Fundamentals of QM

- Physical system
- Quantum states
- Unitary evolution
- Measurement

A <u>quantum</u> <u>(physical)</u> system corresponds to a Hilbert space $\mathcal{H}$.

$\mathcal{H}$ infinite-dimensional — "unbound" systems, e.g. unconfined particles, fields, etc.

$\mathcal{H}$ finite-dimensional ($\mathbb{C}$-space): bound systems — finite degrees of freedom: confined particles

angular momentum
- orbital ang momentum in an atom
- intrinsic spin

Fix $\mathcal{H}$ $\mathbb{C}$-space for a system.

At any given time the system is in a <u>state</u> represented by a unit vector $u \in \mathcal{H}$.

$\|u\| = 1$.

Digression: <u>Dirac Notation (bra-ket)</u>

A unit (column) vector (state) is written as $|\psi\rangle$ (a <u>ket</u>). ψ is some label. Its adjoint $|\psi\rangle^* = \langle\psi|$ (a <u>bra</u>). If $|\psi\rangle$ and $|\varphi\rangle$ are kets, we can form the inner product

$$\langle\varphi|\psi\rangle \in \mathbb{C}$$

("bra-ket")

$|\psi\rangle\langle\varphi|$ is an operator (If $\mathcal{H}$ is n-dimensional, this is an $n \times n$ matrix)

<u>Unitary evolution</u>:

An isolated system evolves according to a (time-dependent) unitary operator:

$$U(t_2, t_1)$$

$t_1, t_2 \in \mathbb{R}$ moments in time and $U(t_2, t_1)$ is unitary. So that, if the system is in state $|\psi\rangle$ at time t_1, it will be in state

$$U(t_2, t_1)\,|\psi\rangle$$

at time t_2. (not necessary that $t_2 > t_1$).

$\| |\psi\rangle \| = 1 \Rightarrow \| U(\cdot\cdot)|\psi\rangle \| = 1$

b/c $U(\cdots)$ is unitary.

So states get mapped to states.

Additional properties (common sensical)

$$U(t,t) = I \qquad (\forall t)$$

$$U(t_2, t_1) = U(t_1, t_2)^* \quad \left[= U(t_1, t_2)^{-1} \right]$$

$\forall t_1, t_2$

$\forall t_1, t_2, t_3$

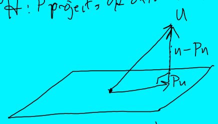
$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$$

Measurement: Projective measurements.

Def:
An operator $P \in \mathcal{L}(\mathcal{H})$ is a projector (orthogonal projection operator) if

1) $P = P^*$ (P is Hermitian)
2) $P = PP$ (P is idempotent)

$\mathcal{H}$: P projects $\mathcal{H}$ onto a subspace



P projects orthogonally:
show that $\langle u-Pu, Pu\rangle = 0$.

$$\langle u-Pu, Pu\rangle = \langle u, Pu\rangle - \langle Pu, Pu\rangle$$
$$= \langle u, Pu\rangle - \langle u, P^*Pu\rangle$$
$$= \langle u, Pu\rangle - \langle u, PPu\rangle$$
$$= \langle u, Pu\rangle - \langle u, Pu\rangle = 0$$

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Note: I is a projector
O " " "

Def: A complete set of orthogonal projectors (csop) is a set $\{P_1, \dots, P_k\} \subseteq \mathcal{L}(\mathcal{H})$ such that
- each P_i is a nonzero projector
- $\sum_{i=1}^{k} P_i = I$

Ex: If $\{P_i\}$ is a csop then $P_iP_j = P_jP_i = 0$ for all $i \neq j$.

Def: Let $\mathcal{H}$ be the $\mathbb{C}$-space of a physical system.
A projective measurement of the system is modeled by a csop $\{P_1, \dots, P_k\} \subseteq \mathcal{L}(\mathcal{H})$

Suppose the system is in some state $|\psi\rangle$ then measuring the system yields one of the outcomes $1, \dots, k$ with probability

$$\Pr[j] = \| P_j |\psi\rangle \|^2$$

Projectors correspond to the possible outcomes of the measurement (a random experiment with sample space $\{1, \dots, k\}$.

Notice that $\forall j, 1 \le j \le k$

$$\Pr[j] = \| P_j |\psi\rangle \|^2$$
$$= \langle P_j|\psi\rangle, P_j|\psi\rangle\rangle$$

Supposing the outcome is some j $(1 \le j \le k)$, the post-measurement state is

$$\frac{P_j|\psi\rangle}{\| P_j|\psi\rangle\|}$$

$$= \frac{P_j|\psi\rangle}{\sqrt{\Pr[j]}}$$

Note $\left\| \frac{P_j|\psi\rangle}{\|P_j|\psi\rangle\|} \right\|$

$$= \|P_j|\psi\rangle\|^{-1} \|P_j|\psi\rangle\| = 1$$

Check: $\Pr[j] \ge 0$ because it is a squared norm.

Need: $\sum_{j=1}^{k} \Pr[j] = 1$:

$$\sum_{j=1}^{k} \Pr[j] = \sum_{j=1}^{k} \langle\psi|P_j|\psi\rangle$$

$$= \langle\psi|\left(\sum_j P_j\right)|\psi\rangle$$

$$= \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle$$

$$= \||\psi\rangle\|^2 = 1.$$

$\therefore$ probabilities sum to 1.

One qubit:

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ bit value 0

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ bit value 1

CSOP on one qubit can be

$P_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$

"the bit is 0" $= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$P_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$

"the bit is 1" $= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$P_0 + P_1 = I$

measures the value of the bit. Measuring in the <u>computational basis</u>.

Let P be a projector, U unitary $(P, U \in \mathcal{L}(\mathcal{H}))$

$\Rightarrow UPU^*$ is a projector.

<u>Proof</u>:

$$(UPU^*)^* = U^{**}P^*U^*$$

$$= UPU^* \text{ and}$$

$$(UP\underbrace{U^*)(U}PU^*)$$

$$= UPPU^* = UPU^*$$