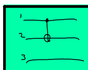
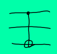
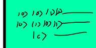


$C\text{-}X \otimes I$

$= (P_0 \otimes I + P_1 \otimes X) \otimes I$



$= P_0 \otimes I \otimes I + P_1 \otimes I \otimes X$



$c \in \{0,1\}$

$$\underbrace{e^{\varepsilon(A+B)}} = \underbrace{e^{\varepsilon A} e^{\varepsilon B} + \frac{\varepsilon^2}{2}[A,B]} + \underbrace{O(\varepsilon^3)}$$

I will check this.

Grover's Algo for quantum search

Setup: Array $A[1..N]$ of values, exactly one of which is a target value ℓ.

Goal: find the index of ℓ in A.

Only assumption: know the target value if it is seen.

Worst-case classical algo: $\Theta(N)$ probes to A (linear search).

Grover: $\Theta(\sqrt{N})$ probes suffice.

$N = 2^n$ (some n) have a black-box function $f : \{0,1\}^n \to \{0,1\}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x = \lambda \\ 0 & \text{o.w.} \end{cases}$$

values in A are binary strings (n bits)

Assume a probe is an application of the n-qubit unitary gate I_f s.t. $\forall z \in \{0,1\}^n$

$$I_f |z\rangle = (-1)^{f(z)} |z\rangle$$

If target is at index w, then

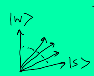
$I_f |w\rangle = -|w\rangle$

$I_f |z\rangle = |z\rangle$ if $z \neq w$.

$$I_f = \begin{array}{r} \\ 2^n \\ w \rightarrow \end{array} \underbrace{\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & 0 & & & -1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \; 1 \end{bmatrix}}_{2^n}$$

$I_f = I - 2|w\rangle\langle w|$

$I_0 = I - 2|0^n\rangle\langle 0^n|$] don't need f for this



Given $|s\rangle$

Let $x := \langle w|s\rangle$

WLOG, $x > 0$ (phase freedom in $|s\rangle$)

$|s\rangle$ is the start state

Let U be unitary such that $U|0^n\rangle = |s\rangle$

Set $G := -U I_0 U^* I_f$

"Grover iterate"

Algo:

1. Start with n qubits in $|0^n\rangle$ state
2. Apply U, get $|s\rangle := U|0^n\rangle$ (start state)
3. Apply G $\underbrace{\left\lfloor \frac{\pi}{4 \sin^{-1} x} \right\rfloor}_{\approx \sqrt{N} \text{ given the right } x}$ many times ($x := \langle w|s\rangle > 0$)
4. Measure the n qubits in the computational basis obtaining $y \in \{0,1\}^n$. Return y.

[if $x = \frac{1}{\sqrt{N}}$, then $\sin^{-1} x \approx x$, ($x \ll 1$) so $\left\lfloor \frac{\pi}{4 \sin^{-1} x} \right\rfloor \approx \frac{\pi}{4x} = \frac{\pi}{4}\sqrt{N}$]

Expand G:

$$G = -U I_0 U^* I_f = -U(I - 2|0^n\rangle\langle 0^n|)U^* (I - 2|w\rangle\langle w|)$$

$$= -(I - 2U|0^n\rangle\langle 0^n|U^*)(I - 2|w\rangle\langle w|)$$

$$= -(I - 2|s\rangle\langle s|)(I - 2|w\rangle\langle w|)$$

$$= -I + 2|s\rangle\langle s| + 2|w\rangle\langle w| - 4|s\rangle\underbrace{\langle s|w\rangle}_{= x = \langle w|s\rangle}\langle w|$$

$$= -I + 2|s\rangle\langle s| + 2|w\rangle\langle w| - 4x|s\rangle\langle w|$$

So,

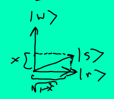
$$G|s\rangle = -|s\rangle + 2|s\rangle + 2x|w\rangle - 4x^2|s\rangle = (1 - 4x^2)|s\rangle + 2x|w\rangle$$

$$G|w\rangle = -2x|s\rangle + |w\rangle$$

$G|s\rangle = -|s\rangle + 2|s\rangle + 2x|w\rangle - 4x^2|s\rangle$
$= (1-4x^2)|s\rangle + 2x|w\rangle$

$G|w\rangle = -2x|s\rangle + |w\rangle$

G maps the (real) plane spanned by $|w\rangle$ & $|s\rangle$ into itself. [actually a rotation through some angle]



Where

$$|r\rangle = \frac{|r'\rangle}{\||r'\rangle\|} \quad \text{where } |r'\rangle = |s\rangle - x|w\rangle$$

$$\||r'\rangle\|^2 = \langle r'|r'\rangle = (\langle s| - x\langle w|)(|s\rangle - x|w\rangle)$$
$$= \langle s|s\rangle - x\underbrace{\langle w|s\rangle}_{x} - x\underbrace{\langle s|w\rangle}_{x} + x^2\langle w|w\rangle$$
$$= 1 - x^2 - x^2 + x^2 = 1 - x^2$$

$$\||r'\rangle\| = \sqrt{1-x^2}$$

So

$$|r\rangle = \frac{|s\rangle - x|w\rangle}{\sqrt{1-x^2}}$$

Check: $\langle r|r\rangle = 1$ & $\langle r|w\rangle = 0$.

Express $|s\rangle$ in the $\{|r\rangle, |w\rangle\}$ basis:

$$|s\rangle = \sqrt{1-x^2}\,|r\rangle + x|w\rangle$$
$$= \cos\theta|r\rangle + \sin\theta|w\rangle$$
$$= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$\boxed{\text{let } \theta \text{ be s.t. } x = \sin\theta,\ 0 < \theta < \pi/2}$

Express G with respect to the $\{|r\rangle, |w\rangle\}$ basis as a 2×2 (real) matrix.

Restricted to this plane:

$$I = P = |r\rangle\langle r| + |w\rangle\langle w|$$

projects orthogonally onto this plane

Then (restricted to the plane)

$$G = -P + 2|s\rangle\langle s| + 2|w\rangle\langle w| - 4x|s\rangle\langle w|$$
$$= -|r\rangle\langle r| - |w\rangle\langle w| + 2(\cos\theta|r\rangle + \sin\theta|w\rangle)(\cos\theta\langle r| + \sin\theta\langle w|)$$
$$+ 2|w\rangle\langle w| - 4\sin\theta(\cos\theta|r\rangle + \sin\theta|w\rangle)\langle w|$$
$$= (2\cos^2\theta - 1)|r\rangle\langle r| + 2\cos\theta\sin\theta|w\rangle\langle r|$$
$$+ (1 - 2\sin^2\theta)|w\rangle\langle w|$$
$$- 2\cos\theta\sin\theta|r\rangle\langle w|$$

$$= \begin{array}{cc} & \begin{array}{cc} r & w \end{array} \\ \begin{array}{c} r \\ w \end{array} & \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin\theta & \cos 2\theta \end{bmatrix} \end{array}$$

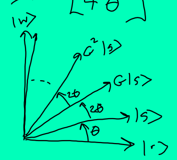
counterclockwise rotation of the plane through angle 2θ

Let $U = H^{\otimes n}$

$$U|0^n\rangle = \frac{1}{2^{n/2}} \sum_{z \in \{0,1\}^n} |z\rangle$$

$$x = \langle w|s\rangle = \langle w|U|0^n\rangle = \frac{1}{2^{n/2}} \sum_z \langle w|z\rangle$$
$$= \frac{1}{2^{n/2}} = \frac{1}{\sqrt{N}}$$

Now $\left\lfloor \frac{\pi}{4\sin^{-1} x} \right\rceil = \left\lfloor \frac{\pi}{4\theta} \right\rceil$



Applying G k times rotates from $|r\rangle$ by angle $(2k+1)\theta$

want $(2k+1)\theta \approx \frac{\pi}{2}$

Solve for k: $\left\lfloor \frac{\pi}{4\theta} \right\rceil$ ← closest integer