$\mathcal{L}(\mathcal{H}, \mathcal{J})$ is a $\mathbb{C}$-space

Dirac notation

Density operators

Back to 1 qubit states

$\dim(\mathcal{H}) = n \quad \dim(\mathcal{J}) = n$

Then $\mathcal{L}(\mathcal{H}, \mathcal{J}) \cong \mathbb{C}^{m \times n}$

Given $A, B \in \mathbb{C}^{m \times n}$

Define Hermitian inner product

$$\langle A, B \rangle = \mathrm{tr}(A^* B)$$

$$\mathrm{tr}(A^* B) = \sum_{j=1}^{n} [A^* B]_{jj}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} [A^*]_{ji} [B]_{ij}$$

$$= \sum_{i,j} [A]_{ij}^* [B]_{ij}$$

Standard basis:

$$\{ E_{kl} : 1 \le k \le n, 1 \le l \le m \}$$

$$[E_{kl}]_{ij} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{o.w.} \end{cases}$$

$$E_{kl} = \begin{array}{c} \\ k \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} & & \\ & 1 & \\ \text{0's elsewhere} & & \end{bmatrix}$$
$$\qquad\qquad\qquad \uparrow l$$

Fact:

$$E_{kl} E_{rs} = \begin{cases} E_{ks} & \text{if } l = r \\ 0 & \text{otherwise} \end{cases}$$

Dirac notation

$|\psi\rangle$ — denotes a column vector with norm 1.

(recall $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

ψ is just a label

$(\psi \neq |\psi\rangle)$

$|\psi\rangle$ is called a ket vector

$\langle\psi| := |\psi\rangle^*$ (row vector adjoint)

Inner product of $|\psi\rangle, |\varphi\rangle$:

is $|\psi\rangle^* |\varphi\rangle = \langle\psi|\varphi\rangle$

$\langle\psi|$ is a bra vector

$\langle\psi|\varphi\rangle$ is a bracket (bra-ket)

$|\varphi\rangle\langle\psi|$ is an operator;

Apply to some vector $|\lambda\rangle$:

$$(|\varphi\rangle\langle\psi|)|\lambda\rangle = |\varphi\rangle \underbrace{\langle\psi|\lambda\rangle}_{\text{scalar}}$$

$$= (\langle\psi|\lambda\rangle)|\varphi\rangle$$

$|\psi\rangle\langle\psi|$ is a projector onto the 1-dim subspace spanned by $|\psi\rangle$: $\{ z|\psi\rangle : z \in \mathbb{C} \}$

Check: $(|\psi\rangle\langle\psi|)^* = |\psi\rangle\langle\psi|$
Hermitian ✓

$$(|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|)$$

$$= |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{\substack{=1 \\ = \| |\psi\rangle \|^2}} \langle\psi|$$

$$= |\psi\rangle\langle\psi|$$

$$(|\psi\rangle\langle\psi|)|\lambda\rangle = z|\psi\rangle$$
$$(z = \langle\psi|\lambda\rangle)$$

Density operators

So far a state is a unit vector $|\psi\rangle$

Phase freedom:

$|\psi\rangle$ and $e^{i\theta}|\psi\rangle$

both represent the same physical state ($|\psi\rangle \propto |\varphi\rangle$ means same state.)

Def: A pure state in $\mathcal{H} = \mathbb{C}^n$ is a one-dimensional projector, equivalently, a pure state is of the form

$$\rho = |\psi\rangle\langle\psi| \quad \text{— density operator}$$

for some unit vector $|\psi\rangle$.

Fact
$|\psi\rangle \propto |\varphi\rangle$ iff $|\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi|$

($\Rightarrow$) if $|\varphi\rangle = e^{i\theta}|\psi\rangle$ then

$$|\varphi\rangle\langle\varphi| = \left(e^{i\theta}|\psi\rangle\right)\left(e^{i\theta}|\psi\rangle\right)^*$$
$$= e^{i\theta}|\psi\rangle e^{-i\theta}\langle\psi|$$
$$= |\psi\rangle\langle\psi|.$$

($\Leftarrow$) $|\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi|$ means both project onto the same 1-dim subspace, which contains both $|\psi\rangle$ and $|\varphi\rangle$

so $|\psi\rangle \propto |\varphi\rangle$. //

Translating axioms of QM to density operator view:

previous

Unitary evolution

$|\psi\rangle \mapsto U|\psi\rangle$
(U unitary)

new

$$|\psi\rangle\langle\psi| = |\psi\rangle|\psi\rangle^*$$
$$\mapsto U|\psi\rangle\left(U|\psi\rangle\right)^*$$
$$= U|\psi\rangle\langle\psi|U^*$$

$\left[\rho \mapsto U\rho U^*\right.$ unitary conjugation $\left.\right]$

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previous

proj. measurement with a csop $\{P_1, \dots, P_k\}$
$\forall j \in \{1, \dots, k\}$

$$\Pr[j] = \| P_j|\psi\rangle \|^2$$
$$= \langle\psi|P_j^* P_j|\psi\rangle$$
$$= \langle\psi|P_j|\psi\rangle$$

new

$$\rho = |\psi\rangle\langle\psi|$$
$$\Pr[j] = \langle\psi|P_j|\psi\rangle$$
$$= tr\left(\langle\psi|\,\underbrace{P_j|\psi\rangle}\right)$$
$$= tr\left(P_j|\psi\rangle\langle\psi|\right)$$
$$= tr\left(P_j\rho\right)$$
$$= \langle P_j, \rho\rangle$$

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previous

post measurement state given outcome j:

$$|\varphi\rangle := \frac{P_j|\psi\rangle}{\| P_j|\psi\rangle \|}$$

new

Given $\rho = |\psi\rangle\langle\psi|$
Want post-meas state to be $|\varphi\rangle\langle\varphi|$, given outcome j:

$$|\varphi\rangle\langle\varphi| = \frac{1}{\| P_j|\psi\rangle\|^2} P_j|\psi\rangle\langle\psi|P_j$$
$$= \frac{P_j\rho P_j}{\langle P_j, \rho\rangle}$$

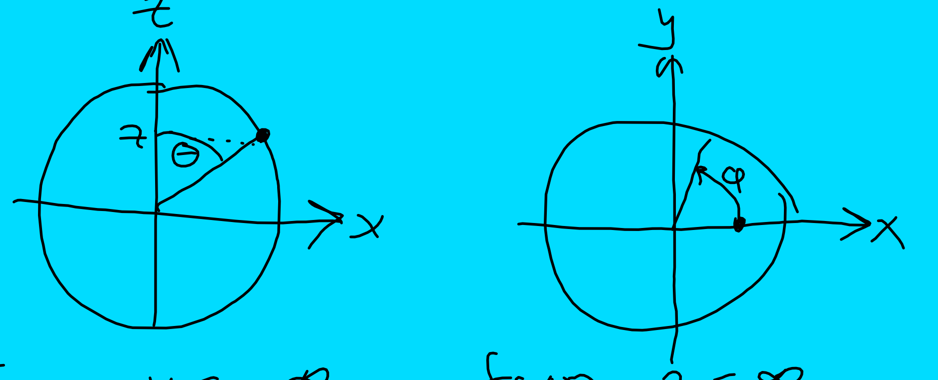
1-qubit again

Recall: arbitrary 1-qubit state is rep by vector

$$\boxed{|\psi\rangle := \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle}$$
$$= \begin{bmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{bmatrix}$$

for some $0 \le \theta \le \pi$
and $0 \le \varphi < 2\pi$

(θ, φ) are the spherical coords of the spin direction:
point on the unit sphere (Bloch sphere)



from $y = -\infty$ from $z = \infty$

$\hat{n} = (x, y, z)$ — cartesian coords

$x, y, z \in \mathbb{R}$ $(x^2 + y^2 + z^2 = 1)$

$$\boxed{\begin{aligned} z &= \cos\theta \\ x &= \sin\theta\cos\varphi \\ y &= \sin\theta\sin\varphi \end{aligned}}$$

Pauli operators (Pauli spin matrices):

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =: \sigma_1$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: \sigma_0$$

$$Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} =: \sigma_2$$

$$Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} =: \sigma_3$$

$$\boxed{|\psi\rangle := \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle}$$

in density operator formalism:

$$\rho := |\psi\rangle\langle\psi| = \begin{bmatrix} \cos(\theta/2) \\ e^{i\varphi}\sin(\theta/2) \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & e^{-i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\left(\frac{\theta}{2}\right) & e^{-i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ e^{i\varphi}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & \sin^2\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\cos\theta}{2} & e^{-i\varphi}\frac{\sin\theta}{2} \\ e^{i\varphi}\frac{\sin\theta}{2} & \frac{1-\cos\theta}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+\cos\theta & (\cos\varphi - i\sin\varphi)\sin\theta \\ (\cos\varphi + i\sin\varphi)\sin\theta & 1-\cos\theta \end{bmatrix}$$

$$= \frac{1}{2}\left(I + (\sin\theta\cos\varphi)X + (\sin\theta\sin\varphi)Y + (\cos\theta)Z \right)$$

$$= \frac{1}{2}\left(I + xX + yY + zZ \right)$$

Express $\rho = |\psi\rangle\langle\psi|$ in the Pauli basis (I, X, Y, Z) — get cartesian coordinates of the spin.

Basic properties:

$$I^2 = X^2 = Y^2 = Z^2 = I$$

$$XY = -YX = iZ$$
$$YZ = -ZY = iX$$
$$ZX = -XZ = iY$$

cyclic shifts

$X \to Y \to Z$

$\mathrm{tr}\, I = 2$, $\mathrm{tr}\, X = \mathrm{tr}\, Y = \mathrm{tr}\, Z = 0$

$\forall i,j \in \{0, \dots, 3\}$

$$\langle \sigma_i, \sigma_j \rangle = \begin{cases} 2 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$$