Complex numbers & arithmetic thereon.
Exponential map
Linear algebra

<u>Def</u>: A complex number is a number of the form $\underbrace{x+iy}_{z}$, $x, y \in \mathbb{R}$

x, y uniquely determined by z

$x = \mathrm{Re}(z)$
$y = \mathrm{Im}(z)$
i — imaginary unit
$i^2 = -1$

Addition: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

$\left\{ \underbrace{x+iy}_{z} \text{ — cartesian form of } z \right\}$

Mult: $(x_1 + iy_1)(x_2 + iy_2)$
$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

$-(x+iy) = (-x) + i(-y)$

Complex conjugate:

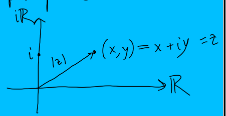
$z = x + iy$
$z^* = x - iy = \bar{z}$

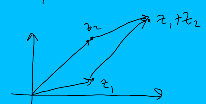
$zz^* = (x+iy)(x-iy)$
$= x^2 - (iy)^2 = \underbrace{x^2 + y^2}_{\in \mathbb{R}} \geq 0$

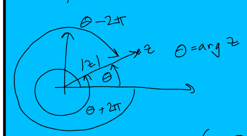
$zz^* = 0$ iff $z = 0$.

Norm (absolute value) of z is $\sqrt{zz^*} = |z| \geq 0$
$= 0$ iff $z = 0$.

Complex plane:

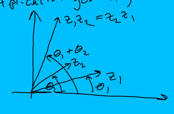






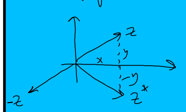
$\arg z = \theta + 2\pi n \quad (n \in \mathbb{Z})$
(if range restriction is needed, usually $0 \leq \theta < 2\pi$)

multiplication geometry



$|z_1 z_2| = |z_1| \cdot |z_2|$

$\arg(z_1 + z_2) = \arg z_1 + \arg z_2$
(up to a multiple of 2π)



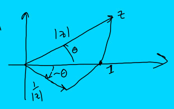
[Exponential map over $\underbrace{\mathbb{C}}_{\text{set of complex numbers (field)}}$
exp

<u>Reciprocal</u>: $1 = 1 + 0i$ mult identity
$1z = z1 = z$

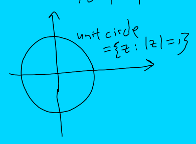
$$z^{-1} = \frac{1}{z} = \frac{z^*}{|z|^2} = \frac{x-iy}{x^2+y^2}$$

$$z z^{-1} = \frac{z z^*}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$$

$|z^{-1}| = \frac{1}{|z|}$



$$|z| = 1 \Rightarrow z^{-1} = z^*$$
$$|z^{-1}| = 1$$



Exponential map: $\forall z \in \mathbb{C}$

$$\exp(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

$$e^0 = 1, \quad e^{-z} = \frac{1}{e^z}$$

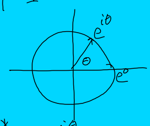
$$z = x + iy$$

$$e^z = e^{x+iy} = \underbrace{e^x}\underbrace{e^{iy}}$$

Euler's formula: $\forall \theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$|e^{i\theta}| = 1$$



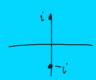
$$(e^{i\theta})^* = e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\Downarrow$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\frac{1}{i} = -i$$



Polar form of z:

$$z = re^{i\theta} \quad (r \geq 0, \theta \in \mathbb{R})$$

$$\boxed{\begin{aligned} r &= |z| \\ &= \sqrt{x^2+y^2} \\ \theta &= \arg z \end{aligned}}$$

cartesian $\to$ polar

$= \arctan\frac{y}{x}$ $\left(\begin{array}{l} +\pi \\ \text{if } x<0 \end{array}\right)$

polar $\to$ cartesian

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Double angle fmla

$$\left.\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1 \end{aligned}\right\}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \left(\begin{array}{l} + \text{ if} \\ 0 \leq \theta \leq 2\pi \end{array}\right)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$(+ \text{ if } -\pi \leq \theta \leq \pi)$$

Linear algebra

Def: A $\mathbb{C}$-space is a finite-dimensional vector space over $\mathbb{C}$ together with a map $\langle \circ, \circ \rangle$ taking any 2 vectors & returning a scalar ("inner product") satisfying the following rules:

$\forall a \in \mathbb{C}$, $\forall v, w, x$ in $\mathcal{H}$

($\mathcal{H}$ is the $\mathbb{C}$-space)

$$\left.\begin{aligned} \langle v, w+x \rangle &= \langle v, w \rangle + \langle v, x \rangle \\ \langle v, aw \rangle &= a \langle v, w \rangle \end{aligned}\right]$$

"linear in the 2nd argument"

$\langle v, w \rangle = \langle w, v \rangle^*$ conjugate symmetric

$\left[\langle v, v \rangle = \langle v, v \rangle^* \ \therefore \langle v, v \rangle \in \mathbb{R} \right]$

$\langle v, v \rangle \geq 0$ & $\langle v, v \rangle = 0 \Rightarrow v = 0$.

positive definite

("$z \geq 0$ or $z > 0$" implicitly implies $z \in \mathbb{R}$)

Note: A $\mathbb{C}$-space is just a finite-dimensional Hilbert space ("complex Euclidean space")

$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ is a <u>Hermitian inner product</u>.

<u>Fact</u>: $\langle v+w, x \rangle = \langle x, v+w \rangle^*$

$= \left(\langle x, v \rangle + \langle x, w \rangle \right)^*$

$\left[(z+w)^* = z^* + w^* \right]$

$= \langle x, v \rangle^* + \langle x, w \rangle^*$

$= \langle v, x \rangle + \langle w, x \rangle$

$\langle av, w \rangle = \langle w, av \rangle^*$

$= \left(a \langle w, v \rangle \right)^*$

$\left[(zw)^* = z^* w^* \right]$

$= a^* \langle w, v \rangle^*$

$= a^* \langle v, w \rangle$

"conjugate linear in the 1st arg"

<u>The most important example</u>

Let $n > 0$ integer

$\mathbb{C}^n$ is a n-dim vector space.

$v \in \mathbb{C}^n$, $v = (v_1, \ldots, v_n)$

$(v_1, \ldots, v_n \in \mathbb{C})$.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{column vectors}$$

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

Define

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j^* w_j$$

One should verify that this satisfies all the rules of a Hermitian inner product.

An $m \times n$ matrix over $\mathbb{C}$ is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{21} & \ddots & & \\ \vdots & & & a_{mn} \end{bmatrix}$$

$[A]_{ij} := a_{ij}$ | (i,j)th entry of A

Def: Let A be an $m \times n$ matrix. The adjoint (or Hermitian conjugate) of A is the $n \times m$ matrix A^* which is the conjugate transpose of A, that is A^* is A^T but with all entries conjugated:

$\forall i,j$

$$[A^*]_{ij} = [A]_{ji}^*$$

Ex:

$$A = \begin{bmatrix} 2+3i & 4-6i \\ -3+4i & 5i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2-3i & -3-4i \\ 4+6i & -5i \end{bmatrix}$$

Notice that if

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \text{ then } v^* = \begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix}$$

$$\langle v, w \rangle = \sum_{j=1}^{n} v_j^* w_j$$

$$= \begin{bmatrix} v_1^* & \cdots & v_n^* \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= v^* w$$