Chap 7 — Basic Quantum Info

Spectral decomp of a normal operator:

Thm: Let $A \in \mathcal{L}(\mathcal{H})$ be normal, ($n := \dim(\mathcal{H})$)
There exist a unique csop $\{P_1, \ldots, P_k\}$ and unique distinct $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ s.t.

$$A = \sum_{j=1}^{k} \lambda_j P_j$$

Further, $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A and $P_1, \ldots, P_k$ project onto the corresponding eigenspaces.
$(\dim(\mathrm{img}(P_j)) = \mathrm{tr}\, P_j = \text{multiplicity of } \lambda_j)$
Must have $k \le n$.

Lifting scalar functions to functions on operators

Def: Let $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$.
Given an operator $A \in \mathcal{L}(\mathcal{H})$, can define $f(A) \in \mathcal{L}(\mathcal{H})$ in 2 possible ways:

1. If $f(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$ $c_j \in \mathbb{C}$
   (power series expansion about $z_0 \in \mathbb{C}$)
   Can define
   $$f(A) = \sum_{j=0}^{\infty} c_j (A - z_0 I)^j$$
   Ex: $\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}$
2. If A is normal, then let $A = \sum_{j=1}^{k} \lambda_j P_j$ be the spectral decomp of A. Define
   $$f(A) = \sum_{j=1}^{k} \underline{f(\lambda_j)} P_j$$
   well-defined iff $\mathrm{spec}(A) \subseteq \Omega$.

If both def's apply then they agree.

Recall: $A \in \mathcal{L}(\mathcal{H})$ is positive [semi-definite], written $A \ge 0$, if $\langle v, Av \rangle \ge 0$ for all $v \in \mathcal{H}$

$A \ge 0 \Rightarrow A$ is Hermitean, hence normal.

Prop: TFAE for $A \in \mathcal{L}(\mathcal{H})$
1. $A \ge 0$.
2. A is normal and all eigenvals of A are ≥ 0.
3. $\exists B$, $A = B^* B$.

Pf: See notes.

For $x \ge 0$, let $f(x) := \sqrt{x} \ge 0$
Can define $\sqrt{A}$ via f.

Prop: For $A \ge 0$, $\sqrt{A}$ is the unique $B \ge 0$ such that $B^2 = A$

Norms of operators

1. L_2-norm of A is defined as $\|A\|_2 := \sqrt{\langle A, A \rangle}$
2. Operator norm (L_∞-norm):
   $$\|A\| := \|A\|_\infty = \max_{v \ne 0} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|$$
3. Trace norm (L_1-norm)
   $$\|A\|_1 = \mathrm{tr}\Big(\underbrace{\sqrt{A^* A}}_{\ge 0}\Big) \ge 0$$

Properties of norms:
For any $p \in [1, \infty)$
Define L_p-norm
$$\|A\|_p := \left[\mathrm{tr}\left(\sqrt{A^* A}^{\,p}\right)\right]^{1/p}$$

$$\|A\|_\infty := \lim_{p \to \infty} \|A_p\|$$

$p = 1, 2, \infty$, this matches the prev definition.

$\forall A \in \mathcal{L}(\mathcal{H})$, define

If $A = \sum_{j=1}^{k} \lambda_j P_j$ spec. decomp

then $|A| = \sum_{j=1}^{k} |\lambda_j| P_j$

$|A| \geq 0$ for all A.

All L_p-norms satisfy:

1. $\|aA\|_p = |a| \cdot \|A\|_p$
2. $\|A\|_p \in \mathbb{R}$ & $\|A\|_p \geq 0$ with equality iff $A = 0$.
3. $\|A+B\|_p \leq \|A\|_p + \|B\|_p$
4. $1 \leq p < q < \infty$, then

$$\|A\|_p \geq \|A\|_q \geq \|A\|_\infty \geq n\|A\|_1 \quad n := \dim(\mathcal{H})$$

<u>Def</u>: The L_p-distance of A to B is $\|A-B\|_p$. Metric on $\mathcal{L}(\mathcal{H})$.

Can use to define convergence & absolute convergence of infinite sequences of operators.

<u>States</u>: Fix $\mathbb{C}$-space $\mathcal{H}$ $(n := \dim(\mathcal{H}))$. So far, we've viewed a quantum state as a unit vector in $\mathcal{H}$.

<u>Def</u>: A <u>state</u> of $\mathcal{H}$ is an operator $\rho \in \mathcal{L}(\mathcal{H})$ such that

1. $\rho \geq 0$
2. $\operatorname{tr} \rho = 1$

ρ is also called a <u>density operator</u>.

$\rho \in \mathcal{L}(\mathcal{H})$ is a state: let

$\rho = \sum_{j=1}^{k} \lambda_j P_j$ be the spectral decomp of ρ.

Know that $\lambda_j \geq 0$ for all j.

$$1 = \operatorname{tr} \rho = \sum_{j=1}^{k} \lambda_j \operatorname{tr} P_j$$

$\therefore$ The sum of the eigenvals (with multiplicity) is 1.

Let $\{p_1, \ldots, p_n\}$ be the eigenvals with multiplicity, then $p_i \geq 0$ and $\sum_{j=1}^{n} p_j = 1$

$\therefore$ $\operatorname{spec}(\rho)$ is a probability distribution.

$\rho = \sum_{i=1}^{n} p_i Q_i$ where $\{Q_1, \ldots, Q_n\}$ is a csop of 1-dim projectors.

<u>Special case</u>: $p_1 = 1$, $p_2, \ldots, p_n = 0$

Then $\rho = Q_1$ (1-dim projector)

$Q_1 = |\psi\rangle\langle\psi|$ for any $|\psi\rangle \in \operatorname{img}(Q)$

ρ is a <u>pure state</u> in this case.

Otherwise, ρ is a probabilistic "mixture" of pure states.

ρ is a <u>mixed state</u>.

"$\rho = Q_i$ with probability p_i."

Let $\{q_1, \ldots, q_k\}$ be any prob. dist, and let $\rho_1, \ldots, \rho_k$ be any states. Then

$$\rho := \underbrace{\sum_{j=1}^{k} q_j \rho_j}_{\text{convex combination}} \text{ is a state:}$$

$\rho \geq 0$ b/c $q_j \geq 0$ and each $\rho_j \geq 0$

$$\operatorname{tr} \rho = \sum_{j=1}^{k} q_j \underbrace{\operatorname{tr}(\rho_j)}_{1} = \sum q_j = 1$$

Translating what we did before:

<u>Unitary evolution</u>:

For initial state ρ & unitary U, Evolving ρ by U result in

$$U\rho U^* \quad \text{(a state)}$$

$\rho = |\psi\rangle\langle\psi|$ pure state;

$$U\rho U^* = U|\psi\rangle\langle\psi|U^* = |\varphi\rangle\langle\varphi|$$

where $|\varphi\rangle := U|\psi\rangle$

<u>Projective measurement</u>:

csop $\{P_1, \ldots, P_k\}$ & state ρ,

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Post-measurement state is

$$\frac{P_j \rho P_j}{\Pr[j]}$$

If $\rho = |\psi\rangle\langle\psi|$ pure state, then

$$\Pr[j] = \langle P_j, \rho \rangle = \mathrm{tr}(P_j^* \rho)$$

$$= \mathrm{tr}(P_j |\psi\rangle\langle\psi|)$$

$$\cancel{= \mathrm{tr}(P_j P_j |\psi\rangle\langle\psi|)}$$

$$= \mathrm{tr}(\langle\psi| P_j |\psi\rangle)$$

$$= \langle\psi| P_j |\psi\rangle = \| P_j |\psi\rangle \|^2$$

Check that the post-meas state agrees with before.

If $|\psi\rangle = e^{i\theta} |\varphi\rangle$
then

$$|\psi\rangle\langle\psi| = e^{i\theta} |\varphi\rangle\langle\varphi| e^{-i\theta}$$

$$= |\varphi\rangle\langle\varphi|$$

(The converse is also true.)