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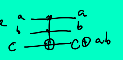
Toffoli gate



Clifford gate set. $\{H, S, CNOT\}$

T-gate

$S|0\rangle = |0\rangle$ $Z|0\rangle = |0\rangle$

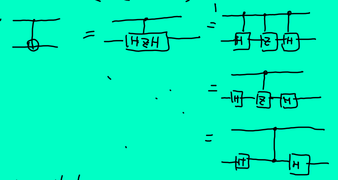
$S|1\rangle = i|1\rangle$ $Z|1\rangle = -|1\rangle$

* Note: Pauli X, Y, Z can be ~~co~~ defined from Clifford gates. e.g. $Z = S^2$, $\boxed{X = HZH}$

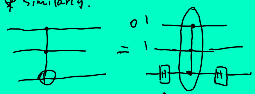
$|0\rangle \xrightarrow{H} |+\rangle \xrightarrow{Z} |-\rangle \xrightarrow{H} |1\rangle$

* $HXH = Z$ (side note)

CX.



* similarly.



* So if we can define CCZ using Clifford + T, then we have Toffoli.

$a, b, c \in \{0,1\}$.

* $CCZ|a,b,c\rangle = (-1)^{a \cdot b \cdot c} |a,b,c\rangle$

$= e^{i\pi abc} |a,b,c\rangle$

$T|0\rangle = |0\rangle$

$T|1\rangle = e^{i\frac{\pi}{4}}|1\rangle$

$CX|a,b\rangle = |a, a\oplus b\rangle$

$T|a\rangle = e^{i\frac{\pi}{4}a}|a\rangle$

$T^\dagger|0\rangle = |0\rangle$

$T^\dagger|1\rangle = e^{-i\frac{\pi}{4}}|1\rangle$

$T^\dagger|a\rangle = e^{-i\frac{\pi}{4}a}|a\rangle$

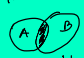
$T^2 = S$, $S^2 = Z$ so $T^\dagger = T^7$

$Z^2 = I$. $T^8 = I$.

* $\rightarrow e^{i\pi abc} = e^{i\frac{\pi}{4}(4abc)}$

* Side note. (inclusion-exclusion)

$|A \cap B| = |A| + |B| - |A \cup B|$



$2\, x \cdot y \stackrel{?}{=} x + y - x \oplus y$.

binary version of incl. exc.

$4abc = 2(2ab)c$

$= 2(a + b - a\oplus b)\cdot c$

$= 2(ac + bc - (a\oplus b)\cdot c)$

$= 2ac + 2bc - 2(a\oplus b)\cdot c$

$= a + c - a\oplus c + b + c - b\oplus c - (a\oplus b + c - a\oplus b\oplus c)$

$= a + c - a\oplus c + b + \cancel{c} - b\oplus c - a\oplus b - \cancel{c} + a\oplus b\oplus c$

$= a + b + c - a\oplus c - b\oplus c - a\oplus b + a\oplus b\oplus c$.

$(a+b)\oplus c$

$= a\oplus c + b\oplus c$

$= \bar{a} + \bar{b}$

* So $CCZ|a,b,c\rangle =$

$e^{i\frac{\pi}{4}(4abc)} |a,b,c\rangle =$

$e^{i\frac{\pi}{4}(a+b+c-a\oplus c-b\oplus c-a\oplus b+a\oplus b\oplus c)} |a,b,c\rangle$

$= e^{i\frac{\pi}{4}a} \cdot e^{i\frac{\pi}{4}b} \cdot e^{i\frac{\pi}{4}c} \cdot e^{-i\frac{\pi}{4}(a\oplus c)} \cdot e^{-i\frac{\pi}{4}(b\oplus c)} \cdot e^{-i\frac{\pi}{4}(a\oplus b)} \cdot e^{i\frac{\pi}{4}(a\oplus b\oplus c)}$ $|a,b,c\rangle$

$e^{i\frac{\pi}{4}a}$