More on adjoints
Hermitian & Unitary operators
Orthonormal sets (bases)
Qubits & Bloch sphere

Recall: A matrix
A^* ($A^\dagger$) the conjugate transpose of A.

<u>Properties</u>:

$$(A+B)^* = A^* + B^*$$
$$(aA)^* = a^* A^*$$
(conjugate linear)
$$(AB)^* = B^* A^*$$
$$A^{**} = A$$

$\mathcal{H}$ $\mathbb{C}$-space

An <u>orthonormal set</u> $\mathcal{B} \subseteq \mathcal{H}$ is such that $\forall u, v \in \mathcal{B}$,

$$\langle u, v \rangle = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

$\langle u, v \rangle = 0$ (u, v are orthogonal or perpendicular)

so $\|u\| = 1$ for all $u \in \mathcal{B}$ (u is a <u>unit vector</u>).

If $\mathcal{H}$ is n-dimensional, then any orthonormal set has size $\leq n$. If $|\mathcal{B}| = n$ then say that $\mathcal{B}$ is an <u>orthonormal basis</u> ($\mathcal{B}$ spans $\mathcal{H}$)

(Almost) all our bases are orthonormal.

$\mathcal{B} = \{u_1, \ldots, u_n\}$

then $\langle u_i, u_j \rangle = \left\{\begin{matrix} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{matrix}\right\} = \delta_{ij}$

(δ_{ij} is the Kronecker δ-fcn).

The <u>standard basis</u> of $\mathbb{C}^n$: $\{e_1, \ldots, e_n\}$

where $\forall i$,

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{'th position}$$

Standard inner product on $\mathbb{C}^n$:

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^n v_i e_i$$

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \sum_{i=1}^n w_i e_i$$

$$\langle v, w \rangle = \left\langle \sum_{i=1}^n v_i e_i, \sum_{j=1}^n w_j e_j \right\rangle$$
$$= \sum_j w_j \left\langle \sum_i v_i e_i, e_j \right\rangle$$
$$= \sum_j w_j \sum_i v_i^* \langle e_i, e_j \rangle$$
$$= \sum_{i,j} v_i^* w_j \delta_{ij} = \sum_i v_i^* w_i$$
$$= v^* w$$

Can do the same thing with respect to any ortho basis $\{u_1, \ldots, u_n\}$

$$v = \sum_i v_i u_i, \quad w = \sum_j w_j u_j$$

Can express an operator A with respect to any ortho basis $\{u_i\}$ as a matrix:

$$[A]_{ij} = \langle u_i, Au_j \rangle$$
$$= u_i^* A u_j$$

Let $\mathcal{H}, \mathcal{J}$ be $\mathbb{C}$-spaces

Let $A : \mathcal{H} \to \mathcal{J}$.

There exists a unique operator $A^* : \mathcal{J} \to \mathcal{H}$ such that, $\forall u \in \mathcal{H}$, $\forall v \in \mathcal{J}$,

$$\langle v, Au \rangle_{\mathcal{J}} = \langle A^* v, u \rangle_{\mathcal{H}}$$

When expressing A with respect to the standard ortho. bases of $\mathcal{H}, \mathcal{J}$, the matrix of A^* is as before (conjugate transpose).

Notation: $\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces.

$\mathcal{L}(\mathcal{H}, \mathcal{J})$ = set of linear maps from $\mathcal{H}$ to $\mathcal{J}$

$\mathcal{L}(\mathcal{H}, \mathcal{J})$ is a vector space over $\mathbb{C}$.

We write $\mathcal{L}(\mathcal{H})$ for $\mathcal{L}(\mathcal{H}, \mathcal{H})$

"square operators"

<u>Def</u>: $\mathcal{H}$ $\mathbb{C}$-space

$A \in \mathcal{L}(\mathcal{H})$

A is <u>Hermitian</u> if $A = A^*$

A is <u>unitary</u> if $AA^* = I$

(I = identity) $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

(equivalently, $A^*A = I$)

That is $A^* = A^{-1}$.

$\forall u, v \in \mathcal{H}$,

$$\langle Au, Av \rangle = \langle A^*Au, v \rangle$$

$$= \langle u, v \rangle \text{ if } A \text{ is unitary.}$$

$\star$ Unitary operators preserve the inner product.

So: Applying unitary A to an ortho set yields an ortho set.

Change of (ortho) basis are accomplished by unitary operators:

Let $\{u_1, \ldots, u_n\}$ be an ortho basis of $\mathcal{H}$.

Let $U \in \mathcal{L}(\mathcal{H})$ be the unique operator such that

$Ue_i = u_i \quad i = 1, \ldots, n.$

Let $A \in \mathcal{L}(\mathcal{H})$

Let $\{u_1, \ldots, u_n\}$ be an ortho basis of $\mathcal{H}$.

Let $U \in \mathcal{L}(\mathcal{H})$ be the unique operator such that

$U e_i = u_i \qquad i = 1, \ldots, n.$

U is <u>unitary</u>

Let $A \in \mathcal{L}(\mathcal{H})$

Express A as a matrix with respect to $\{u_1, \ldots, u_n\}$

$$\left[A^{(u)}\right]_{ij} = u_i^* A\, u_j$$

$$= (U e_i)^* A\, (U e_j)$$

$$= e_i^* (U^* A\, U) e_j$$

$$= \left[U^* A\, U\right]_{ij}$$

with respect to $\{e_1, \ldots, e_n\}$

<u>Unitary</u> <u>conjugation</u>:

$A \longmapsto U^* A U$

some unitary U.

Equivalent to change of ortho basis.

Likewise, If U is unitary then so is U^*.

so $A \longmapsto U A U^*$

is also unitary conjugation.

A <u>qubit</u> is a quantum system described by a 2-dimension $\mathbb{C}$-space. $(\mathbb{C}^2)$.

The <u>state</u> of a qubit is a unit vector in $\mathbb{C}^2$.

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = u \quad (\alpha, \beta \in \mathbb{C})$$

$$1 = \|u\|^2 = \langle u, u \rangle$$

$$= u^* u = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

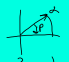
$$= \alpha^* \alpha + \beta^* \beta = \boxed{\begin{matrix} |\alpha|^2 + |\beta|^2 \\ \| \\ 1 \end{matrix}}$$

<u>Physically</u> if u is a quantum state (unit vector), then $v = e^{ip} u$ (any $p \in \mathbb{R}$) is the same physical state.

$|e^{ip}| = 1$ is called a <u>phase</u> factor. Vectors differing by a phase factor represent the same physical state ("phase freedom")

Given a state $u = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$,

by phase freedom, we can assume that $\alpha \geq 0$ (real)



$$|\alpha|^2 + \underbrace{|\beta|^2}_{\geq 0} = 1 \Rightarrow 0 \leq \alpha \leq 1$$

Let θ be such that $0 \leq \theta \leq \pi$ and $\alpha = \sin(\theta/2)$.

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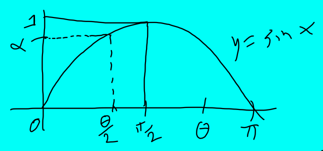
Let θ be such that $0 \leq \theta \leq \pi$ and $\alpha = \sin(\theta/2)$.



Note: θ uniquely determined by α. So:

$1 = |\alpha|^2 + |\beta|^2 = \alpha^2 + |\beta|^2$

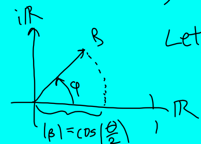
$= \sin^2\left(\frac{\theta}{2}\right) + |\beta|^2$

$\therefore |\beta|^2 = 1 - \sin^2\left(\frac{\theta}{2}\right)$

$= \cos^2\left(\frac{\theta}{2}\right)$

$\therefore |\beta| = \cos\left(\frac{\theta}{2}\right)$

$\therefore$



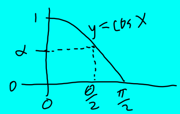
Let $\varphi := \arg \beta$.

Then $\beta = |\beta| e^{i\varphi} = \cos\left(\frac{\theta}{2}\right) e^{i\varphi}$

Can assume $0 \leq \varphi < 2\pi$

So $u = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$

(sorry $\longrightarrow$ switch sin, cos)



2 real parameters to describe u. (θ, φ)

Same parameters specify a point on the unit sphere in $\mathbb{R}^3$:

$$\underbrace{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)}_{\text{spherical coordinates}}$$

$$\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\varphi} \end{bmatrix} \longleftrightarrow (\qquad)$$

Represent a qubit state as a point on the sphere in $\mathbb{R}^3$ (Bloch sphere).