CSCE 785, Fall 2024
Homework 5, due Wednesday December 4

1 Written Exercises

Do Exercises 7.2, 7.9, 7.10, 7.12.
Do Exercises 8.4, 8.7, 8.11.

2 Programming Exercise

This exercise involves building and testing circuits on IonQ's quantum platform. You may try any one of the four options. If you want to do more than one, great, but it's not required. Also if you'd like another option besides these, please let me know and I may include it. These exercises are somewhat open-ended.

1. Implement the quantum circuit for the Toffoli gate (see Exercise 3.15(3)).
2. Implement the teleportation circuit of Figure 3.5 on page 99.
3. Implement encoding and error-correction circuit (on the same circuit) for the 3-qubit bit-flip code.
4. Implement the phase estimation circuit with three control qubits.

Item (2.) needs some explanation. The circuit is a dynamic circuit that uses classically controlled Pauli gates. This is handled by adding an if-conditional attribute to the gate, rather than making a fully quantum controlled gate. You also need to produce an EPR pair on the second and third qubits. This is done by a little circuit with one Hadamard gate and one controlled NOT gate. NOTE: Dynamic circuits should run on the simulator, but maybe not on the QPU. (This was true with IBM, but I'm not yet sure about IonQ.)

For (3.), if you include the complete circuit, including Pauli corrections, then this is a dynamic circuit, so it may not run on a QPU, but if you leave off those corrections and just report the error syndrome as the final measurement, then this part of the circuit *can* be run on a QPU.

In all cases, be aware that all qubits are in the $|0\rangle$ state to start with.

Testing

Testing for correctness is not a trivial task, so please devote some thought to this.

The Toffoli gate

You should test your circuit by running on all possible computational basis states as inputs. All qubits are in state $|0\rangle$ initially, so to set a qubit to $|1\rangle$, insert an X-gate on that qubit at the beginning. Measure all three qubits at the end, and check how closely the probabilities match what you'd expect from a Toffoli gate. After debugging on the simulator (first with the ideal model, then with the aria-1 noise model), run it on the QPU with 200 shots for each run.

The teleportation circuit

The goal is to see if an *arbitary* 1-qubit is teleported correctly. To this end, you will add a $U(\cdots)$ gate on the first qubit at the beginning, to rotate $|0\rangle$ to an arbitrary state. $U(\cdots)$ takes three real parameters, which are the Euler angles for an arbitrary 1-qubit unitary (up to an overall phase factor).

How the U-gate works: I did not find the documentation for this gate helpful at all. After much trial and error, I discovered for myself that

$$U(\theta,\phi,\lambda) = e^{i(\phi+\lambda)/2} R_z(\phi) R_y(\theta) R_z(\lambda) = e^{i(\phi+\lambda)/2} e^{-iZ\phi/2} e^{-iY\theta/2} e^{-iZ\lambda/2} \ ,$$

where Y and Z are the usual Pauli operators. You should choose random values for θ, ϕ, and λ. Choose $\theta \in [0,\pi]$ and $\phi,\lambda \in [0,2\pi)$. Do this several (four or five) times for independently random triples of numbers. To verify that the qubit was correctly teleported in each case, you should place the *adjoint* of the U-gate on the third qubit at the end, then measure the third qubit. If the circuit works correctly, you should see 0 with probabilty 1 (or something close to it, given round-off error and noise). The adjoint of $U(\theta,\phi,\lambda)$ is

$$U(\theta,\phi,\lambda)^* = U(-\theta,-\lambda,-\phi) \ .$$

Note that λ and ϕ swap positions as well as being negated.

Since this is a dynamic circuit, it may not be implementable on an IonQ QPU. If not, just just the simulator.

The bit-flip code

Run the circuit on unencoded inputs $|0\rangle$ and $|1\rangle$, measuring the three qubits at the end. You should get $|000\rangle$ and $|111\rangle$, respectively.

Now do the same as above, but insert an X gate on one of the three qubits between the encoding and error recovery parts of the circuit, simulating a bit flip error on that qubit. You should see the same results. Try with an X gate on each of the three qubits individually.

Since the error recovery is a dynamic circuit, the same caveat applies here as with the teleportation circuit. However in this case, you *can* test the syndrome measurement on a QPU, just not the classically controlled recovery.

Phase estimation

This will require you to build and test the QFT_3 circuit you designed in the last homework. You should run your QFT_3 circuit on all eight computational basis states as inputs, as in the Toffoli gate testing above. If your circuit is correct, then for any $x \in \mathbb{Z}_8$, the output qubit state is given by the right-hand side of (4.2). You will measure each qubit, but only after rotating each qubit back to $|0\rangle$ by first applying a phase gate with phase $e_k(-x)$ for $k = 1, 2, 3$ (to obtain the state $|+\rangle$ on that qubit) then applying a Hadamard gate (to obtain the state $|0\rangle$ on that qubit). Then each measurement should give 0 with probability 1 (or close to it, given noise). Note that you need a different trio of phase gates for each of the eight values of x. Also remember that, without adding explicit swap gates, the qubits come out in reverse order at the end.

For the phase estimation circuit, use $U := R_z(\theta)$ for various θ. $|\psi\rangle = |0\rangle$ is an eigenvector of this gate with eigenvalue $e^{-i\theta/2}$.