Scenario: Alice & Bob

Alice: Chooses Plan A or Plan B

Plan A: Alice flips a fair coin
if heads, prepares the state

$|\psi_+\rangle = \frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle$

if tails, prepares the state

$|\psi_-\rangle = \frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle$.

Sends this state to Bob.
Repeated n times (independently) (n large).

Plan B: Alice flips a biased coin, prepares

$|0\rangle$ with prob $\frac{16}{25}$

and $|1\rangle$ with " $\frac{9}{25}$

sends this state to Bob.
Repeated n times independently.

Bob: figure out which plan Alice is using (her plan does not switch.)

Bob knows what the two plans are & that Alice is using the same plan each time, but nothing else.

Plan A:

$$\rho_A = \frac{1}{2}|\psi_+\rangle\langle\psi_+| + \frac{1}{2}|\psi_-\rangle\langle\psi_-|$$

$$= \frac{1}{2}\left(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle\right)\left(\frac{4}{5}\langle 0| + \frac{3}{5}\langle 1|\right) + \frac{1}{2}\left(\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle\right)\left(\frac{4}{5}\langle 0| - \frac{3}{5}\langle 1|\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}\begin{bmatrix} \frac{4}{5} & \frac{3}{5} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 16/25 & -\frac{12}{25} \\ -\frac{12}{25} & 9/25 \end{bmatrix}$$

$$= \begin{bmatrix} 16/25 & 0 \\ 0 & 9/25 \end{bmatrix} = \frac{16}{25}|0\rangle\langle 0| + \frac{9}{25}|1\rangle\langle 1| =: \rho_B$$

Bob gets the same mixed state in either case, so no POVM Bob uses can distinguish (even probabilistically) which plan Alice is using.

More channels

(Completely) depolarizing channel:

$$\Xi(X) = (\operatorname{tr} X)\frac{I}{\dim \mathcal{H}},$$

where $\Xi \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$

For any state ρ,

$$\Xi(\rho) = \frac{I}{\dim \mathcal{H}}$$

completely incoherent (decohered) state (no useful info)

(completely) dephasing channel

$\Delta : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$

$\Delta(A)$ sets all off-diagonal entries to 0, leaving the diagonal entries unchanged.
$n \times n$ matrix

A state is **classical** if it is a diagonal matrix.
[probability dist. over classical pure states]
1-dim projectors in the standard basis.

Note: $\Delta \circ \Delta = \Delta$

Def: A measurement channel (most generally) is a map

$$\mathcal{M} : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_D \otimes \mathbb{C}^n)$$

for some positive integer n such that

$$(\mathbb{1}_{\mathcal{L}(\mathcal{H}_D)} \otimes \Delta) \circ \mathcal{M} = \mathcal{M}.$$

Equivalently, a measurement channel is one of the form

$\mathcal{M} := (\mathbb{1}_{\mathcal{L}(\mathcal{H}_D)} \otimes \Delta) \circ \Phi$

where $\Phi \in \mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_D) \otimes \mathcal{L}(\mathbb{C}^n))$ is a channel.

$\mathbb{C}^n$ is called the meter
standard basis corresponds to the classical outcomes of the measurement (labeled $1, \dots, n$)

$\mathcal{H}_S$ is space of states to be measured, & $\mathcal{H}_D$ is the space of post-measurement states.

Ex: $\rho \;—\boxed{\nearrow}= \Delta(\rho)$

One way of Augmenting an arbitrary channel to a measurement channel:

$$\Phi : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_D)$$

$$\Phi(A) = \sum_{j=1}^{m} K_j A K_j^*$$

$\left(\sum_{j=1}^{m} K_j^* K_j = \mathbb{1}\right)$

$$\Phi : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_D)$$

$$\Phi(A) = \sum_{j=1}^{m} K_j A K_j^*$$

$\underline{\left(\sum_{j=1}^{m} K_j^* K_j = \mathbb{1}\right)}$

Think of $\Phi(\rho)$ as applying a random Kraus operator K_j to ρ with probability $tr(K_j \rho K_j^*) = \langle K_j, K_j \rangle \geq 0$

Augment to a measurement that tells us which K_j was applied

Measurement

$$\mathcal{M}(A) := \sum_{j=1}^{m} (K_j \otimes e_j) A (K_j \otimes e_j)^*$$

State of the meter, given input state ρ

$$tr_{\mathcal{H}_D}(\mathcal{M}(\rho))$$

$$= \sum_{j=1}^{m} tr_{\mathcal{H}_D} (K_j \otimes e_j) \rho (K_j \otimes e_j)^*$$

$$= \sum_{j=1}^{m} \overbrace{tr(K_j \rho K_j^*)}^{p_j} E_{jj}$$

$$= \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_m \end{bmatrix}$$

<u>Verify</u>: Both sides are linear in ρ, so only need to check equality on a spanning subset of $\mathcal{L}(\mathcal{H}_S)$ e.g., tensor products:

$$= \sum_{j=1}^{m} tr_{\mathcal{H}_D} (K_j \otimes e_j) \overbrace{(\rho \otimes \mathbb{1})}^{\rho} (K_j \otimes e_j)^*$$

$$= \sum_j tr_{\mathcal{H}_D} \left(\underline{K_j \rho K_j^*} \otimes \underbrace{e_j e_j^*}_{E_{jj}} \right)$$

$$= \sum_j tr(K_j \rho K_j^*) E_{jj}$$

$= r.h.s.$

Info-free measurement: trace out the meter

POVM: trace out the post-measurement state (get ops for the POVM from Kraus ops from the measurement channel).

Error channels:
(1-qubit channels)

Bit-flip channel:

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$$

(X is Pauli X operator), & $0 \leq p \leq 1$

Phase-flip channel:

$$\mathcal{F}(\rho) = (1-p)\rho + pZ\rho Z$$

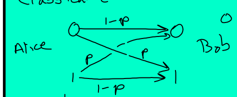
Partially depolarizing channel (convex combination of the identity channel & the completely depolarizing channel):

$$\mathcal{D}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

p = "error probability"

(good worst-case error model).

Classical error correction



$0 \leq p \leq 1$

noisy channel (binary symmetric channel)

Majority-of-3 code

$0 \longmapsto 000$

$1 \longmapsto 111$

Assume $p \leq \frac{1}{2}$

Bob: $b_1 b_2 b_3 \longmapsto \text{majority}(b_1, b_2, b_3)$