CSCE 785, Fall 2025
Homework 1
due 9/8/2025

Reading

By "course notes" I will always mean the latest version of the course notes that is available on Blackboard (not on the web). This is significantly updated from previous semesters.

Submit all exercises via Blackboard, now and in the future. (Scanned hand-written answers are fine, as long as they are neat and legible. You can also typeset some or all of your answers.)

Collaboration on the homework is allowed, and collaborating groups may submit a single homework for the group. (Choose a representative to submit for the group. Make sure the names of all members are listed on the submission.)

Exercises

Read through enough of Chapter 1 of the course notes so that you can do the exercises below.

The following are all from the course notes but are reproduced here for your convenience (and some of the hints given here are improvements on those in the notes). The course notes have links from the exercises to the places in the text where they are most relevant.

Ex 1.3 Verify Equation (1.3) on page 4. [Hint: Compare the power series for $e^{i\theta}$ with those for $\sin\theta$ and $\cos\theta$. Other proofs exist.]

Ex 1.4 Using Euler's formula (from the previous exercise), find $e^{i\pi/2}$ and $e^{-i\pi/3}$. Express each answer in the form $x + iy$ for real x, y.

Ex 1.6 Let Ω be a probability space and $A, B \subseteq \Omega$ events. Show that if A and B are independent, then $\Omega \setminus A$ and B are independent. [Hint: Use separation (Proposition 1.1(3)).]

Ex 1.13 Let $\ell : F^{n\times n} \times F^{n\times n} \to F^{n\times n}$ be a bilinear map such that $\ell(A, A) = 0$ for all $A \in F^{n\times n}$. Show that ℓ is antisymmetric, that is, $\ell(A, B) = -\ell(B, A)$ for all $A, B \in F^{n\times n}$.

Ex 1.14 Let A and B be invertible matrices. Prove that if A commutes with B, then A commutes with B^{-1}. (So applying this fact repeatedly, we get that all of A, B, A^{-1}, and B^{-1} commute with each other pairwise.)

Ex 1.15 Prove that tr is the unique function mapping $n \times n$ matrices to scalars satisfying the properties (1)–(3) given in Section 1.4.11.

Ex 1.25 For every integer $n \geq 1$, let s_n be the probability that a uniformly randomly chosen $n \times n$ matrix over $\mathbb{Z}_2$ is invertible (equivalently, has rank n). For example, $s_1 = 1/2$ and $s_2 = 3/8$.

1. Find an expression for s_n that holds for all $n \geq 1$. (It won't be closed form.)
2. Using the expression you got in (1), show that $s_1 > s_2 > s_3 > \cdots$.
3. (Harder) Let $s := \lim_{n\to\infty} s_n$. Show that $s > 0$. In fact, show that $s > 1/4$.

Ex 1.28 Show that any orthogonal set S of nonzero vectors in a $\mathbb{C}$-space is linearly independent. [Hint: Let v be any linear combination of elements of S, let $u \in S$ be arbitrary, and consider $\langle u, v\rangle$. You'll need the fact that $\langle \cdot, \cdot \rangle$ is positive definite.]

Ex 1.37 Show that, for any orthonormal basis $\mathcal{B} := \{\beta_1, \ldots, \beta_n\} \subseteq \mathcal{H}$, where $\mathcal{H}$ is a $\mathbb{C}$-space, the identity operator I on $\mathcal{H}$ satisfies

$$I = \sum_{j=1}^{n} \beta_j \beta_j^* \ .$$

This generalizes Eq. (1.41). [Hint: Since both sides are linear maps $\mathcal{H} \to \mathcal{H}$, it suffices to show that both sides are equal when applied to any β_k.]