POVMs

1-qubit states & Bloch sphere

Quantum Channels

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Positive Operator-Valued Measures (POVMs). Most general possible measurement where post-measurement state is ignored/discarded.

Let $\{M_j : j \in \mathcal{J}\}$ ← the POVM

be a (finite or countable) collection of positive operators such that

$$\sum_{j \in \mathcal{J}} M_j = I$$

$\mathcal{J}$ is the set of possible outcomes (sample space), and

$\Pr_\rho[j]$:= probability of seeing outcome j when applying the POVM to a system in state $\rho \in \mathcal{L}(\mathcal{H})$

$$\Pr_\rho[j] = \langle M_j, \rho \rangle = \langle \rho, M_j \rangle$$

Check that if $A, B \geq 0$, then $\langle A, B \rangle = \langle B, A \rangle$:

$$\langle A, B \rangle = \operatorname{tr}(A^* B) = \operatorname{tr}(AB)$$
$$= \operatorname{tr}(BA) = \operatorname{tr}(B^* A) = \langle B, A \rangle$$

Check: (a) $\Pr_\rho[j] \geq 0$ &

(b) $\sum_{j \in \mathcal{J}} \Pr_\rho[j] = 1$.

for (a),

$$\langle M_j, \rho \rangle = \operatorname{tr}(M_j \rho)$$
$$= \operatorname{tr}\left(\sqrt{M_j}\sqrt{M_j}\sqrt{\rho}\sqrt{\rho}\right)$$
$$= \operatorname{tr}\left(\sqrt{\rho}\sqrt{M_j}\sqrt{M_j}\sqrt{\rho}\right)$$
$$= \operatorname{tr}\left(\left(\sqrt{M_j}\sqrt{\rho}\right)^*\left(\sqrt{M_j}\sqrt{\rho}\right)\right)$$
$$= \langle \sqrt{M_j}\sqrt{\rho}, \sqrt{M_j}\sqrt{\rho} \rangle \geq 0$$

↑ pos def inner product

$$\sum_{j \in \mathcal{J}} \Pr_\rho[j] = \sum_j \langle M_j, \rho \rangle$$
$$= \left\langle \sum_j M_j, \rho \right\rangle$$
$$= \langle I, \rho \rangle = \operatorname{tr}(I^* \rho)$$
$$= \operatorname{tr} \rho = 1$$

[Partial POVM: $\underbrace{\sum_j M_j \leq I}_{I - \sum_j M_j \geq 0}$.

Can add one more operator:

$$M_{none} := I - \sum_j M_j$$

1-qubit states:

Pure state is of the form

$$\rho = \frac{1}{2}\left(I + a_x X + a_y Y + a_z Z\right)$$

where $(a_x, a_y, a_z) \in \mathbb{R}^3$ a point on the sphere.

ρ_1, ρ_2 1-qubit pure states $\lambda_1, \lambda_2 \geq 0$ s.t. $\lambda_1 + \lambda_2 = 1$

Then

$$\rho = \lambda_1\rho_1 + \lambda_2\rho_2 = \frac{1}{2}\Big(I + (\lambda_1 a_{x,1} + \lambda_2 a_{x,2})X + (\lambda_1 a_{y,1} + \lambda_2 a_{y,2})Y + (\lambda_1 a_{z,1} + \lambda_2 a_{z,2})Z\Big)$$

point in $\mathbb{R}^3$ with coords

$\lambda_1(a_{x,1}, a_{y,1}, a_{z,1}) + \lambda_2(a_{x,2}, a_{y,2}, a_{z,2})$

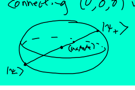
Mixed states on 1-qubit corresp. to points in the interior of the Bloch sphere.

If $\rho = \frac{1}{2}\left(I + a_x X + a_y Y + a_z Z\right)$

where $r := \|(a_x, a_y, a_z)\| = (a_x^2 + a_y^2 + a_z^2)^{1/2}$, then

eigenvals are $\frac{1+r}{2}$, $\frac{1-r}{2}$

eigenvectors are points on the sphere intersect the line connecting $(0,0,0)$ with (a_x, a_y, a_z)



<u>Quantum Channels</u> (Quantum operations)

Most general action one can perform.

<u>Def</u>: A <u>superoperator</u> is a linear map $\Phi \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{J}))$ where $\mathcal{H}$ & $\mathcal{J}$ are $\mathbb{C}$-spaces.

<u>Def</u>: Let $\Phi \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{J}))$ be a superoperator.

1. Φ is <u>positive</u> if $\Phi(A) \geq 0$ for all $A \in \mathcal{L}(\mathcal{H})$ s.t. $A \geq 0$.
2. Φ is <u>trace-preserving</u> if $\forall A \in \mathcal{L}(\mathcal{H})$, $tr(\Phi(A)) = tr\, A$.

Can combine superators as usual:

1. $\Phi_1 \otimes \Phi_2$ such that
$$(\Phi_1 \otimes \Phi_2)(A_1 \otimes A_2) = \Phi_1(A_1) \otimes \Phi_2(A_2)$$
for A_1, A_2 appropriately typed operators.
2. $\mathbb{1}_{\mathcal{L}(\mathcal{H})}(A) = A$ <u>identity</u> superoperator
3. Given $\mathbb{C}$-space $\mathcal{H}, \mathcal{J}, \mathcal{K}$,
$\Phi \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{J}))$
$\Psi \in \mathcal{L}(\mathcal{L}(\mathcal{J}), \mathcal{L}(\mathcal{K}))$
then
$$\Psi \circ \Phi = \Psi\Phi \in \mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{K}))$$
suchthat
$$(\Psi \circ \Phi)(A) = \Psi(\Phi(A)) \quad \forall A \in \mathcal{L}(\mathcal{H}).$$
composition.

<u>Def</u>: Φ is <u>completely positive</u> if $\Phi \otimes \mathbb{1}_{\mathcal{J}}$ is positive for all $\mathbb{C}$-spaces $\mathcal{J}$.

[The transpose operator
$$A \longmapsto A^T$$
is not completely positive, even for 2×2 matrices A.]

<u>Def</u>: A <u>quantum channel</u>
$\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{J})$
is a superoperator in $\mathcal{L}(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{J}))$ that is trace-preserving and completely positive.

Representations of Q. Channels:

1. <u>Operator sum</u>:
Fix $\mathbb{C}$-spaces $\mathcal{H}, \mathcal{J}$.
Φ superop from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{J})$ is a q. channel iff $\exists n$ natural number and operators $K_1, \ldots, K_n \in \mathcal{L}(\mathcal{H}, \mathcal{J})$ such-that, $\forall A \in \mathcal{L}(\mathcal{H})$,
$$\Phi(A) = \sum_{j=1}^{n} K_j A K_j^* \in \mathcal{L}(\mathcal{J})$$
and $\boxed{\sum_{j=1}^{n} K_j^* K_j = I_{\mathcal{H}}}$
K_j called <u>Kraus operators</u>.

Check trace preservation:
$$\begin{aligned} tr\,\Phi(A) &= tr\left(\sum_j K_j A K_j^*\right) \\ &= \sum_j tr(K_j A K_j^*) \\ &= \sum_j tr(K_j^* K_j A) \\ &= tr\Big(\underbrace{\sum_j K_j^* K_j}_{I} A\Big) \\ &= tr\, A \end{aligned}$$

<u>Partial trace</u> (channel)

$\mathcal{H}, \mathcal{J}$ $\mathbb{C}$-spaces
$$tr_{\mathcal{H}} \in \mathcal{L}(\mathcal{L}(\mathcal{H} \otimes \mathcal{J}), \mathcal{L}(\mathcal{J}))$$
is such-that $\forall A \in \mathcal{L}(\mathcal{H})$
$\forall B \in \mathcal{L}(\mathcal{J})$
$$tr_{\mathcal{H}}(A \otimes B) = (tr\,A) B$$
"<u>tracing out $\mathcal{H}$</u>"

<u>Ex</u>: If $\rho \in \mathcal{L}(\mathcal{H})$ & $\sigma \in \mathcal{L}(\mathcal{J})$ states, then
$$tr_{\mathcal{H}}(\rho \otimes \sigma) = \sigma$$
"forget the system $\mathcal{H}$"

Similarly,

$$\mathrm{tr}_{\mathcal{Y}}(A \otimes B) = (\mathrm{tr}\, B) A$$

Prop: For any state $\rho \in \mathcal{L}(\mathcal{H})$, there exists a pure state $\sigma \in \mathcal{L}(\mathcal{H} \otimes \mathcal{Y})$ s.t.

$$\rho = \mathrm{tr}_{\mathcal{Y}}\, \sigma$$

σ is a purification of ρ.

Unitary evolution channel:

$$\Phi(A) = UAU^*$$

U is unitary. Kraus operator $\{U\}$

$$U^*U = I$$

Projective measurement (ignore the classical info)

csop $\{P_1, \ldots, P_n\}$

$$\Phi(A) = \sum_{j=1}^{n} P_j A P_j .$$