow u_2$
  - $u_1$ is disturbance, $g_d = g_{21}$
  
  $$g_{I2} = -\frac{g_{21}}{g_{22}}$$

- Can always reorder outputs to make it look like option 1
16.25 Numerical Optimization

Numerical Optimization
Finding the best feasible solution

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Lecture 25

Review: Unconstrained Optimization

- “Objective Function” or “Cost Function”
  - May be function of many variables
- Ex: min $x^2 + y^2$
  - 2D problem defines manifold (surface) in 3D space
  - $z = x^2 + y^2$
- Must find point where gradient of function $= 0$

$$\nabla f(x, y) = \begin{bmatrix} \frac{\delta f}{\delta x} \\ \frac{\delta f}{\delta y} \end{bmatrix}$$
Question 1

\[ f(x, y) = x^3 y^2 \]

**Question:** What is equivalent to the gradient of the above function evaluated at the point \( x = 1, y = 1 \)

- [A] \( \begin{bmatrix} 3x \\ 2y \end{bmatrix} \)
- [B] \( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \)
- [C] \( \begin{bmatrix} 3x^2 y^2 \\ 2x^3 y \end{bmatrix} \)

- [D] All of the above.
- [E] I have no clue.

**Answer:**

???

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Review: Unconstrained Optimization

- For minimization, \(-\nabla f\) points “downhill” in steepest direction

- Trick for maximization problems:

\[
\max_x f(x) = \min_x -f(x)
\]

- Steepest Descent Method:
  - Start at initial guess, \(\underline{x}_0\)
  - Evaluate \(-\nabla f\) at current point
  - Perform line search in improving direction
  - Update current best point and repeat until \(-\nabla f\) is “small”
Review: Constrained Optimization

- Standard form with $N$ constraints:
  \[
  \min_{\mathbf{x}} f(\mathbf{x}) \\
  \text{subject to } g_1(\mathbf{x}) \leq 0 \\
  g_2(\mathbf{x}) \leq 0 \\
  \vdots \\
  g_N(\mathbf{x}) \leq 0
  \]

- Bounds on variables in standard form:
  \[
  1 \leq x \leq 10 \\
  1 - x \leq 0 \\
  x - 10 \leq 0
  \]

- Equality constraints in “standard” form (inequalities):
  \[
  x = y^2 \\
  0 \leq x - y^2 \leq 0 \\
  x - y^2 \leq 0 \\
  -x + y^2 \leq 0
  \]
Review: Linear vs. Nonlinear

- Linear examples and general form for sets of linear equality and inequality constraints:

\[
\begin{align*}
2x + 3y &= 4 \\
x - y &\leq 2 \\
A\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} b \end{bmatrix} \\
A\begin{bmatrix} x \\ y \end{bmatrix} &\leq \begin{bmatrix} b \end{bmatrix}
\end{align*}
\]

- Nonlinear examples and general form for nonlinear constraints:

\[
\begin{align*}
2x^2 + y^2 &= 4 \\
y &= e^x \\
g(x) &\leq 0 \\
h(x) &= 0
\end{align*}
\]

Review: Convex vs. Nonconvex

- Eigenvalues of Hessian must be \( \geq 0 \) to be convex function
- Graphically: In a convex set you may pick any two points and the line between the two points contains only points inside the set.
- Get problem in standard form, check Hessian eigenvalues:

\[
\begin{align*}
y &\geq x^2 \\
x^2 - y &\leq 0
\end{align*}
\]

\[
\nabla f(x) = \begin{bmatrix} 2x \\ -1 \end{bmatrix} \quad H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}
\]

Eigenvalues are 2, 0 so \( y \geq x^2 \) is a convex constraint.
Review: Convex vs. Nonconvex

- Unconstrained example

\[\min_{x, y} x^2 + y^2\]

\[\nabla f(x) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, \quad H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\]

Eigenvalues are 2, 0 so \(x^2 + y^2\) is a convex constraint.

---

Question 2

\[x^2 + y^2 \geq 9\]

- **Question:** Is the constraint convex?
  - [A] Yes
  - [B] No
  - [C] What?
  - [D] All of the above.
  - [E] I have no clue.

- **Answer:**
  - ???

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Review: Convex vs. Nonconvex

- General form for constrained optimization:
  \[
  \min_{\mathbf{x}} f(\mathbf{x})
  \]
  subject to \( g_1(\mathbf{x}) \leq 0 \)
  \( g_2(\mathbf{x}) \leq 0 \)
  \( h_1(\mathbf{x}) = 0 \)
  \( h_2(\mathbf{x}) = 0 \)

- Rules. Nonconvex problem if any of following are true:
  - \( f(\mathbf{x}) \) is a nonconvex function in the domain of \( \mathbf{x} \)
  - Any \( g_i(\mathbf{x}) \) inequality constraint is a nonconvex function in the domain of \( \mathbf{x} \)
  - Any \( h_i(\mathbf{x}) \) equality constraint is nonlinear

- Convex problems have a single solution

Review: KKT Conditions

- Check to see if a point \( \mathbf{x}^* \) is optimal in constrained optimization
  - Put problem in standard form with only inequality constraints
  - Find any active constraints, \( g_i(\mathbf{x}^*) = 0 \)
  - Solve for Lagrange multipliers
    \[
    - \nabla f(\mathbf{x}^*) = \sum \lambda_i g_i(\mathbf{x}^*)
    \]
    \[\lambda_i \geq 0\]

- Lagrange multipliers represent how “hard” the problem “pushes” against the constraints
- Lagrange multipliers must all be positive for the point to be a KKT point
Review: KKT Conditions

Feasible space

\[
\begin{bmatrix}
\frac{3}{3}
\end{bmatrix} = \lambda_1 \begin{bmatrix}
\frac{2}{1}
\end{bmatrix} + \lambda_2 \begin{bmatrix}
\frac{1}{2}
\end{bmatrix}
\]

\[\lambda_1 = \lambda_2 - 1\]

\[
\begin{bmatrix}
\frac{2}{3}
\end{bmatrix} = \lambda_1 \begin{bmatrix}
\frac{2}{1}
\end{bmatrix}
\]

\[\lambda_1 = ??\]

\[
\begin{bmatrix}
\frac{2}{3}
\end{bmatrix} = \lambda_1 \begin{bmatrix}
\frac{2}{1}
\end{bmatrix} + \lambda_2 \begin{bmatrix}
\frac{0}{2}
\end{bmatrix}
\]

\[\lambda_1 = 1, \lambda_2 = -1\]

Infeasible space

Question 3

Question: Why do we care about optimization?

[A] Gatzke told me to
[B] Model Predictive Control (MPC)
[C] Parameter estimation
[D] It is a useful tool for many problems
[E] All of the above

Answer:

???
Motivation: Process Design

- **Objective function:** Maximize profits, minimize cost
- **Decision variables:**
  - Number and size of components
  - Flow rates, temperatures, pressures
- **Constraints**
  - Mass and energy balances / design equations
  - Environmental limits, product limits

Motivation: Process Modeling

- **Objective function:** minimize the error between data and model
- **Decision variables:**
  - Model parameters (kinetic parameters)
- **Constraints:**
  - Model equations
  - Assumed limits on parameters
  - Physical limits on variables (concentrations positive)
Motivation: Process Scheduling

- **Objective function:** minimize the cost for your process
- **Decision variables:**
  - When to make products
  - What equipment to use
- **Constraints:**
  - Limits on batch sizes
  - Limits on storage
  - Order fulfillment requirements

Motivation: Process Control

- **Objective function:** minimize future deviation from setpoint
- **Decision variables:**
  - Future process input values
- **Constraints:**
  - Dynamic model equations for prediction
  - Limits on the inputs
  - Process variable limits
Problems: Linear Programming (LP)

- All linear objective function and constraints

\[
\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \\
\mathbf{A} \mathbf{x} \leq \mathbf{b} \\
\mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U
\]

- Compact form. Methods scale to very large problems. **Convex!**

\[
\min_{x,y} 3x - 4y \\
y - x \leq 2 \\
y + x \leq 3 \\
0 \leq x, y \leq 3
\]

Problems: Quadratic Programming (QP)

- Almost linear. Quadratic objective function, linear constraints

\[
\min_{\mathbf{x}} \mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{c}^T \mathbf{x} \\
\mathbf{A} \mathbf{x} \leq \mathbf{b} \\
\mathbf{x}^L \leq \mathbf{x} \leq \mathbf{x}^U
\]

- Simple compact form. Convex if eigenvalues of \( \mathbf{H} \) are non-negative

\[
\min_{x,y} 3x^2 + 4y^2 - x \\
y - x \leq 2 \\
y + x \leq 3 \\
0 \leq x, y \leq 3
\]
Problems: Nonlinear Programming (NLP)

- NLP: Nonlinear objective and/or nonlinear constraints

\[
\min_{x} f(x) \\
g(x) \leq 0 \\
x^L \leq x \leq x^U
\]

- May be **convex** or **nonconvex**

\[
\min_{x, y} x^3 + \frac{x}{y^2} + e^x \\
y - x^2 - 1 \leq 0 \\
y + e^x - 1 \leq 0 \\
0 \leq x, y \leq 3
\]

---

Problems: Mixed-Integer Linear Programming (MILP)

- All linear objective function and constraints, binary variables

\[
\min_{x} c^T x \\
Ax \leq b \\
x^L \leq x \leq x^U \\
x_i \in \{0, 1\}
\]

- Compact form. Solution can be difficult. **Nonconvex**!

\[
\min_{x, y} 3x - 4y \\
y - x \leq 2 \\
y + x \leq 3 \\
0 \leq x, y \leq 3 \\
x, y \in \{0, 1\}
\]
Problems: Mixed-Integer Nonlinear Programming (MINLP)

- NLP: Nonlinear objective and/or nonlinear constraints
  \[
  \min f(x) \\
  g(x) \leq 0 \\
  x^L \leq x \leq x^U \\
  x_i \in \{0, 1\}
  \]

- Nonconvex and difficult to solve
  \[
  \min_{x,y} x^3 + \frac{x}{y^2 + e^x} \\
  y - x^2 - 1 \leq 0 \\
  y + e^x - 1 \leq 0 \\
  0 \leq x, y \leq 3 \\
  x, y \in \{0, 1\}
  \]

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Model Predictive Control

- Setpoint
- Model Prediction
- Corrected Model Estimate
- Current Model Estimate
- Future Input Trajectory

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Model Predictive Control

- Also known as “Dynamic Matrix Control” or just MPC
- Procedure
  - Get process measurements
  - Formulate constrained optimization problem
    - Use prediction horizon $P$
  - Solve constrained optimization problem
    - Solve over move horizon $M$ for optimal input changes
  - Implement only first input move value
  - Repeat

Features:
- Includes actuator constraints (limits)
- Accounts for multivariable interaction
- Can be applied in non-square MIMO systems
- Can accommodate known future setpoint changes
- Can use measured disturbance + model (feedforward)

Issues:
- Requires accurate dynamic multivariable model
  - Usually linear discrete time models (ARMA or SS)
- Requires repeated online solution of optimization problem
- Can be difficult to tune $(P, M, \Gamma_y, \Gamma_{\Delta u})$
MPC Formulation

\[
\min \sum_{k=1}^{P} e(k)^T \Gamma_y e(k) + \sum_{k=1}^{M} \Delta u(k)^T \Gamma \Delta u(k)
\]

\[y_j(k) = d_j + \sum_{i=0}^{N_a} a_i u(k-i) + \sum_{j=1}^{N_b} b_j y(k-j) \quad \forall i = 1...P, j = 1...N_y\]

\[e_j(k) = y_j SP(k) - y_j(k) \quad \forall k = 1...P\]

\[u^L \leq u(k) \leq u^U \quad \forall k = 1...M\]

\[\Delta u(k) = u(k+1) - u(k) \quad \forall k = 0...M\]

\[d_j = y_j \text{ meas} - y_j \text{ model}\]

- Quadratic objective function based on error vector \(e\) and \(\Delta u\)
- Dynamic model equations \(P\) steps into future (ARMA with disturbance update \(d_j\))
- Error values based on known setpoint and model prediction
- Bounds on input values (upper and lower bounds)
- Velocity of input changes (\(\Delta u\)) calculated
- Disturbance update based on measurement and model

---

Question 4

What is the most confusing optimization topic?

- [A] Convexity of functions / constraints
- [B] KKT conditions
- [C] Problem classification
- [D] Model Predictive Control
- [E] Other

Answer:

???

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