Majority-of-3 code

Alice

$0 \mapsto 000 = 0_L$

$1 \mapsto 111 = 1_L$

$\longrightarrow$ Bob: majority(x,y,z)

where x,y,z are what Bob receives.

Binary symmetric error channel with error prob $p \leq \frac{1}{2}$

physical

$0 \mapsto 0$ w/ prob $1-p$

$0 \mapsto 1$ " p

$1 \mapsto 0$ " p

$1 \mapsto 1$ " $1-p$

$000 \mapsto \overbrace{100}^{p(1-p)^2}$ or $\overbrace{010}^{p(1-p)^2}$ or $\overbrace{001}^{p(1-p)^2}$ or $\underbrace{000}_{(1-p)^3}$ — Bob can correct

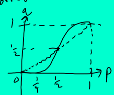
111 (similar)

$\therefore$ Bob can correct with prob

$(1-p)^3 + 3p(1-p)^2$

$= 1 - \cancel{3p} + \cancel{3p^2} - p^3 + \cancel{3p} - \cancel{6p^2} + 3p^3$

$= 1 - \underbrace{(3p^2 - 2p^3)}_{\text{prob that Bob does not correct} = q}$



$q = O(p^2)$ as $p \to 0$

Quantum bit-flip channel with error prob p:

$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$

Kraus operators are

$\{\sqrt{1-p}\, I, \sqrt{p}\, X\}$

Alice: $|\psi\rangle := \alpha|0\rangle + \beta|1\rangle$

$\mapsto |\psi_L\rangle := \alpha|0_L\rangle + \beta|1_L\rangle$

$= \alpha|000\rangle + \beta|111\rangle$

sends to Bob, but Bob can't just measure the 3 qubit values because this will collapse the state, destroying the superposition.

Want Bob to recover $|\psi_L\rangle$

Idea: Bob measures the error syndrome, that is, what type of error probably occurred in the channel without getting any classical info about $|\psi_L\rangle$.

Alice: $\rho := |\psi\rangle\langle\psi| \in \mathcal{L}(\mathbb{C}^2)$

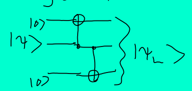
$\rho \xrightarrow{\text{encode}} \rho_L \xmapsto[\text{noisy channel}]{\mathcal{E}^{\otimes 3}} \sigma \xrightarrow{\text{decode}} \tau$

$\rho_L = |\psi_L\rangle\langle\psi_L|$

$\mathcal{E}$ acting on each qubit independently

Bob error corrects

Encoding (Alice)



$\left[\text{GHZ state} = \frac{|000\rangle + |111\rangle}{\sqrt{2}}\right]$

Bob gets σ, some 3-qubit state.

1) measures the error syndrome by computing the parity of adjacent qubits.

Projective measurement with

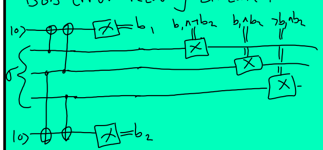
no error: $P_0 := (|000\rangle + |111\rangle)(\langle 000| + \langle 111|)$ (sop)

1st qubit flipped: $P_1 := (|100\rangle + |011\rangle)(\langle 100| + \langle 011|)$

2nd qubit flipped: $P_2 := (|010\rangle + |101\rangle)(\langle 010| + \langle 101|$

$P_3 := (|001\rangle + |110\rangle)(\langle 001| + \langle 110|$

Bob error recovery channel circuit:



$b_1, b_2 \in \{0,1\}$

$b_1 = b_2 = 0$: P_0

$b_1 = 0, b_2 = 1$: P_3

$b_1 = 1, b_2 = 0$: P_1

$b_1 = b_2 = 1$: P_2

Bob's error-recovery channel $\mathcal{R}$:

$$\mathcal{R}(\sigma) = P_0 \sigma P_0 + X_1 P_1 \sigma P_1 X_1 + X_2 P_2 \sigma P_2 X_2 + X_3 P_3 \sigma P_3 X_3$$

Kraus operators are

$\{P_0, X_1 P_1, X_2 P_2, X_3 P_3\}$

Phase-flip channel (error prob p)

$$\mathcal{Z}(\rho) = (1-p)\rho + pZ\rho Z$$

Notice:

$$\mathcal{Z}(H\rho H) = (1-p)H\rho H + pZH\rho HZ$$

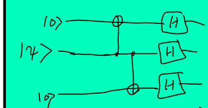
$[HZ = XH,\ HX = ZH]$

$$= (1-p)H\rho H + pHX\rho XH$$

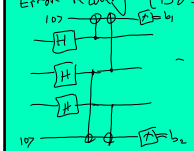
$$= H\big((1-p)\rho + pX\rho X\big)H$$

$$= H\mathcal{E}(\rho)H$$

Alice's encoding (for phase-flip channel):



Error recovery (Bob)



- - - (same)

Bit & phase flip:

$$\mathcal{G}(\rho) = (1-p)\rho + ZX\rho XZ$$
$$= (1-p)\rho + Y\rho Y$$

Partially depolarizing channel includes all 3 possibilities (with equal prob):

$$\mathcal{D}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

Shor code (9 qubits)

Thm: (Discretation of errors)

If you can recover from a channel with Kraus ops $K_1, \ldots, K_n$, then same code/recovery recovers from any channel whose Kraus ops are linear combinations of $K_1, \ldots, K_n$

Cor: Shor code recovers from arbitrary 1-qubit errors.